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ON A NONLINEAR WAVE EQUATION ASSOCIATED WITH THE NONHOMOGENEOUS BOUNDARY CONDITIONS INVOLVING CONVOLUTION

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Abstract. The paper deals with the initial-boundary value problem for the nonlinear wave equation

$$\begin{cases} u_{tt} - \frac{\partial}{\partial x}\left(\mu\left(x,t\right)u_x\right) + K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t = F\left(x,t\right),\ 0<x<1,\ 0<t<T, \\ \mu\left(0,t\right)u_x\left(0,t\right) = g_0\left(t\right) + \int_0^t k_0\left(t-s\right)u\left(0,s\right)ds, \\ -\mu\left(1,t\right)u_x\left(1,t\right) = g_1\left(t\right) + \int_0^t k_1\left(t-s\right)u\left(1,s\right)ds, \\ u\left(x,0\right) = \tilde{u}_0\left(x\right),\ u_t\left(x,0\right) = \tilde{u}_1\left(x\right), \end{cases} \tag{1}$$

where $K \geqslant 0$, $\lambda > 0$ is given constants, p, $q \geqslant 2$ and F, μ, g_0, g_1, k_0, $k_1, \tilde{u}_0, \tilde{u}_1$ are given functions. In this paper, we consider two main parts. In Part 1, under conditions $(\tilde{u}_0, \tilde{u}_1, g_0, g_1, k_0, k_1) \in H^2 \times H^1 \times \left(H^2\left(0,T\right)\right)^2 \times \left(W^{2,1}\left(0,T\right)\right)^2$, $\mu \in C^1\left(\overline{Q}_T\right)$, $\mu_{tt} \in L^1\left(0,T;L^\infty\right)$, $\mu\left(x,t\right) \geq \mu_0 > 0$ a.e. $(x,t) \in Q_T$, we prove a theorem of existence and uniqueness of a weak solution u of (1). The proof is based on the Faedo-Galerkin method associated with the weak compact method. In Part 2, we obtain an asymptotic expansion of the solution u of (1) up to order $N+1$ in two small parameters K, λ.

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1. Introduction

In this paper, we consider the initial-boundary value problem for the nonlinear wave equation:

$$u_{tt} - \frac{\partial}{\partial x}\left(\mu\left(x,t\right)u_x\right) + f\left(u,u_t\right) = F\left(x,t\right),\ 0<x<1,\ 0<t<T, \quad (1.1)$$

$$\mu\left(0,t\right)u_x\left(0,t\right) = g_0\left(t\right) + \int_0^t k_0\left(t-s\right)u\left(0,s\right)ds, \quad (1.2)$$

$$-\mu\left(1,t\right)u_x\left(1,t\right) = g_1\left(t\right) + \int_0^t k_1\left(t-s\right)u\left(1,s\right)ds, \quad (1.3)$$

$$u\left(x,0\right) = \tilde{u}_0\left(x\right),\ u_t\left(x,0\right) = \tilde{u}_1\left(x\right), \quad (1.4)$$

where $f\left(u,u_t\right) = K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t$, with $K \geqslant 0$, $\lambda > 0$, p, $q \geqslant 2$ and F, μ, g_0, g_1, k_0, k_1, $\tilde{u}_0$, $\tilde{u}_1$ are given functions satisfying conditions specified later.

In [1], An and Trieu studied a special case of problem (1.1) and (1.4) associated with the following boundary conditions:

$$u_x\left(0,t\right) = g_0\left(t\right) + h_0u\left(0,t\right) + \int_0^t k_0\left(t-s\right)u\left(0,s\right)ds, \quad (1.5)$$

$$u\left(1,t\right) = 0, \quad (1.6)$$

with $\mu \equiv 1$, $\tilde{u}_0 = \tilde{u}_1 \equiv 0$, and $f\left(u,u_t\right) = Ku + \lambda u_t$ with $K \geqslant 0, \lambda \geqslant 0$ given constants, and g_0, k_0 are given functions. In the latter case the problem (1.1), (1.4), (1.5) and (1.6) is a mathematical model describing the shock of a rigid body and a linear viscoelastic bar resting on a rigid base [1].

In [2], Bergounioux, Long and Dinh studied problem (1.1), (1.4) with the mixed boundary conditions (1.2) and (1.3) stand for

$$u_x\left(0,t\right) = g\left(t\right) + hu\left(0,t\right) - \int_0^t k\left(t-s\right)u\left(0,s\right)ds, \quad (1.7)$$

$$u_x\left(1,t\right) + K_1u\left(1,t\right) + \lambda_1u_t\left(1,t\right) = 0, \quad (1.8)$$

where $f\left(u,u_t\right) = Ku + \lambda u_t$, with $K \geqslant 0, \lambda \geqslant 0$, $h \geqslant 0$, $K_1 \geqslant 0$, $\lambda_1 > 0$ are given constants and g, k are given functions.

In [9], Long, Dinh and Diem studied problem (1.1), (1.4), (1.7) and (1.8) for the case of $f\left(u,u_t\right) = K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t$, where K, $\lambda \geqslant 0$; $p,q \geqslant 2$ and $\left(\tilde{u}_0,\ \tilde{u}_1\right) \in H^2 \times H^1$.

In [12], Ngoc, Hang and Long gave the unique existence, stability and asymptotic expansion of the problem (1.1)-(1.4) for the case of $f\left(u,u_t\right) = F\left(u\right) + \lambda u_t$, where λ is a constant, $F \in C^1\left(\mathbb{R}\right)$ satisfies the following conditions $\int_0^z F\left(s\right)ds \geq -C_1z^2 - C_1'$ for all $z \in \mathbb{R}$, C_1, $C_1' > 0$ given constants.

In this paper, we consider two main parts. In Part 1, under conditions $(\tilde{u}_0, \tilde{u}_1, g_0, g_1, k_0, k_1) \in H^2 \times H^1 \times \left(H^2(0,T)\right)^2 \times \left(W^{2,1}(0,T)\right)^2$, $\mu \in C^1\left(\overline{Q}_T\right)$, $\mu_{tt} \in L^1(0,T;L^\infty)$, $\mu(x,t) \geq \mu_0 > 0$ a.e. $(x,t) \in Q_T$, we prove a theorem of existence and uniqueness of a weak solution u of problem (1.1)-(1.4) corresponding to $f(u,u_t) = K|u|^{p-2}u + \lambda|u_t|^{q-2}u_t$, with $K \geqslant 0$, $\lambda > 0$, p, $q \geqslant 2$. The proof is based on the Faedo-Galerkin method associated with the weak compact method. Finally, in Part 2 we obtain an asymptotic expansion of the solution u of the problem (1.1)-(1.4) up to order $N+1$ in two small parameters K, λ. The result here may be considered as the generalizations of this in [1, 2, 7–12].

2. The existence and uniqueness theorem

Put $\Omega = (0,1)$, $Q_T = \Omega \times (0,T)$, $T > 0$. We omit the definitions of usual function spaces: $C^m\left(\overline{\Omega}\right)$, $L^p(\Omega)$, $W^{m,p}(\Omega)$.

We denote

$$W^{m,p} = W^{m,p}(\Omega), \quad L^p = W^{0,p}(\Omega), \quad H^m = W^{m,2}(\Omega),$$

$1 \leq p \leq \infty$, $m = 0, 1, \cdots$. The norm in L^2 is denoted by $\|\cdot\|$. We also denote by $\langle\cdot,\cdot\rangle$ the scalar product in L^2 or pair of dual scalar products of continuous linear functionals with an element of a function space. The norm in L^∞ is denoted by $\|\cdot\|_\infty$. We denote by $\|\cdot\|_X$ the norm of a Banach space X and by X' the dual space of X.

We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$, the Banach space of the real functions $u: (0,T) \to X$ measurable such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty$$

or

$$\|u\|_{L^\infty(0,T;X)} = ess\sup \|u(t)\|_X \quad \text{for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

On H^1 we will use the following norm

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}. \tag{2.1}$$

Then we have the following lemma.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0([0,1])$ is compact and*

$$\|v\|_{C^0([0,1])} \leq \sqrt{2}\,\|v\|_{H^1}, \quad \text{for all } v \in H^1. \tag{2.2}$$

Proof. The proof of this lemma is straightforward, and we omit the details. □

We make the following assumptions:

(H_1) $\tilde{u}_0 \in H^2,\ \tilde{u}_1 \in H^1,$

(H_2) $g_0,\ g_1 \in H^2,$

(H_3) $k_0,\ k_1 \in W^{2,1},$

(H_4) $\mu \in C^1\left(\overline{Q}_T\right),\ \mu_{tt} \in L^1\left(0,T;L^\infty\right),\ \mu\left(x,t\right) \geq \mu_0 > 0,$

(H_5) $K \geq 0,\ \lambda > 0,\ p \geq 2,\ q \geq 2,$

(H_6) $F,\ F_t \in L^2\left(Q_T\right),$

(H_7) $\mu\left(0,0\right)\tilde{u}_{0x}\left(0\right) = g_0\left(0\right), -\mu\left(1,0\right)\tilde{u}_{0x}\left(1\right) = g_1\left(0\right).$

Then we have the following theorem.

