Nonlinear Functional Analysis and Applications
Vol. 17, No. 2 (2012), pp. 213-224

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

STRONG CONVERGENCE FOR SEMIGROUP OF ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS

Balwant Singh Thakur[1] and Mohammad Saeed Khan[2]

[1]School of Studies in Mathematics
Pt. Ravishankar Shukla University
Raipur (C.G.) 492010, India
e-mail: balwantst@gmail.com

[2]Department of Mathematics and Statistics
Sultan Qaboos University
P.O. Box 36, PCode 123 Al-Khod, Muscat
Sultanate of Oman, Oman
e-mail: mohammad@squ.edu.om

Abstract. In this paper, we propose a viscosity iteration process for semigroup of asymptotically pseudocontractive mappings, and prove a strong convergence theorem in uniformly convex Banach space for the proposed iteration process.

1. Introduction

Let E be a real Banach space, E^* be its dual space, K a nonempty closed convex subset of E and $J : E \to 2^{E^*}$ the normalized duality mapping defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \left\|f^2\right\|, \|f\| = \|x\|\}, \quad \text{for all } x \in E,$$

where $\langle \cdot, \cdot \rangle$ denote the duality pairing between E and E^*. The single-valued normalized duality mapping is denoted by j.
A mapping $T : K \to K$ is said to be

[0]Received Received March 4, 2012. Revised May 25, 2012.
[0]2000 Mathematics Subject Classification: 47H20, 47H06, 47H10.
[0]Keywords: Semigroup of asymptotically pseudocontractive mappings, strongly pseudocontractive mapping, viscosity iteration process, iterative approximation, variational inequality.

- nonexpansive, if
$$\|Tx - Ty\| \leq \|x - y\|$$
for all $x, y \in K$,
- pseudocontractive, if there exists some $j(x-y) \in J(x-y)$ such that
$$\langle Tx - Ty, j(x-y)\rangle \leq \|x - y\|^2$$
for all $x, y \in K$,
- strongly pseudocontractive, if there exists a constant $\alpha \in (0,1)$ such that
$$\langle Tx - Ty, j(x-y)\rangle \leq \alpha \|x - y\|^2$$
for all $x, y \in K$,
- asymptotically nonexpansive [9], if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that
$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$
for all $x, y \in K$ and $n \in \mathbb{N}$.
- asymptotically pseudocontractive [13], if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that
$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n \|x - y\|^2$$
for all $x, y \in K$, and $n \in \mathbb{N}$.

It can be seen from the above definitions that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is asymptotically pseudocontractive. A mapping T is called uniformly $L-$ Lipschitzian, if there exists $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$, for all $x, y \in K$ and for each integer $n \geq 1$. Uniformly asymptotically regular if $\|T^{n+1}x - T^n x\| \to 0$ as $n \to \infty$ for all $x \in K$.

Let K be a closed convex subset of a Banach space E and $\mathbb{R}^+$ the set of nonnegative real numbers. $\mathcal{T} := \{T(t) : t \in \mathbb{R}^+\}$ is said to be *strongly continuous semigroup of asymptotically pseudocontractive mappings* from K in to K if the following conditions are satisfied [5]:

(1) $T(0)x = x$ for all $x \in K$;
(2) $T(s+t) = T(s) \circ T(t)$ for all $s, t \in \mathbb{R}^+$;
(3) there exist $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j(x-y) \in J(x-y)$ such that
$$\langle (T(t_n))^n x - (T(t_n))^n y, j(x-y)\rangle \leq k_n \|x - y\|^2 \ , \ \forall\, t_n > 0,\ x, y \in K\,;$$
(4) for each $x \in K$, the mapping $T(\cdot)x$ from $\mathbb{R}^+$ into K is continuous.

If in the above definition, condition (3) is replaced by the following condition:

(3)* there exist $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|(T(t_n))^n x - (T(t_n))^n y\| \le k_n \|x - y\| , \ \forall t_n > 0, \ x, y \in K$$

then $\mathcal{T}$ is called strongly continuous semi-group of asymptotically nonexpansive mappings on K.

$\mathcal{T}$ is said to have a fixed point if there exists $x_0 \in K$ such that $T(t)x_0 = x_0$ for all $t \ge 0$. We denote by F the set of fixed point of $\mathcal{T}$, i.e. $F := \bigcap_{t\in\mathbb{R}^+} F(T(t))$.

Numerous problems in mathematics and physical sciences can be recast in terms of a fixed point problem for nonexpansive mappings. Due to practical importance of these problems, algorithms for finding fixed points of nonexpansive mappings continue to be a flourishing topic of interest in fixed point theory.

The most straightforward attempt to solve the fixed point problem for nonexpansive mappings is by Picard iteration :

$$x_{n+1} = Tx_n , \quad \forall n \ge 0 \ \ (x_0 \in K) \tag{1.1}$$

Unfortunately, algorithm (1.1) may fail to produce a norm convergence sequence $\{x_n\}$.

In view of celebrated Banach contraction principle, the attempt to approximate fixed point of nonexpansive self mappings seems very promising :
For given $u \in K$ and each $t \in (0, 1)$ define a contraction $T_t : K \to K$ by

$$T_t x = tu + (1 - t)Tx \quad \forall x \in K .$$

Clearly T_t is $(1 - t)$ contraction, so by Banach contraction principle, it has a unique fixed point $z_t \in K$, i.e. z_t is the unique solution of equation

$$z_t = tu + (1 - t)Tz_t , \tag{1.2}$$

here z_t is defined implicitly.

In 1967, Browder [2] proved that z_t defined by (1.2) converges strongly to a fixed point of T as $t \to 0$. In the same year, Halpern [10] devised an explicit iteration method which converges in norm to a fixed point of T, the iteration process is known as Halpern iterative method and defined as below :
For a sequence $\{\alpha_n\}$ in $(0, 1)$, obtain the modified version of (1.1)

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)Tx_n , \ \ n \ge 0 \tag{1.3}$$

Further, it is proved that the sequence $\{x_n\}$ defined by (1.3) converges strongly to a fixed point of T if $\{\alpha_n\}$ satisfies certain conditions.

It is an interesting problem to extend results related to nonexpansive, asymptotically nonexpansive, pseudocontractive, asymptotically pseudocontractive mappings to semigroup of respective mappings.

Suzuki [14] proved the following result for strongly continuous semigroup of nonexpansive mappings:

Theorem S. *Let K be a closed convex subset of a Hilbert space H. Let $\{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous semigroup of nonexpansive mappings on K such that $F = \bigcap_{t\in\mathbb{R}^+} F(T(t)) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_n t_n = \lim_n \alpha_n/t_n = 0$. Fix $u \in K$ and define a sequence $\{u_n\}$ in K by*

$$u_n = \alpha_n u + (1 - \alpha_n)T(t_n)u_n$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to the element of F nearest to u.

Chidume [5] proved following result for strongly continuous semigroup of asymptotically pseudocontractive mappings in the setting of Banach space:

Theorem C. *Let K be a closed convex and bounded subset of a real uniformly convex Banach space E having uniformly Gâteaux differential norm, $L < N(E)^{1/2}$. Let $\mathcal{T} := \{T(t) : t \in R^+\}$ be a strongly continuous uniformly asymptotically regular and uniformly $L-$Lipschitzian semigroup of asymptotically pseudocontractive mappings from K into K with a sequence $\{k_n\} \subset [1,\infty)$. Then for $u \in K$, $t_n > 0$ and $s_n \in (0,1)$, there exists a sequence $\{x_n\} \in K$ satisfying the following condition:*

$$x_n = \alpha_n u + (1 - \alpha_n)\left(T(t)\right)^n x_n\,,$$

where $\alpha_n := (1 - s_n/k_n)$. Moreover, if $\lim t_n = \lim(\alpha_n/t_n) = 0$, $\frac{(k_n-1)}{k_n-s_n} \to 0$ as $n \to \infty$, and

$$\|x_n - (T(t))^m x\|^2 \leq \langle x_n - (T(t))^m x, j(x_n - x)\rangle\,,$$

$\forall m, n \geq 1$, $\forall x \in C$, $t \in \mathbb{R}^+$, where $C := \{x \in K : \Phi(y) = \min_{z\in K} \Phi(z)\}$ where $\Phi(z) := LIM\,\|x_n - z\|^2$ for each $z \in K$. Then $\{x_n\}$ converges strongly to a fixed point of $\mathcal{T}$.