Theorem 2.2. *Let* $(H_1) - (H_7)$ *hold. Then, for every* $T > 0$, *there exists a weak unique solution* u *of problem* (1.1)–(1.4) *such that*

$$u \in L^\infty\left(0,T;H^2\right),\quad u_t \in L^\infty\left(0,T;H^1\right),\quad u_{tt} \in L^\infty\left(0,T;L^2\right). \tag{2.3}$$

Proof. **Step 1**. *The Faedo–Galerkin approximation.*

Let $\{w_j\}_{j\in\mathbb{N}}$ be a denumerable base of H^2. We find the approximation solution of problem (1.1)–(1.4) in the form

$$u_m\left(t\right) = \sum_{j=1}^{m} c_{mj}\left(t\right) w_j, \tag{2.4}$$

where the coefficient functions c_{mj} satisfy the system of ordinary differential equations

$$\begin{aligned} &\left\langle u_m''\left(t\right), w_j\right\rangle + \left\langle \mu\left(t\right) u_{mx}\left(t\right), w_{jx}\right\rangle + P_m\left(t\right) w_j\left(0\right) + Q_m\left(t\right) w_j\left(1\right) \\ &\quad + K\left\langle \left|u_m\right|^{p-2} u_m, w_j\right\rangle + \lambda\left\langle \left|u_m'\right|^{q-2} u_m', w_j\right\rangle \\ &= \left\langle F\left(t\right), w_j\right\rangle,\quad 1 \leq j \leq m, \end{aligned} \tag{2.5}$$

$$\begin{aligned} P_m\left(t\right) &= g_0\left(t\right) + \int_0^t k_0\left(t-s\right) u_m\left(0,s\right) ds, \\ Q_m\left(t\right) &= g_1\left(t\right) + \int_0^t k_1\left(t-s\right) u_m\left(1,s\right) ds, \end{aligned} \tag{2.6}$$

$$u_m\left(0\right) = \tilde{u}_0,\ u_m'\left(0\right) = \tilde{u}_1. \tag{2.7}$$

From the assumptions of Theorem 2.2, system (2.5)–(2.7) has a solution u_m on an interval $[0,T_m] \subset [0,T]$. The following estimates allow one to take $T_m = T$ for all m.

Step 2. *A priori estimates I.*

Substituting (2.6) into (2.5), then multiplying the j^{th} equation of (2.5) by $c'_{mj}(t)$ and summing with respect to j, and afterwards integrating with respect to the time variable from 0 to t, we get after some rearrangements

$$\begin{aligned} S_m(t) &= \bar{S}_0 + \int_0^t ds \int_0^1 \mu'(x,s)\, u_{mx}^2(x,s)\, dx - 2\int_0^t P_m(s)\, u'_m(0,s)\, ds \\ &\quad -2\int_0^t Q_m(s)\, u'_m(1,s)\, ds + 2\int_0^t \left\langle F(s), u'_m(s)\right\rangle ds \\ &= \bar{S}_0 + \sum\nolimits_{j=1}^4 I_j, \end{aligned} \tag{2.8}$$

where

$$\left\{ \begin{aligned} S_m(t) &= \|u'_m(t)\|^2 + \left\|\sqrt{\mu(t)}u_{mx}(t)\right\|^2 + \tfrac{2K}{p}\|u_m(t)\|_{L^p}^p \\ &\quad +2\lambda \textstyle\int_0^t \|u'_m(s)\|_{L^q}^q\, ds, \\ \bar{S}_0 &= \|\tilde{u}_1\|^2 + \left\|\sqrt{\mu(0)}\tilde{u}_{0x}\right\|^2. \end{aligned} \right. \tag{2.9}$$

We will estimate the following four integrals in the right-hand side of (2.8).

First integral I_1. By means of the following inequality from (2.9)

$$\|u_{mx}(t)\|^2 \le \frac{1}{\mu_0} S_m(t), \tag{2.10}$$

it follows that

$$I_1 = \int_0^t ds \int_0^1 \mu'(x,s)\, u_{mx}^2(x,s)\, dx \le \frac{1}{\mu_0}\left\|\mu'\right\|_{L^\infty(Q_T)} \int_0^t S_m(s)\, ds. \tag{2.11}$$

Second integral I_2. By using integration by parts, it follows that

$$\begin{aligned} I_2 &= -2\int_0^t P_m(s)\, u'_m(0,s)\, ds \qquad (2.12) \\ &= 2g_0(0)\,\tilde{u}_0(0) - 2g_0(t)\, u_m(0,t) + 2\int_0^t g'_0(s)\, u_m(0,s)\, ds \\ &\quad +2k_0(0)\int_0^t u_m^2(0,s)\, ds - 2u_m(0,t)\int_0^t k_0(t-s)\, u_m(0,s)\, ds \\ &\quad +2\int_0^t \left[\int_0^s k'_0(s-r)\, u_m(0,r)\, dr\right] u_m(0,s)\, ds. \end{aligned}$$

By Lemma 2.1 and the following inequality

$$2ab \le \beta a^2 + \frac{1}{\beta} b^2, \quad \text{for all } a,\ b \in \mathbb{R},\ \beta > 0, \tag{2.13}$$

it follows from (2.12) that

$$\begin{aligned} I_2 &\leq 2\left|g_0(0)\tilde{u}_0(0)\right| + \frac{2}{\beta}g_0^2(t) + 2\left\|g_0'\right\|_{L^2(0,T)}^2 + 2\beta\left\|u_m(t)\right\|_{H^1}^2 \\ &\quad + \left(1 + 4\left|k_0(0)\right| + \frac{4}{\beta}\left\|k_0\right\|_{L^2(0,T)}^2 + \left\|k_0'\right\|_{L^1(0,T)}\right)\int_0^t \left\|u_m(s)\right\|_{H^1}^2 ds \\ &\leq C_T + 2\beta\left\|u_m(t)\right\|_{H^1}^2 + C_T\int_0^t \left\|u_m(s)\right\|_{H^1}^2 ds, \end{aligned} \tag{2.14}$$

for all $\beta > 0$, where C_T always indicates a constant depending on T.

Third integral I_3. Similarly, we obtain

$$\begin{aligned} I_3 &= -2\int_0^t Q_m(s)\,u_m'(1,s) \leq C_T + 2\beta\left\|u_m(t)\right\|_{H^1}^2 \\ &\quad + C_T\int_0^t \left\|u_m(s)\right\|_{H^1}^2 ds. \end{aligned} \tag{2.15}$$

Fourth integral I_4. By means of the inequality (2.13), we have

$$\begin{aligned} I_4 &= 2\int_0^t \left\langle F(s), u_m'(s)\right\rangle ds \\ &\leq \frac{1}{\beta}\int_0^t \left\|F(s)\right\|^2 ds + \beta\int_0^t \left\|u_m'(s)\right\|^2 ds \\ &\leq \frac{1}{\beta}\left\|F\right\|_{L^2(Q_T)}^2 + \beta\int_0^t S_m(s)\,ds. \end{aligned} \tag{2.16}$$

We will use the following inequalities from lemma 2.1 and (2.10)

$$\begin{aligned} \left\|u_m(t)\right\|^2 &\leq \left(\left\|u_m(0)\right\|^2 + \int_0^t \left\|u_m'(s)\right\| ds\right)^2 \\ &\leq 2\left\|\tilde{u}_0\right\|^2 + 2t\int_0^t S_m(s)\,ds \\ &\leq C_0 + 2t\int_0^t S_m(s)\,ds \end{aligned} \tag{2.17}$$

and

$$\left\|u_m(t)\right\|_{H^1}^2 = \left\|u_m(t)\right\|^2 + \left\|u_{mx}(t)\right\|^2 \leq C_0 + 2t\int_0^t S_m(s)\,ds + \frac{1}{\mu_0}S_m(t), \tag{2.18}$$

where C_0 always indicates a constant depending on $\tilde{u}_0$. Combining (2.8), (2.11), (2.14)–(2.16) and choose $\beta = \frac{1}{8}\mu_0$, we obtain after some rearrangements

$$S_m(t) \leq M_T + 2\bar{S}_0 + N_T\int_0^t S_m(s)\,ds, \tag{2.19}$$

where

$$\begin{cases} M_T = 2\left(2C_T + 4\beta C_0 + 2TC_0C_T + \frac{1}{\beta}\|F\|^2_{L^2(Q_T)}\right), \\ N_T = 2\left(\beta + 8\beta T + 4T^2C_T + \frac{2C_T}{\mu_0} + \frac{1}{\mu_0}\|\mu'\|_{L^\infty(Q_T)}\right). \end{cases} \tag{2.20}$$

From assumptions (H_1)-(H_4), (H_6), (H_7) and Lemma 2.1, there exist a positive constant $\bar{M}_T$ depending on $\tilde{u}_0,\ \tilde{u}_1,\ k_0,\ k_1,\ g_0,\ g_1,\ F,\ \mu$ such that

$$S_m(t) \le \bar{M}_T + N_T\int_0^t S_m(s)\,ds, \quad \forall\ m,\ \ \forall\ t \in [0,T]. \tag{2.21}$$

By Gronwall's Lemma, we deduce from (2.21) that

$$S_m(t) \le \bar{M}_T \exp(tN_T) \le C_T, \quad \forall\ t \in [0,T]. \tag{2.22}$$

A priori estimates II.

Now differentiating (2.5) with respect to t, we have

$$\begin{aligned} &\left\langle u_m'''(t), w_j\right\rangle + \left\langle \mu(t)\, u_{mx}'(t), w_{jx}\right\rangle + \left\langle \mu'(t)\, u_{mx}(t), w_{jx}\right\rangle \\ &\quad + P_m'(t)\, w_j(0) + Q_m'(t)\, w_j(1) + K(p-1)\left\langle |u_m|^{p-2} u_m', w_j\right\rangle \\ &\quad + \lambda(q-1)\left\langle \left|u_m'\right|^{q-2} u_m'', w_j\right\rangle \\ &= \left\langle F'(t), w_j\right\rangle, \quad 1 \le j \le m. \end{aligned} \tag{2.23}$$

Multiplying the j^{th} equation of (2.23) by $c_{mj}''(t)$ and summing with respect to j, and afterwards integrating with respect to the time variable from 0 to t, we get after some rearrangements

$$\begin{aligned} &X_m(t) \\ &= X_m(0) + \int_0^t ds \int_0^1 \mu'(x,s)\left|u_{mx}'(x,s)\right|^2 dx \\ &\quad -2\int_0^t ds\int_0^1 \mu'(x,s)\, u_{mx}(x,s)\, u_{mx}''(x,s)\,dx + 2\int_0^t \left\langle F(s), u_m'(s)\right\rangle ds \\ &\quad -2K(p-1)\int_0^t \left\langle |u_m|^{p-2} u_m', u_m''\right\rangle ds - 2\int_0^t P_m'(s)\, u_m''(0,s)\,ds \\ &\quad -2\int_0^t Q_m'(s)\, u_m''(1,s)\,ds, \end{aligned} \tag{2.24}$$

where

$$\begin{aligned} X_m(t) \;\; &= \;\; \left\|u_m''(t)\right\|^2 + ||\sqrt{\mu(t)}u_{mx}'(t)\,||^2 \\ &\quad +2\lambda(q-1)\int_0^t ds\int_0^1 \left|u_m'(x,s)\right|^{q-2}\left|u_m''(x,s)\right|^2 dx. \end{aligned} \tag{2.25}$$

Integrating by parts, we have

$$\begin{aligned}
&\int_0^t ds \int_0^1 \mu'(x,s)\, u_{mx} u''_{mx} dx \\
&= \left\langle \mu'(t)\, u_{mx}(t), u'_{mx}(t) \right\rangle - \left\langle \mu'(0)\, \tilde{u}_{0x}, \tilde{u}_{1x} \right\rangle \\
&\quad - \int_0^t \left\langle \mu''(s)\, u_{mx}(s), u'_{mx}(s) \right\rangle ds \\
&\quad - \int_0^t ds \int_0^1 \mu'(x,s) \left| u'_{mx}(x,s) \right|^2 dx, \qquad (2.26)
\end{aligned}$$

so we can rewrite (2.24) as follows

$$\begin{aligned}
&X_m(t) \\
&= X_m(0) + 2\left\langle \mu'(0)\, \tilde{u}_{0x}, \tilde{u}_{1x} \right\rangle + 3\int_0^t ds \int_0^1 \mu'(x,s) \left| u'_{mx}(x,s) \right|^2 dx \\
&\quad -2\left\langle \mu'(t)\, u_{mx}(t), u'_{mx}(t) \right\rangle + 2\int_0^t \left\langle \mu''(s)\, u_{mx}(s), u'_{mx}(s) \right\rangle ds \\
&\quad +2\int_0^t \left\langle F'(s), u''_m(s) \right\rangle ds - 2K(p-1)\int_0^t \left\langle |u_m|^{p-2} u'_m, u''_m \right\rangle ds \\
&\quad -2\int_0^t P'_m(s)\, u''_m(0,s)\, ds - 2\int_0^t Q'_m(s)\, u''_m(1,s)\, ds \\
&= X_m(0) + 2\left\langle \mu'(0)\, \tilde{u}_{0x}, \tilde{u}_{1x} \right\rangle + \sum_{i=1}^7 J_i. \qquad (2.27)
\end{aligned}$$

From the assumptions (H_1), (H_4), (H_6), (H_7), (2.25) and the imbedding $H^1(\Omega) \hookrightarrow C^0\left(\overline{\Omega}\right)$, there exists a positive constant D_0 depending on $\tilde{u}_0$, $\tilde{u}_1$, μ, F, such that

$$\begin{aligned}
&X_m(0) + 2\left\langle \mu'(0)\, \tilde{u}_{0x}, \tilde{u}_{1x} \right\rangle \\
&= \left\| u''_m(0) \right\|^2 + ||\sqrt{\mu(0)}\tilde{u}_{1x}||^2 + 2\left\langle \mu'(0)\, \tilde{u}_{0x}, \tilde{u}_{1x} \right\rangle \qquad (2.28) \\
&= \left\| \mu(0)\, \tilde{u}_{0xx} + \mu_x(0)\, \tilde{u}_{0x} - f(\tilde{u}_0, \tilde{u}_1) + F(0) \right\|^2 \\
&\quad + ||\sqrt{\mu(0)}\tilde{u}_{1x}||^2 + 2\left\langle \mu'(0)\, \tilde{u}_{0x}, \tilde{u}_{1x} \right\rangle \\
&\le D_0.
\end{aligned}$$

Using the lemma 2.1, (2.22) and (2.25). We have some inequalities

$$\left\| u'_{mx}(t) \right\|^2 \le \frac{1}{\mu_0} X_m(t), \qquad (2.29)$$

$$\begin{aligned}
|u_m(x,t)| &\le \|u_m(t)\|_{C^0(\overline{\Omega})} \le \sqrt{2}\, \|u_m(t)\|_{H^1} \\
&\le \sqrt{2}\, \|u_m\|_{L^\infty(0,T;H^1)} \le C_T, \qquad (2.30)
\end{aligned}$$

$$\begin{aligned}
\left|u_m'(x,t)\right| &\leq \left\|u_m'(t)\right\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\left\|u_m'(t)\right\|_{H^1} \\
&\leq \sqrt{2}\left(S_m(t)+\frac{1}{\mu_0}X_m(t)\right)^{1/2} \leq \sqrt{2}\left(C_T+\frac{1}{\mu_0}X_m(t)\right)^{1/2} \\
&\leq \sqrt{\frac{2}{\mu_0}}\sqrt{X_m(t)}+D_T. \qquad (2.31)
\end{aligned}$$

We will estimate the following seven integrals in the right-hand side of (2.27).

First integral J_1.

$$\begin{aligned}
J_1 &= 3\int_0^t ds\int_0^1 \mu'(x,s)\left|u_{mx}'(x,s)\right|^2 dx \\
&\leq 3\left\|\mu'\right\|_{L^\infty(Q_T)}\int_0^t \left\|u_{mx}'(s)\right\|^2 ds \\
&\leq \frac{3}{\mu_0}\left\|\mu'\right\|_{L^\infty(Q_T)}\int_0^t X_m(s)\,ds \leq C_T\int_0^t X_m(s)\,ds. \qquad (2.32)
\end{aligned}$$

Second integral J_2.

$$\begin{aligned}
J_2 &= -2\left\langle \mu'(t)\,u_{mx}(t)\,,u_{mx}'(t)\right\rangle \leq 2\left\|\mu'\right\|_{L^\infty(Q_T)}\left\|u_{mx}(t)\right\|\left\|u_{mx}'(t)\right\| \\
&\leq \frac{2}{\mu_0}\left\|\mu'\right\|_{L^\infty(Q_T)}\sqrt{S_m(t)}\sqrt{X_m(t)} \leq \frac{2\sqrt{C_T}}{\mu_0}\left\|\mu'\right\|_{L^\infty(Q_T)}\sqrt{X_m(t)} \\
&\leq \frac{C_T}{\beta\mu_0^2}\left\|\mu'\right\|_{L^\infty(Q_T)}^2+\beta X_m(t) \leq C_T+\beta X_m(t)\,. \qquad (2.33)
\end{aligned}$$

Third integral J_3.