On the other hand viscosity method provide an efficient approach to a large number of ill-posed problems (lack of existence, or uniqueness, or stability of a solution) coming from different branches of mathematics. A major feature of these methods is to provide as a limit of the solution of the approximate problems, a particular (possibly relaxed or generalized) solution of the original problem.

First abstract formulation of the properties of the viscosity approximation have been given by A.N.Tykhonov [15] in 1963 when studying ill-posed problems (see [8] for details).

Let us now make precise the mathematical abstract setting. Let X be a abstract space, given $f : X \to \mathbb{R}^+ \cup \{+\infty\}$ an extended real valued function, let us consider the minimization problem

$$\min\{f(x)\ :\ x \in X\} \tag{$\mathcal{P}$}$$

which is assumed to be ill-posed.
For any $\varepsilon > 0$, let us consider the approximate minimization problem

$$\min\{f(x) + \varepsilon g(x)\ :\ x \in X\} \tag{$\mathcal{P}_\varepsilon$}$$

which is well posed due to nice properties of a nonnegative real valued function $g : X \to \mathbb{R}^+ \cup \{+\infty\}$. So, it is assumed that, for all $\varepsilon > 0$, there exists a solution u_ε of $(\mathcal{P}_\varepsilon)$. The central question is to study the convergence of the sequence $\{u_\varepsilon;\ \varepsilon \to 0\}$ and the characterization of its limit. The function g is called viscosity function.

Using contraction mapping as a viscosity function, Moudafi [12] introduced viscosity approximation method of selecting a particular fixed point of a nonexpansive mapping. Given a real number $t \in (0, 1)$ and a contraction mapping $f : K \to K$ with contraction constant $\alpha \in [0, 1)$. Define a mapping $T_t = T_t^f : K \to K$ by

$$T_t x = tf(x) + (1 - t)Tx, \quad x \in K\,. \tag{1.4}$$

Clearly T_t is a $(1 - t(1 - \alpha))$ contraction, and so has a unique fixed point $x_t = x_t^f \in K$. Thus x_t is the unique solution of the fixed point equation

$$x_t = tf(x_t) + (1 - t)Tx_t\,. \tag{1.5}$$

Xu [16] studied the strong convergence of x_t defined by (1.5) as $t \to 0$. He also introduced the following iterative algorithm to approximate fixed points of nonexpansive mappings: For arbitrary chosen $x_o \in K$, construct a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n\,, \quad \forall n \geq 0\,. \tag{1.6}$$

More recently Cho and Kang [6] proved following theorem for strongly continuous semigroup of nonexpansive mappings:

Theorem CK. *Let K be a closed convex subset of a real uniformly convex Banach space E having uniformly Gâteaux differential norm. Let $\{T(t) : t \in \mathbb{R}^+\}$*

be a strongly continuous $L-$ Lipschitz semigroup of pseudocontractive mappings on K such that $F \neq \emptyset$. Let $f : K \to K$ be a fixed bounded, continuous and strong pseudocontraction with the coefficient $\alpha \in (0,1)$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$. Let $\{x_n\}$ be a sequence generated in the following manner :

$$x_n = \alpha_n f(x_n) + (1-\alpha_n)\,(T(t))\,x_n\,, \quad \forall n \geq 1.$$

Assume that $LIM\,\|T(t)x_n - T(t)x^*\| \leq LIM\,\|x_n - x^*\|$, $\forall x^* \in C$, $t \geq 0$, where $C := \{x^* \in K : \Phi(x^*) = \min_{x\in K}\Phi(x)\}$ where $\Phi(x) := LIM\,\|x_n - x\|^2$ for each $x \in K$. Then $\{x_n\}$ converges strongly to a fixed point of $\mathcal{T}$, which solves the following variational inequality