$$\begin{aligned}
J_3 &= 2\int_0^t \left\langle \mu''(s)\,u_{mx}(s)\,,u_{mx}'(s)\right\rangle ds \\
&\leq 2\int_0^t \left\|\mu''(s)\right\|_\infty \left\|u_{mx}(s)\right\|\left\|u_{mx}'(s)\right\| ds \\
&\leq \frac{2}{\mu_0}\int_0^t \left\|\mu''(s)\right\|_\infty \sqrt{S_m(s)}\sqrt{X_m(s)}ds \\
&\leq \frac{2\sqrt{C_T}}{\mu_0}\int_0^t \left\|\mu''(s)\right\|_\infty \sqrt{X_m(s)}ds \\
&\leq \frac{\sqrt{C_T}}{\mu_0}\int_0^t \left\|\mu''(s)\right\|_\infty (1+X_m(s))ds \\
&\leq \frac{\sqrt{C_T}}{\mu_0}\left\|\mu''\right\|_{L^1(0,T;L^\infty)}+\frac{\sqrt{C_T}}{\mu_0}\int_0^t \left\|\mu''(s)\right\|_\infty X_m(s)\,ds \\
&\leq C_T+C_T\int_0^t \left\|\mu''(s)\right\|_\infty X_m(s)\,ds. \qquad (2.34)
\end{aligned}$$

Fourth integral J_4.

$$\begin{aligned} J_4 &= 2\int_0^t \left\langle F'(s), u_m''(s)\right\rangle ds \le 2\int_0^t \left\|F'(s)\right\| \left\|u_m''(s)\right\| ds \\ &\le \int_0^t \left\|F'(s)\right\|^2 ds + \int_0^t \left\|u_m''(s)\right\|^2 ds \le C_T + \int_0^t X_m(s)\, ds. \quad (2.35)\end{aligned}$$

Fifth integral J_5.

$$\begin{aligned} J_5 &= -2K(p-1)\int_0^t \left\langle |u_m|^{p-2} u_m', u_m''\right\rangle ds \\ &\le 2K(p-1) C_T^{p-2} \int_0^t \left\|u_m'(s)\right\| \left\|u_m''(s)\right\| ds \\ &\le 2K(p-1) C_T^{p-2} \int_0^t \sqrt{S_m(s)}\sqrt{X_m(s)} ds \\ &\le C_T + C_T \int_0^t X_m(s)\, ds. \end{aligned} \tag{2.36}$$

Sixth integral J_6.

$$\begin{aligned} J_6 &= -2\int_0^t P_m'(s)\, u_m''(0,s)\, ds \\ &= 2P_m'(0)\,\tilde{u}_1(0) - 2P_m'(0)\, u_m'(0,t) \\ &\quad -2u_m'(0,t)\int_0^t P_m''(s)\, ds + 2\int_0^t P_m''(s)\, u_m'(0,s)\, ds \\ &= \sum\nolimits_{i=1}^4 J_6^{(i)}. \end{aligned} \tag{2.37}$$

We will estimate integrals in the right-hand side of (2.37) by means of (2.7), (2.30), (2.31):

$$J_6^{(1)} \le 2\left|P_m'(0)\,\tilde{u}_1(0)\right| \le C_0, \tag{2.38}$$

$$\begin{aligned} J_6^{(2)} &\le 2\left|P_m'(0)\, u_m'(0,t)\right| = 2\left|g_0'(0) + k_0(0)\,\tilde{u}_0(0)\right| \; \left\|u_m'(t)\right\|_{C^0(\overline{\Omega})} \\ &\le 2C_0\left(\sqrt{\frac{2}{\mu_0}}\sqrt{X_m(t)} + D_T\right) \le \beta X_m(t) + D_T, \end{aligned} \tag{2.39}$$

$$\begin{aligned} J_6^{(3)} &\le 2\sqrt{2}\left\|u_m'(t)\right\|_{H^1} \int_0^t \left|P_m''(s)\right| ds \\ &\le \beta\left\|u_m'(t)\right\|_{H^1}^2 + \frac{2}{\beta} t \int_0^t \left|P_m''(s)\right|^2 ds, \end{aligned} \tag{2.40}$$

we have

$$\begin{aligned}\left|P_m''(t)\right| &\leq \left|g_0''(t)\right| + \sqrt{2}\left|k_0(0)\right|\left\|u_m'(t)\right\|_{H^1} + \sqrt{2}\left|k_0'(0)\right|\left\|u_m(t)\right\|_{H^1} \\ &\quad + \sqrt{2}\int_0^t \left|k_0''(t-s)\right|\left\|u_m(s)\right\|_{H^1} ds \\ &\leq C_T + \sqrt{2}\left|k_0(0)\right|\left\|u_m'(t)\right\|_{H^1} + \sqrt{2}C_T\left|k_0'(0)\right| \\ &\quad + \sqrt{2}C_T\int_0^t \left|k_0''(s)\right| ds \\ &\leq C_T + \sqrt{2}\left|k_0(0)\right|\left\|u_m'(t)\right\|_{H^1}, \end{aligned} \tag{2.41}$$

so it follows from (2.40) and (2.41) that

$$\begin{aligned}J_6^{(3)} &\leq \beta\left\|u_m'(t)\right\|_{H^1}^2 + \frac{2}{\beta}t\int_0^t \left(C_T + \sqrt{2}\left|k_0(0)\right|\left\|u_m'(s)\right\|_{H^1}\right)^2 ds \\ &\leq C_T + \beta X_m(t) + C_T\int_0^t X_m(s)\, ds, \end{aligned} \tag{2.42}$$

$$\begin{aligned}J_6^{(4)} &\leq 2\int_0^t \left|P_m''(s)\right|\left|u_m'(0,s)\right| ds \\ &\leq 2\sqrt{2}\int_0^t \left(C_T + \sqrt{2}\left|k_0(0)\right|\left\|u_m'(s)\right\|_{H^1}\right)\left\|u_m'(s)\right\|_{H^1} ds \\ &\leq C_T + C_T\int_0^t X_m(s)\, ds. \end{aligned} \tag{2.43}$$

We deduce from (2.38), (2.39), (2.42) and (2.43) that

$$J_6 \leq C_T + 2\beta X_m(t) + 2C_T\int_0^t X_m(s)\, ds. \tag{2.44}$$

Seventh integral J_7. Similarly, the last term in the right-hand side of (2.27) is estimated

$$J_7 = -2\int_0^t Q_m'(s)\, u_m''(1,s)\, ds \leq C_T + 2\beta X_m(t) + 2C_T\int_0^t X_m(s)\, ds. \tag{2.45}$$

Combining (2.27), (2.28), (2.32)–(2.36), (2.44) and (2.45) we obtain after some rearrangements

$$\begin{aligned}X_m(t) &\leq D_0 + 6C_T + 5\beta X_m(t) + (1+6C_T)\int_0^t X_m(s)\, ds \\ &\quad + C_T\int_0^t \left\|\mu''(s)\right\|_\infty X_m(s)\, ds \\ &\leq C_T + 5\beta X_m(t) + C_T\int_0^t \left(1+\left\|\mu''(s)\right\|_\infty\right) X_m(s)\, ds, \end{aligned} \tag{2.46}$$

where C_T is a positive constant depending on T. Choosing $\beta = \frac{1}{10}$, from (2.46), we obtain that

$$X_m(t) \leq 2C_T + 2C_T \int_0^t \left(1 + \left\|\mu''(s)\right\|_\infty\right) X_m(s)\, ds. \tag{2.47}$$

Applying Gronwall's inequality, it follows from (2.47) that

$$\begin{aligned} X_m(t) &\leq 2C_T \exp\left(2C_T \int_0^t \left(1 + \left\|\mu''(s)\right\|_\infty\right) ds\right) \\ &\leq 2C_T \exp\left(2C_T \int_0^T \left(1 + \left\|\mu''(s)\right\|_\infty\right) ds\right) \leq C_T, \end{aligned} \tag{2.48}$$

for all $t \in [0, T]$.

On the other hand, from the assumptions (H_2), (H_3) and (2.6), (2.22), (2.48) we deduce that

$$\|P_m\|_{W^{2,\infty}(0,T)} \leq C_T, \tag{2.49}$$

$$\|Q_m\|_{W^{2,\infty}(0,T)} \leq C_T, \tag{2.50}$$

where C_T is a positive constant depending on T.

Step 3. *Limiting process.*

From (2.22) and (2.48)–(2.50), we deduce the existence of a subsequence of $\{(u_m, P_m, Q_m)\}$ still also so denoted, such that

$$\begin{cases} u_m \to u & \text{in } L^\infty\left(0, T; H^1\right) \text{ weak}^*, \\ u'_m \to u' & \text{in } L^\infty\left(0, T; H^1\right) \text{ weak}^*, \\ u'_m \to u' & \text{in } L^q(Q_T) \text{ weakly}, \\ u''_m \to u'' & \text{in } L^\infty\left(0, T; L^2\right) \text{ weak}^*, \\ u_m(0, \cdot) \to u(0, \cdot) & \text{in } W^{1,\infty}(0, T) \text{ weak}^*, \\ u_m(1, \cdot) \to u(1, \cdot) & \text{in } W^{1,\infty}(0, T) \text{ weak}^*, \\ P_m \to P & \text{in } W^{1,\infty}(0, T) \text{ weak}^*, \\ Q_m \to Q & \text{in } W^{1,\infty}(0, T) \text{ weak}^*. \end{cases} \tag{2.51}$$

By the compactness Lemma of Lions [8, p.57], we can deduce from (2.51) the existence of a subsequence still denoted by $\{(u_m, P_m, Q_m)\}$ such that

$$\begin{cases} u_m \to u & \text{strongly in } L^2(Q_T), \text{ and a.e. in } Q_T, \\ u'_m \to u' & \text{strongly in } L^2(Q_T), \text{ and a.e. in } Q_T, \\ u_m(0, \cdot) \to u(0, \cdot) & \text{strongly in } C^0([0, T]), \\ u_m(1, \cdot) \to u(1, \cdot) & \text{strongly in } C^0([0, T]), \\ P_m \to P & \text{strongly in } C^1([0, T]), \\ Q_m \to Q & \text{strongly in } C^1([0, T]). \end{cases} \tag{2.52}$$

From $(2.6)_1$ and $(2.52)_3$ we have

$$P_m(t) \to g_0(t) + \int_0^t k_0(t-s)\, u(0,s)\, ds \equiv P(t), \tag{2.53}$$

strongly in $C^0([0,T])$. Similarly, we have also

$$Q_m(t) \to g_1(t) + \int_0^t k_1(t-s)\, u(1,s)\, ds \equiv Q(t), \tag{2.54}$$

strongly in $C^0([0,T])$. Using the inequality

$$\left| |x|^{p-2} x - |y|^{p-2} y \right| \le (p-1) R^{p-2} |x-y|, \quad \forall\, x,y \in [-R,\ R], \tag{2.55}$$

for all $R > 0$ and all $p \ge 2$, it follows from (2.30) and $(2.52)_1$ that

$$|u_m|^{p-2} u_m \to |u|^{p-2} u, \tag{2.56}$$

strongly in $L^2(Q_T)$. Similarly, we can also obtain from (2.31), (2.48), $(2.52)_2$ and inequality (2.55) that

$$\left|u_m'\right|^{q-2} u_m' \to \left|u'\right|^{q-2} u', \tag{2.57}$$

strongly in $L^2(Q_T)$. Passing to the limit in (2.5), (2.7) by $(2.51)_{1,2,4}$, (2.53), (2.54) and (2.56), (2.57), we have u satisfying the problem

$$\begin{aligned} &\left\langle u''(t), v\right\rangle + \langle \mu(t)\, u_x(t), v_x\rangle + P(t)\, v(0) + Q(t)\, v(1) \\ &\quad + K \left\langle |u|^{p-2} u, v\right\rangle + \lambda \left\langle \left|u'\right|^{q-2} u', v\right\rangle \\ &= \langle F(t), v\rangle, \end{aligned} \tag{2.58}$$

$$u(0) = \tilde{u}_0, \quad u'(0) = \tilde{u}_1, \tag{2.59}$$

for all $v \in H^1$, where

$$\begin{aligned} P(t) &= g_0(t) + \int_0^t k_0(t-s)\, u(0,s)\, ds, \\ Q(t) &= g_1(t) + \int_0^t k_1(t-s)\, u(1,s)\, ds. \end{aligned} \tag{2.60}$$

On the other hand, we have from $(2.51)_{1,2,4}$, (2.58) and $(H_4), (H_6)$, that

$$\begin{aligned} u_{xx} &= \frac{1}{\mu(x,t)} \left(u'' + K |u|^{p-2} u + \lambda \left|u'\right|^{q-2} u' - \mu_x u_x - F \right) \\ &\in L^\infty\left(0,T; L^2\right). \end{aligned} \tag{2.61}$$

So $u \in L^\infty\left(0,T; H^2\right)$ and the existence is proved completely.

Step 4. *Uniqueness of the solution.*

Let u_1, u_2 be two weak solutions of problem (1.1)–(1.4) such that

$$u_j \in L^\infty\left(0,T;H^2\right),\ u_j' \in L^\infty\left(0,T;H^1\right),\ u_j'' \in L^\infty\left(0,T;L^2\right), \tag{2.62}$$

for $j = 1, 2$. Then $u = u_1 - u_2$ verifies the variational problem

$$\begin{cases} \langle u''(t), v\rangle + \langle \mu(t)\, u_x(t), v_x\rangle + P(t)\, v(0) + Q(t)\, v(1) \\ \qquad +K\left\langle |u_1|^{p-2} u_1 - |u_2|^{p-2} u_2, v\right\rangle \\ \qquad +\lambda\left\langle |u_1'|^{q-2} u_1' - |u_2'|^{q-2} u_2', v\right\rangle = 0 \quad \text{for all } v \in H^1, \\ u(0) = u'(0) = 0, \end{cases} \tag{2.63}$$

where

$$P(t) = \int_0^t k_0(t-s)\, u(0,s)\, ds,\ Q(t) = \int_0^t k_1(t-s)\, u(1,s)\, ds. \tag{2.64}$$

We take $v = u'$ in (2.63) and integrate with respect to t, we obtain

$$\begin{aligned} Z(t) &= \int_0^t \left\langle \mu'(s), u_x^2(s)\right\rangle ds - 2\int_0^t u'(0,s)\, ds \int_0^s k_0(s-r)\, u(0,r)\, dr \\ &\quad -2\int_0^t u'(1,s)\, ds \int_0^s k_1(s-r)\, u(1,r)\, dr \\ &\quad -2K\int_0^t \left\langle |u_1|^{p-2} u_1 - |u_2|^{p-2} u_2, u'\right\rangle ds, \end{aligned} \tag{2.65}$$

where

$$\begin{aligned} Z(t) &= \left\|u'(t)\right\|^2 + ||\sqrt{\mu(t)} u_x(t)||^2 \\ &\quad +2\lambda\int_0^t \left\langle \left|u_1'\right|^{q-2} u_1' - \left|u_2'\right|^{q-2} u_2', u'\right\rangle ds. \end{aligned} \tag{2.66}$$

Put

$$\begin{cases} \bar{N}_T = \frac{2}{\mu_0}\|\mu'\|_{L^\infty(Q_T)} + 2C_T[k_0] + 2C_T[k_1] + 4K(p-1)\, C_1^{p-2}, \\ C_T[k_0] = \frac{1}{4}\mu_0 T + \left(4|k_0(0)| + \frac{16}{\mu_0}\|k_0\|_{L^2(0,T)}^2 + \|k_0'\|_{L^1(0,T)}\right)\left(T^2 + \frac{1}{\mu_0}\right), \\ C_T[k_1] = \frac{1}{4}\mu_0 T + \left(4|k_1(0)| + \frac{16}{\mu_0}\|k_1\|_{L^2(0,T)}^2 + \|k_1'\|_{L^1(0,T)}\right)\left(T^2 + \frac{1}{\mu_0}\right), \\ C_1 = \max_{j=1,2}\|u_j\|_{L^\infty(0,T;H^2)}. \end{cases} \tag{2.67}$$

Then, it follows from that (2.65)–(2.67) that

$$Z(t) \le \bar{N}_T \int_0^t Z(s)\, ds, \quad \forall\, t \in [0,T]. \tag{2.68}$$

Using Gronwall's Lemma, we deduce that $Z \equiv 0$, i.e., $u_1 = u_2$ and Theorem 2.2 is completely proved. □

3. Asymptotic expansion of the solution with respect to two parameters (K, λ)

In this part, we assume that $p, q \geq N+1$, $N \geq 2$, and $(\tilde{u}_0, \tilde{u}_1, g_0, g_1, k_0, k_1, F)$ satisfy the assumptions (H_1)-(H_4), (H_6), (H_7). Let $\vec{\varepsilon} = (K, \lambda) \in \mathbb{R}_+^2$. By Theorem 2.2, the problem (1.1)–(1.4) has a unique weak solution $u = u_{\vec{\varepsilon}}$ depending on $\vec{\varepsilon} = (K, \lambda)$.