$$\langle (I-f)x^*, j(x^*-x)\rangle \leq 0\,, \ \ \forall x \in F\,.$$

Motivated by the above results and a viscosity iteration defined by Ceng, Xu and Yao [3], in this paper we propose a viscosity iteration method (VIM) for strongly continuous semigroup of asymptotically pseudocontractive mappings and prove a strong convergence theorem for proposed VIM.

2. Preliminaries

Let E be a real normed space of dimension ≥ 2. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in S_1(0) := \{x \in E : \|x\| = 1\}$ the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exist uniformly for $x \in S_1(0)$.

Let l^∞ be the Banach space of all bounded real-valued sequences. A Banach limit LIM is a bounded linear functional on l^∞ such that

$$\|LIM\| = 1\,, \quad \liminf_{n\to\infty} t_n \leq LIM t_n \leq \limsup_{n\to\infty} t_n\,,$$

and $LIM t_n = LIM\, t_{n+1}$ for all $t_n \in l^\infty$.

We need following results to prove our main result:

Lemma 2.1. ([1]) *Let K be a nonempty closed convex subset of a uniformly convex Banach space E, $\{x_n\}$ a bounded sequence in K, LIM a Banach limit, and Φ a real valued function on K defined by $\Phi(z) = LIM\,\|x_n - z\|^2$, $z \in K$. Then the set M defined by*

$$M = \left\{ u \in K : LIM\,\|x_n - u\|^2 = \inf_{z\in K} LIM\,\|x_n - z\|^2 \right\}$$

is a nonempty closed convex bounded set and has exactly one point.

Lemma 2.2. ([7]) *Let E be a Banach space, K be a nonempty closed convex subset of E and $T : K \to K$ be a continuous and strong pseudocontraction. Then T has a unique fixed point.*

Lemma 2.3. ([4]) *For any $x, y \in E$ the following holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle\ , \quad \forall j(x+y) \in J(x+y).$$

3. Main Results

Theorem 3.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Let $\mathcal{T} := \{T(t) : t \in \mathbb{R}^+\}$ be a strongly continuous uniformly asymptotically regular and uniformly $L-$ Lipschitzian semigroup of asymptotically pseudocontractive mappings form K into K such that $F := \bigcap_{\mathbb{R}^+} F(T(t)) \neq \emptyset$. Let $f : K \to K$ be a fixed bounded, continuous strong pseudocontraction with constant $\alpha \in (0,1)$. Let $\{x_n\}$ be a sequence generated by*

$$x_n = \left(1 - \frac{1}{k_n}\right) x_n + \frac{1-\alpha_n}{k_n} f x_n + \frac{\alpha_n}{k_n} \left(T(t_n)\right)^n x_n \tag{3.1}$$

where $\{\alpha_n\}$ and $\{t_n\}$ are sequences of real numbers satisfying

$$\begin{cases} 0 < \alpha_n < \dfrac{1-\alpha}{k_n - \alpha} \quad \text{and} \quad \displaystyle\lim_{n\to\infty} \frac{k_n - 1}{1-\alpha_n} = 0\,, \\ t_n > 0 \ (\forall n) \quad \text{and} \quad \displaystyle\lim_{n\to\infty} t_n = 0 = \lim_{n\to\infty} \frac{1-\alpha_n}{t_n}\,. \end{cases} \tag{3.2}$$

Assume that $LIM\,\|x_n - (T(t))^m x^*\| \leq LIM\,\|x_n - x^*\|$, for all $x^* \in M$, $m \geq 1$, $t \in \mathbb{R}^+$, where $M := \{x^* \in K : \Phi(x^*) = \inf_{x\in K} \Phi(x)\}$ with $\Phi(x) := LIM\ \|x_n - x\|^2$, for all $x \in K$. Then $\{x_n\}$ converges to $x^* \in F$ which solves the variational inequality:

$$\langle (f-I)x^*, j(x-x^*)\rangle \leq 0 \quad \forall\, x \in F\,. \tag{3.3}$$