We consider the following perturbed problem, where K, λ are small parameters such that $0 \leq K \leq K_*$, $0 < \lambda \leq \lambda_*$:

$$\left(\tilde{P}_{\vec{\varepsilon}}\right) \begin{cases} Lu \equiv u_{tt} - \frac{\partial}{\partial x}\left(\mu\left(x,t\right) u_x\right) \\ \quad = -K\Psi_p(u) - \lambda\Psi_q(u_t) + F\left(x,t\right), \quad 0 < x < 1, \quad 0 < t < T, \\ L_0 u \equiv \mu\left(0,t\right) u_x\left(0,t\right) - \int_0^t k_0\left(t-s\right) u\left(0,s\right) ds = g_0\left(t\right), \\ L_1 u \equiv -\mu\left(1,t\right) u_x\left(1,t\right) - \int_0^t k_1\left(t-s\right) u\left(1,s\right) ds = g_1\left(t\right), \\ u\left(x,0\right) = \tilde{u}_0\left(x\right), \; u_t\left(x,0\right) = \tilde{u}_1\left(x\right), \end{cases}$$

where $\Psi_r(z) = |z|^{r-2}z, \;\; r \in \{p, q\}$.

We shall study the asymptotic expansion of the solution of problem $\left(\tilde{P}_{\vec{\varepsilon}}\right)$ with respect to $\vec{\varepsilon} = (K, \lambda)$. We use the following notations. For a multi-index $\gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}_+^2$ and $\vec{\varepsilon} = (K, \lambda) \in \mathbb{R}_+^2$, we put

$$\begin{cases} |\gamma| = \gamma_1 + \gamma_2, \quad \gamma! = \gamma_1!\gamma_2!, \\ \vec{\varepsilon}^{\,\gamma} = K^{\gamma_1}\lambda^{\gamma_2}, \quad \|\vec{\varepsilon}\| = \sqrt{K^2 + \lambda^2}, \\ \alpha, \; \beta \in \mathbb{Z}_+^2, \quad \beta \leq \alpha \Longleftrightarrow \beta_i \leq \alpha_i, \; \forall\, i = 1, 2, \\ C_\alpha^\beta = \frac{\alpha!}{\beta!(\alpha-\beta)!}. \end{cases}$$

First, we shall need the following Lemma.

Lemma 3.1. *Let $m, N \in \mathbb{N}$ and $v_\alpha \in \mathbb{R}$, $\alpha \in \mathbb{Z}_+^2$, $1 \leq |\alpha| \leq N$. Then*

$$\left(\sum_{1 \leq |\alpha| \leq N} v_\alpha \vec{\varepsilon}^{\alpha}\right)^m = \sum_{m \leq |\alpha| \leq mN} T^{(m)}[\bar{v}]_\alpha \vec{\varepsilon}^{\alpha}, \tag{3.1}$$

where the coefficents $T^{(m)}[\bar{v}]_\alpha$, $m \leq |\alpha| \leq mN$, depending on $\bar{v} = (v_\alpha)$, $\alpha \in \mathbb{Z}_+^2$, $1 \leq |\alpha| \leq N$, are defined by the recurrence formulas

$$\begin{cases} T^{(1)}[\bar{v}]_\alpha = v_\alpha, \quad 1 \leq |\alpha| \leq N, \\ T^{(m)}[\bar{v}]_\alpha = \sum_{\beta \in I_\alpha^{(m)}} v_{\alpha-\beta} T^{(m-1)}[\bar{v}]_\beta, \quad m \leq |\alpha| \leq mN, \quad m \geq 2, \\ I_\alpha^{(m)} = \{\beta \in \mathbb{Z}_+^2 : \beta \leq \alpha, \; 1 \leq |\alpha-\beta| \leq N, \; m-1 \leq |\beta| \leq (m-1)N\}. \end{cases} \tag{3.2}$$

Proof. The proof of Lemma 3.1 can be found in [11]. □

Let $u_0 \equiv u_{0,0}$ be a unique weak solution of problem $\left(\tilde{P}_{0,0}\right)$ (as in Theorem 2.2) corresponding to $\vec{\varepsilon} = (K, \lambda) = (0,\, 0)$, *i.e.*,

$$\left(\tilde{P}_{0,0}\right) \begin{cases} Lu_0 = F_{0,0} \equiv F(x,t),\ 0 < x < 1,\ 0 < t < T, \\ L_0 u_0 = g_0\left(t\right), \\ L_1 u_0 = g_1\left(t\right), \\ u_0\left(x,0\right) = \tilde{u}_0\left(x\right),\ u_0'\left(x,0\right) = \tilde{u}_1\left(x\right), \\ u_0 \in C^0\left(0,T;H^1\right) \cap C^1\left(0,T;L^2\right) \cap L^\infty\left(0,T;H^2\right), \\ u_0' \in L^\infty\left(0,T;H^1\right),\ u_0'' \in L^\infty\left(0,T;L^2\right). \end{cases} \tag{3.3}$$

Let us consider the sequence of weak solutions u_γ, $\gamma \in \mathbb{Z}_+^2$, $1 \leq |\gamma| \leq N$, are defined by the following problems:

$$\left(\tilde{P}_{\gamma}\right) \begin{cases} Lu_\gamma = F_\gamma,\ 0 < x < 1,\ 0 < t < T, \\ L_0 u_\gamma = L_1 u_\gamma = 0, \\ u_\gamma(x,0) = u_\gamma'\left(x,0\right) = 0, \\ u_\gamma \in C^0\left(0,T;H^1\right) \cap C^1\left(0,T;L^2\right) \cap L^\infty\left(0,T;H^2\right), \\ u_\gamma' \in L^\infty\left(0,T;H^1\right),\ u_\gamma'' \in L^\infty\left(0,T;L^2\right). \end{cases} \tag{3.4}$$

where F_γ, $|\gamma| \leq N$, defined by the recurrence formulas

$$F_\gamma = \begin{cases} F, & |\gamma| = 0, \\ -\Psi_p\left(u_0\right), & \gamma = (1,0), \\ -\sum_{m=1}^{|\gamma|-1} \frac{1}{m!} \Psi_p^{(m)}\left(u_0\right) T^{(m)}[\bar{u}]_{\gamma_1 - 1, 0}, & 2 \leq \gamma_1 \leq N,\ \gamma_2 = 0, \\ -\Psi_q\left(u_0'\right), & \gamma = (0,1), \\ -\sum_{m=1}^{|\gamma|-1} \frac{1}{m!} \Psi_q^{(m)}\left(u_0'\right) T^{(m)}[\bar{u}']_{0, \gamma_2 - 1}, & \gamma_1 = 0,\ 2 \leq \gamma_2 \leq N, \\ -\sum_{m=1}^{|\gamma|-1} \frac{1}{m!} \Big[\Psi_p^{(m)}(u_0)\, T^{(m)}[\bar{u}]_{\gamma_1 - 1, \gamma_2} & +\Psi_q^{(m)}(u_0')\ T^{(m)}[\bar{u}']_{\gamma_1, \gamma_2 - 1}\Big], \\ & \gamma_1 \geq 1,\ \gamma_2 \geq 1, 2 \leq |\gamma| \leq N, \end{cases} \tag{3.5}$$

and here we have used the notations $\bar{u} = \left(u_\gamma\right),\ \bar{u}' = \left(u_\gamma'\right),\ |\gamma| \leq N$.

Let $u = u_{\vec{\varepsilon}}$ be a unique weak solution of problem $\left(\tilde{P}_{\vec{\varepsilon}}\right)$. Then v, with

$$v = u - \sum_{|\gamma|\, \leq\, N} u_\gamma \vec{\varepsilon}^{\,\gamma} \equiv u - h, \tag{3.6}$$

satisfies the problem

$$\begin{cases} Lv = -K\left[\Psi_p(v+h) - \Psi_p(h)\right] - \lambda\left[\Psi_q(v'+h') - \Psi_q(h')\right] + E_N(\vec{\varepsilon}), \\ \qquad 0 < x < 1,\ 0 < t < T, \\ L_0 v = L_1 v = 0, \\ v(x,0) = v'(x,0) = 0, \\ v \in C^0\left(0,T;H^1\right) \cap C^1\left(0,T;L^2\right) \cap L^\infty\left(0,T;H^2\right), \\ v' \in L^\infty\left(0,T;H^1\right),\ v'' \in L^\infty\left(0,T;L^2\right), \end{cases} \tag{3.7}$$

where

$$E_N(\vec{\varepsilon}) = F(x,t) - K\Psi_p(h) - \lambda\Psi_q(h') - \sum_{|\gamma|\le N} F_\gamma \vec{\varepsilon}^{\,\gamma}. \tag{3.8}$$

Then, we have the following lemma.

Lemma 3.2. *Let p, $q \ge N+1$, $N \ge 2$, and (H_1)-(H_4), (H_6) hold. Then*

$$\|E_N(\vec{\varepsilon})\|_{L^\infty(0,T;L^2)} \le \widetilde{C}_{1N}\,\|\vec{\varepsilon}\|^{N+1}, \tag{3.9}$$

for all $\vec{\varepsilon} = (K,\lambda) \in \mathbb{R}^2_+$, $\|\vec{\varepsilon}\| \le \|\vec{\varepsilon}_*\|$ with $\vec{\varepsilon}_* = (K_*, \lambda_*)$, where $\widetilde{C}_{1N}$ is positive constant depending only on the constants $\|\vec{\varepsilon}_*\|$, $\|u_\gamma\|_{L^\infty(0,T;H^1)}$, $\left\|u'_\gamma\right\|_{L^\infty(0,T;H^1)}$, $(|\gamma| \le N)$.

Proof. Put

$$h = u_0 + h_1, \qquad h_1 = \sum_{1\le|\gamma|\le N} u_\gamma \vec{\varepsilon}^{\,\gamma}. \tag{3.10}$$

By using Taylor's expansion of the function $\Psi_p(h) = \Psi_p(u_0 + h_1)$ around the point u_0 up to order $N-1$, we obtain

$$\Psi_p(h) = \Psi_p(u_0) + \sum_{m=1}^{N-1} \frac{1}{m!}\Psi_p^{(m)}(u_0) h_1^m + \frac{1}{N!}\Psi_p^{(N)}(u_0 + \theta_1 h_1) h_1^N, \tag{3.11}$$

where $0 < \theta_1 < 1$. By Lemma 3.1, we obtain from (3.14) after some rearrangements in the order to of $\vec{\varepsilon}^{\,\gamma}$, that

$$\begin{aligned} K\Psi_p(h) \;\; = \;\; & K\Psi_p(u_0) + \sum_{2\le|\gamma|\le N,\ \gamma_1\ge 1} \sum_{m=1}^{|\gamma|-1} \frac{1}{m!}\Psi_p^{(m)}(u_0) T^{(m)}[\bar{u}]_{\gamma_1-1,\gamma_2} \vec{\varepsilon}^{\,\gamma} \\ & + R^{(1)}(p,\vec{\varepsilon}), \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} R^{(1)}(p,\vec{\varepsilon}) &= K\sum_{m=1}^{N-1}\frac{1}{m!}\Psi_p^{(m)}(u_0)\sum_{N\leq|\gamma|\leq mN}T^{(m)}[\bar{u}]_{\gamma}\vec{\varepsilon}^{\,\gamma} \\ &\quad +\frac{1}{N!}\Psi_p^{(N)}(u_0+\theta_1 h_1)Kh_1^N. \end{aligned} \tag{3.13}$$

Similarly, we use Taylor's expansion of the function $\Psi_q(h') = \Psi_q\left(u_0' + h_1'\right)$, up to order $N-1$, we obtain

$$\begin{aligned} \lambda\Psi_q(h') &= \lambda\Psi_q(u_0') + \sum_{2\leq|\gamma|\,\leq\, N,\ \gamma_2\geq 1}\sum_{m=1}^{|\gamma|-1}\frac{1}{m!}\Psi_q^{(m)}(u_0')T^{(m)}[\bar{u}']_{\gamma_1,\gamma_2-1}\vec{\varepsilon}^{\,\gamma} \\ &\quad +R^{(2)}(q,\vec{\varepsilon}), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} R^{(2)}(q,\vec{\varepsilon}) &= \lambda\sum_{m=1}^{N-1}\frac{1}{m!}\Psi_q^{(m)}(u_0')\sum_{N\leq|\gamma|\leq mN}T^{(m)}[\bar{u}']_{\gamma}\vec{\varepsilon}^{\,\gamma} \\ &\quad +\lambda\frac{1}{N!}\Psi_q^{(N)}(u_0'+\theta_2 h_1')\left(h_1'\right)^N, \end{aligned} \tag{3.15}$$

and $0<\theta_2<1$. Combining (3.5), (3.8), (3.12)-(3.15), we then obtain

$$\begin{aligned} E_N(\vec{\varepsilon}) &= F(x,t) - K\Psi_p(u_0) - \lambda\Psi_q(u_0') \\ &\quad - \sum_{2\leq|\gamma|\,\leq\, N,\ \gamma_1\geq 1}\sum_{m=1}^{|\gamma|-1}\frac{1}{m!}\Psi_p^{(m)}(u_0)T^{(m)}[\bar{u}]_{\gamma_1-1,\gamma_2}\vec{\varepsilon}^{\,\gamma} \\ &\quad - \sum_{2\leq|\gamma|\,\leq\, N,\ \gamma_2\geq 1}\sum_{m=1}^{|\gamma|-1}\frac{1}{m!}\Psi_q^{(m)}(u_0')T^{(m)}[\bar{u}']_{\gamma_1,\gamma_2-1}\vec{\varepsilon}^{\,\gamma} \\ &\quad - \sum_{|\gamma|\leq N}F_{\gamma}\vec{\varepsilon}^{\gamma} - R^{(1)}(p,\vec{\varepsilon}) - R^{(2)}(q,\vec{\varepsilon}) \\ &= -R^{(1)}(p,\vec{\varepsilon}) - R^{(2)}(q,\vec{\varepsilon}). \end{aligned} \tag{3.16}$$

We shall estimate the following terms on the right-hand side of (3.16).

Estimating $R^{(1)}(p,\vec{\varepsilon})$.

By the boundedness of the functions $u_\gamma,\ \gamma\in\mathbb{Z}_+^2,\ |\gamma|\le N$ in the function space $L^\infty(0,T;H^1)$, we obtain from (3.13) that

$$\begin{aligned}&\left\|R^{(1)}(p,\vec{\varepsilon})\right\|_{L^\infty(0,T;L^2)}\\ &\le |K|\sum_{m=1}^{N-1}\sum_{N\le|\gamma|\,\le\,mN}\frac{1}{m!}\left\|\Psi_p^{(m)}(u_0)\right\|_{L^\infty(0,T;H^1)}\left\|T^{(m)}[\bar{u}]_\gamma\right\|_{L^\infty(0,T;L^2)}|\vec{\varepsilon}^\gamma|\\ &\quad+\frac{1}{N!}K\left\|\Psi_p^{(N)}(u_0+\theta_1h_1)\right\|_{L^\infty(0,T;H^1)}\|h_1\|^N_{L^\infty(0,T;H^1)}.\end{aligned}\tag{3.17}$$

Using the inequality

$$|\vec{\varepsilon}^{\,\gamma}|\le\|\vec{\varepsilon}\|^{|\gamma|},\qquad\text{for all }\gamma\in\mathbb{Z}_+^2,\tag{3.18}$$

it follows from (3.17), (3.18), that

$$\left\|R^{(1)}(p,\vec{\varepsilon})\right\|_{L^\infty(0,T;L^2)}\le\widetilde{C}_{1N}^{(1)}\ \|\vec{\varepsilon}\|^{N+1},\qquad\|\vec{\varepsilon}\|\le\|\vec{\varepsilon}_*\|,\tag{3.19}$$

where

$$\begin{aligned}&\widetilde{C}_{1N}^{(1)}\\ &=\sum_{m=1}^{N-1}C_{p-1}^m\Big(\sqrt{2}\,\|u_0\|_{L^\infty(0,T;H^1)}\Big)^{p-m-1}\sum_{N\le|\gamma|\le mN}\left\|T^{(m)}[\bar{u}]_\gamma\right\|_{L^\infty(0,T;L^2)}\|\vec{\varepsilon}_*\|^{|\gamma|-N}\\ &\quad+C_{p-1}^N\,\|\vec{\varepsilon}_*\|^{-N}\left(\sum_{|\gamma|\,\le N}\sqrt{2}\,\|u_\gamma\|_{L^\infty(0,T;H^1)}\,\|\vec{\varepsilon}_*\|^{|\gamma|}\right)^{p-1},\end{aligned}\tag{3.20}$$

and $\vec{\varepsilon}_*=(K_*,\lambda_*)\,,\ C_{p-1}^m=\frac{(p-1)(p-2)...(p-m)}{m!}$.

Estimate $R^{(2)}(q,\vec{\varepsilon})$.

We obtain from (3.15) in a manner corresponding to the above part that

$$||R^{(2)}(q,\vec{\varepsilon})||_{L^\infty(0,T;L^2)}\le\widetilde{C}_{1N}^{(2)}\ \|\vec{\varepsilon}\|^{N+1},\qquad\|\vec{\varepsilon}\|\le\|\vec{\varepsilon}_*\|,\tag{3.21}$$

where

$$\begin{aligned}&\widetilde{C}_{1N}^{(2)}\\ &=\sum_{m=1}^{N-1}C_{q-1}^m\Big(\sqrt{2}\,\|u_0'\|_{L^\infty(0,T;H^1)}\Big)^{q-m-1}\sum_{N\le|\gamma|\le mN}\left\|T^{(m)}[\bar{u}']_\gamma\right\|^{|\gamma|-N}_{L^\infty(0,T;L^2)}\|\vec{\varepsilon}_*\|^{|\gamma|-N}\\ &\quad+C_{q-1}^N\,\|\vec{\varepsilon}_*\|^{-N}\left(\sum_{|\gamma|\,\le N}\sqrt{2}\,\|u_\gamma'\|_{L^\infty(0,T;H^1)}\,\|\vec{\varepsilon}_*\|^{|\gamma|}\right)^{q-1},\end{aligned}\tag{3.22}$$

Therefore, it follows from (3.16), (3.19)-(3.22), that

$$\begin{aligned} \|E_N(\vec{\varepsilon})\|_{L^\infty(0,T;L^2)} &\leq \left(\widetilde{C}_{1N}^{(1)} + \widetilde{C}_{1N}^{(2)}\right) \|\vec{\varepsilon}\|^{N+1} \equiv \widetilde{C}_{1N} \|\vec{\varepsilon}\|^{N+1}, \\ &\quad \|\vec{\varepsilon}\| \leq \|\vec{\varepsilon}_*\|. \end{aligned} \tag{3.23}$$

The proof of Lemma 3.2 is complete. □

Next, we obtain the following theorem.