Proof. First we show that the sequence $\{x_n\}$ generated by (3.1) is well defined. For each $n \in \mathbb{N}$, define a mapping $\widetilde{T_n}$ as follows

$$\widetilde{T_n} x := \left(1 - \frac{1}{k_n}\right) x + \frac{1-\alpha_n}{k_n} f x + \frac{\alpha_n}{k_n} \left(T(t_n)\right)^n x\,, \ \text{ for all } x \in K,$$

then

$$\begin{aligned}
&\left\langle \widetilde{T_n}x-\widetilde{T_n}y, j(x-y)\right\rangle \\
&=\left\langle\left(1-\frac{1}{k_n}\right)(x-y)+\frac{1-\alpha_n}{k_n}(fx_n-fy)+\frac{\alpha_n}{k_n}\left((T(t_n))^n x-(T(t_n))^n y\right), j(x-y)\right\rangle \\
&\leq\left(1-\frac{1}{k_n}\right)\|x-y\|^2+\frac{1-\alpha_n}{k_n}\|x-y\|^2+\alpha_n\|x-y\|^2 \\
&=\left(1-\frac{1}{k_n}+\frac{\alpha(1-\alpha_n)}{k_n}+\alpha_n\right)\|x-y\|^2
\end{aligned}$$

and $\left(1-\frac{1}{k_n}+\frac{\alpha(1-\alpha_n)}{k_n}+\alpha_n\right)<1$ by choice of α_n. So $\widetilde{T_n}$ is continuous and strongly pseudocontractive mapping, therefore from Lemma 2.2, the mapping $\widetilde{T_n}$ has a unique fixed point say $x_n\in K$, that is, the equation

$$x_n=\left(1-\frac{1}{k_n}\right)x_n+\frac{1-\alpha_n}{k_n}fx_n+\frac{\alpha_n}{k_n}\left(T(t_n)\right)^n x_n$$

has a unique solution for each $n\in\mathbb{N}$.

Next we show that $\{x_n\}$ is bounded. For any fixed $p\in F$, from Lemma 2.3, we have

$$\begin{aligned}
&\|x_n-p\|^2 \\
&=\left\langle\left(1-\frac{1}{k_n}\right)(x_n-p)+\frac{1-\alpha_n}{k_n}(fx_n-p)+\frac{\alpha_n}{k_n}\left((T(t_n))^n x_n-p\right), j(x_n-p)\right\rangle \\
&=\left(1-\frac{1}{k_n}\right)\langle x_n-p, j(x_n-p)\rangle+\frac{1-\alpha_n}{k_n}\langle fx_n-fp, j(x_n-p)\rangle \\
&\qquad+\frac{1-\alpha_n}{k_n}\langle fp-p, j(x_n-p)\rangle+\frac{\alpha_n}{k_n}\left\langle (T(t_n))^n x_n-p, j(x_n-p)\right\rangle \\
&\leq\left(1-\frac{1}{k_n}\right)\|x_n-p\|^2+\frac{(1-\alpha_n)\alpha}{k_n}\|x_n-p\|^2 \\
&\qquad+\frac{1-\alpha_n}{k_n}\langle fp-p, j(x_n-p)\rangle+\alpha_n\|x_n-p\|^2 \\
&=\left[1-\frac{1}{k_n}+\frac{(1-\alpha_n)\alpha}{k_n}+\alpha_n\right]\|x_n-p\|^2+\frac{1-\alpha_n}{k_n}\langle fp-p, j(x_n-p)\rangle \\
&=\left[1-\frac{1-\alpha(1-\alpha_n)-\alpha_n k_n}{k_n}\right]\|x_n-p\|^2+\frac{1-\alpha_n}{k_n}\langle fp-p, j(x_n-p)\rangle \\
&=(1-\eta_n)\|x_n-p\|^2+\frac{1-\alpha_n}{k_n}\langle fp-p, j(x_n-p)\rangle,
\end{aligned}$$