Theorem 3.3. *Let $p,\ q \geq N+1,\ N \geq 2$, and (H_1)-(H_4), (H_6), (H_7) hold. Then, for every $\vec{\varepsilon} = (K, \lambda) \in \mathbb{R}_+^2$, with $0 \leq K \leq K_*,\ 0 \leq \lambda \leq \lambda_*$, problem $\left(\tilde{P}_{\vec{\varepsilon}}\right)$ has a unique weak solution $u = u_{\vec{\varepsilon}}$ satisfying the asymptotic estimations up to order $N+1$ as follows*

$$\left\| u' - \sum_{|\gamma| \leq N} u'_\gamma \vec{\varepsilon}^{\,\gamma} \right\|_{L^\infty(0,T;L^2)} + \left\| u - \sum_{|\gamma| \leq N} u_\gamma \vec{\varepsilon}^{\,\gamma} \right\|_{L^\infty(0,T;H^1)} \leq \widetilde{C}_N^* \|\vec{\varepsilon}\|^{N+1}, \tag{3.24}$$

for all $\vec{\varepsilon} \in \mathbb{R}_+^2$, $\|\vec{\varepsilon}\| \leq \|\vec{\varepsilon}_*\|$, $\widetilde{C}_N^*$ is a positive constant independent of $\vec{\varepsilon}$, the functions u_γ is the weak solutions of problems $\left(\tilde{P}_\gamma\right)$, $\gamma \in \mathbb{Z}_+^2$, $|\gamma| \leq N$.

Proof. First, we note that, if the data $\vec{\varepsilon} = (K,\ \lambda)$ satisfy

$$0 \leq K \leq K_*, \quad 0 \leq \lambda \leq \lambda_*, \tag{3.25}$$

where K_*, λ_* are fixed positive constants, then, the a priori estimates of the sequence $\{u_m\}$ in the proof of Theorem 2.2 satisfy

$$\left\|u'_m(t)\right\|^2 + ||\sqrt{\mu(t)} u_{mx}(t)||^2 + \frac{2K}{p} \|u_m(t)\|_{L^p}^p + 2\lambda \int_0^t \left\|u'_m(s)\right\|_{L^q}^q ds \leq C_T, \tag{3.26}$$

$$\left\|u''_m(t)\right\|^2 + ||\sqrt{\mu(t)} u'_{mx}(t)||^2 + 2\lambda(q-1) \int_0^t ds \int_0^1 \left|u'_m(x,s)\right|^{q-2} \left|u''_m(x,s)\right|^2 dx \leq C_T, \tag{3.27}$$

for all $t \in [0,T]$, where C_T is a constant depending only on T, $\tilde{u}_0$, $\tilde{u}_1$, g_0, g_1, k_0, k_1, F, p, q, K_*, λ_* (independent of $\vec{\varepsilon}$). Hence, the limit u in suitable function spaces of the sequence $\{u_m\}$ defined by (2.5)–(2.7) is a weak solution of the problem (1.1)–(1.4) satisfying the a priori estimates (3.26), (3.27).

By multiplying the two sides of $(3.7)_1$ by v', after integration in t, we find without difficulty from Lemma 3.2 that

$$\begin{aligned}\sigma(t) \quad &\leq \quad 2\widetilde{C}_{1N}^2 \left\|\vec{\varepsilon}\right\|^{2N+2} \\ &+2\left(2+\frac{1}{\mu_0}\left\|\mu'\right\|_{L^\infty(Q_T)}+C_T(\beta,k_0)+C_T(\beta,k_1)\right)\int_0^t \sigma(s)ds \\ &+8K^2\int_0^t \left\|\Psi_p(v+h)-\Psi_p(h)\right\|^2 ds, \end{aligned} \tag{3.28}$$

where $\beta=\frac{\mu_0}{4}$ and

$$\sigma(t)=\left\|v'(t)\right\|^2+||\sqrt{\mu\left(t\right)}v_x(t)||^2+2\lambda\int_0^t \left\langle \Psi_q(v'+h')-\Psi_q(h'),v'\right\rangle ds. \tag{3.29}$$

By using the same arguments as in the above part we can show that weak solution u of problem $\left(\tilde{P}_{\vec{\varepsilon}}\right)$ satisfies

$$\left\|u'(t)\right\|^2+\left\|u_x(t)\right\|^2 \leq C_T, \qquad \forall\, t\in[0,T], \tag{3.30}$$

where C_T is a constant independent of K, λ. On the other hand,

$$\left\|h\right\|_{L^\infty(0,T;H^1)} \leq \sum_{|\gamma|\,\leq\, N}\left\|u_\gamma\right\|_{L^\infty(0,T;H^1)}\left\|\vec{\varepsilon}_*\right\|^{|\gamma|} \equiv \frac{1}{\sqrt{2}}R_1. \tag{3.31}$$

We again use inequality (2.55) with $R=R_2\equiv\max\{R_1,\ \sqrt{2\left(1+T^2\right)C_T}\}$, then, it follows from (3.29) to (3.31) that

$$8K^2\int_0^t \left\|\Psi_p(v+h)-\Psi_p(h)\right\|^2 ds \leq 8K^2(p-1)^2R_2^{2p-4}\int_0^t \sigma(s)ds. \tag{3.32}$$

Combining (3.28), (3.31) and (3.32), we then obtain

$$\sigma(t)\leq K_T^{(1)}\left\|\vec{\varepsilon}\right\|^{2N+2}+K_T^{(2)}\int_0^t \sigma(s)ds, \tag{3.33}$$

for all $t\in[0,T]$, where

$$\begin{cases} K_T^{(1)}=2\widetilde{C}_{1N}^2, \\ K_T^{(2)}=2\left(2+\frac{1}{\mu_0}\left\|\mu'\right\|_{L^\infty(Q_T)}+C_T(\beta,k_0)+C_T(\beta,k_1)\right) \\ \qquad\quad +8K^2(p-1)^2R_2^{2p-4}. \end{cases} \tag{3.34}$$

By Gronwall's lemma, we obtain from (3.33), that

$$\sigma(t)\leq K_T^{(1)}\left\|\vec{\varepsilon}\right\|^{2N+2}\exp\left(TK_T^{(2)}\right)\equiv\widetilde{D}_T^{(1)}\left\|\vec{\varepsilon}\right\|^{2N+2}, \qquad \forall\, t\in[0,T], \tag{3.35}$$

for all $\vec{\varepsilon}\in\mathbb{R}_+^2$, $\left\|\vec{\varepsilon}\right\|\leq\left\|\vec{\varepsilon}_*\right\|$. It follows that

$$\left\|v'(t)\right\|^2+\mu_0\left\|v_x(t)\right\|^2\leq\sigma(t)\leq\widetilde{D}_T^{(1)}\left\|\vec{\varepsilon}\right\|^{2N+2}. \tag{3.36}$$

Hence

$$\left\|v'\right\|_{L^\infty(0,T;L^2)} + \|v\|_{L^\infty(0,T;H^1)} \le \widetilde{C}_N^* \left\|\vec{\varepsilon}\right\|^{N+1} \tag{3.37}$$

or

$$\begin{aligned}&\left\|u' - \sum_{|\gamma|\,\le\,N} u'_\gamma \vec{\varepsilon}^{\,\gamma}\right\|_{L^\infty(0,T;L^2)} + \left\|u - \sum_{|\gamma|\,\le\,N} u_\gamma \vec{\varepsilon}^{\gamma}\right\|_{L^\infty(0,T;H^1)} \\ &\le \widetilde{C}_N^* \left\|\vec{\varepsilon}\right\|^{N+1},\end{aligned} \tag{3.38}$$

for all $\vec{\varepsilon} \in \mathbb{R}_+^2$, $\|\vec{\varepsilon}\| \le \|\vec{\varepsilon}_*\|$, where $\widetilde{C}_N^*$ is a constant independent of $\vec{\varepsilon}$. The proof of Theorem 3.3 is complete. □

Remark 3.4. In [9], as a special case of problem (1.1) - (1.4), that is $p = q = 2$, the result about the assymptotic expansion of the solutions with respect to two parameters (K, λ) up to order $N + 1$ was obtained.

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