where $\eta_n = \frac{1-\alpha(1-\alpha_n)-\alpha_n k_n}{k_n}$. Therefore,

$$\|x_n - p\|^2 \leq \frac{1-\alpha_n}{k_n \eta_n} \langle fp - p, j(x_n - p) \rangle \ , \tag{3.4}$$

since using (3.2), we have

$$\frac{1-\alpha_n}{k_n \eta_n} = \frac{1}{\left(1 - \frac{k_n - 1}{1-\alpha_n}\, \alpha_n - \alpha\right)} \quad \longrightarrow \quad \frac{1}{1-\alpha} \tag{3.5}$$

thus $\{x_n\}$ is bounded and so $\{f(x_n)\}$ and $\{(T(t_n))^n x_n\}$ are bounded.

Now for any given $t > 0$, we have

$$\begin{aligned}
\|x_n - (T(t))^n x_n\| &\leq \sum_{k=0}^{\left[\frac{t}{t_n}\right]-1} \|(T((k+1)t_n))^n x_n - (T(kt_n))^n x_n\| \\
&\quad + \left\| \left(T \left(\left[\frac{t}{t_n} \right] t_n \right) \right)^n x_n - (T(t))^n x_n \right\| \\
&\leq \left[\frac{t}{t_n} \right] L \, \|(T(t_n))^n x_n - x_n\| \\
&\quad + L \left\| \left(T \left(t - \left[\frac{t}{t_n} \right] t_n \right) \right)^n x_n - x_n \right\| \\
&= \left[\frac{t}{t_n} \right] (1-\alpha_n) L \, \|(T(t_n))^n x_n - f(x_n)\| \\
&\quad + L \max \{ \|(T(s))^n x_n - x_n\| : 0 \leq s \leq t_n \} \\
&\leq t \left(\frac{1-\alpha_n}{t_n} \right) L \, \|(T(t_n))^n x_n - f(x_n)\| \\
&\quad + L \max \{ \|(T(s))^n x_n - x_n\| : 0 \leq s \leq t_n \}
\end{aligned}$$

for $n \in \mathbb{N}$, which gives that

$$\|x_n - (T(t))^n x_n\| \to 0 \quad \text{as} \quad n \to \infty \, . \tag{3.6}$$

Thus,

$$\begin{aligned}
\|x_n - (T(t))\, x_n\| &\leq \|x_n - (T(t))^n x_n\| + \left\| (T(t))^n x_n - (T(t))^{n+1} x_n \right\| \\
&\quad + \left\| (T(t))^{n+1} x_n - (T(t))\, x_n \right\| \\
&\leq (1+L)\, \|x_n - (T(t))^n x_n\| + \left\| (T(t))^n x_n - (T(t))^{n+1} x_n \right\|
\end{aligned}$$

therefore, from (3.6) and uniform asymptotic regularity of $T(t)$, we have

$$\|x_n - (T(t))\, x_n\| \to 0 \quad \text{as} \quad n \to \infty \, . \tag{3.7}$$

On the other hand, since K is closed, we see from Lemma 2.1 that M is a nonempty closed convex bounded subset of K and M is singleton. For any $t \geq 0$, $x^* \in M$, we obtain by assumption that

$$\Phi\left((T(t))^m x^*\right) = LIM \left\|x_n - (T(t))^m x^*\right\| \leq LIM \left\|x_n - x^*\right\| = \Phi(x^*)\,.$$

That is $(T(t))^m M \subseteq M$, since M is singleton, we have $(T(t))^m x^* = x^*$, by continuity of $T(t)$ we have $T(t)x^* = x^*$, i.e. there exists a unique $x^* \in M$ such that $x^* \in F$.

Now, for any $x \in F$, from (3.1), we have

$$\begin{aligned}
&\langle fx_n - x_n, j(x_n - x)\rangle \\
&= \frac{1}{1-\alpha_n}\left\langle x_n - (T(t_n))^n x_n, j(x_n - x)\right\rangle \\
&= \frac{1}{1-\alpha_n}\left[\langle x_n - x, j(x_n - x)\rangle - \langle (T(t))^n x_n - x, j(x_n - x)\rangle\right] \\
&\geq \frac{1}{1-\alpha_n}\left(\|x_n - x\|^2 - k_n\|x_n - x\|^2\right) \\
&= -\left(\frac{k_n - 1}{1-\alpha_n}\right)\|x_n - x\|^2\,. \qquad (3.8)
\end{aligned}$$

From (3.8), we have

$$LIM\,\langle x_n - fx_n, j(x_n - x)\rangle \leq LIM\left(\frac{k_n - 1}{1-\alpha_n}\right)\|x_n - x\|^2 \to 0\,, \qquad (3.9)$$

as $n \to \infty$. On the other hand, for any $s \in (0,1)$, it follows from Lemma 2.3, that

$$\begin{aligned}
&\|x_n - x^* - s\,(fx^* - x^*)\|^2 \\
&\leq \|x_n - x^*\|^2 + 2\left\langle -s\,(fx^* - x^*)\,, j\left(x_n - x^* - s\,(fx^* - x^*)\right)\right\rangle \\
&= \|x_n - x^*\|^2 - 2s\,\langle fx^* - x^*, j(x_n - x^*)\rangle \\
&\qquad - 2s\left\langle fx^* - x^*, j\left(x_n - x^* - s\,(fx^* - x^*)\right) - j(x_n - x^*)\right\rangle.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\langle fx^* - x^*, j(x_n - x^*)\rangle \\
&\leq \frac{1}{2s}\left[\|x_n - x^*\|^2 - \|x_n - x^* - s\,(fx^* - x^*)\|^2\right] \\
&\qquad - \left\langle fx^* - x^*, j\left(x_n - x^* - s\,(fx^* - x^*)\right) - j(x_n - x^*)\right\rangle\,. \qquad (3.10)
\end{aligned}$$

Since E has uniform Gâteaux differential norm, so j is norm-to-weak* uniformly continuous on bounded subsets of E. For any $\varepsilon > 0$, there exists $\delta > 0$

such that for all $s \in (0, \delta)$, we have

$$\langle fx^* - x^*, j(x_n - x^*) \rangle \leq \frac{1}{2s} \left[\|x_n - x^*\|^2 - \|x_n - x^* - s\,(fx^* - x^*)\|^2 \right] + \varepsilon\,.$$

Taking Banach limit LIM on the above inequality, we have

$$\begin{aligned} &LIM\,\langle fx^* - x^*, j(x_n - x^*) \rangle \\ &\leq \frac{1}{2s} \left[LIM\,\|x_n - x^*\|^2 - LIM\,\|x_n - x^* - s\,(fx^* - x^*)\|^2 \right] + \varepsilon \\ &< \varepsilon\,. \end{aligned}$$

Now, since ε is arbitrary, this implies that

$$LIM\,\langle fx^* - x^*, j(x_n - x^*) \rangle \leq 0\,. \tag{3.11}$$

Again by inequality (3.4), we have

$$LIM\,\|x_n - x^*\|^2 \leq LIM \frac{1 - \alpha_n}{k_n \eta_n} \langle fx^* - x^*, j(x_n - x^*) \rangle \leq 0\,,$$

and hence

$$LIM\,\|x_n - x^*\|^2 = 0 \tag{3.12}$$

therefore, there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ which converges strongly to x^*. Using (3.8), for any $x \in F$, we have

$$\left\langle x_{n_j} - fx_{n_j}, j(x_{n_j} - x) \right\rangle \leq \left(\frac{k_{n_j} - 1}{1 - \alpha_{n_j}} \right) \left\|x_{n_j} - x\right\|^2\,, \tag{3.13}$$

taking limit in (3.13), we get

$$\langle x^* - fx^*, j(x^* - x) \rangle \leq 0\,, \quad \text{for any } x \in F\,. \tag{3.14}$$

Now, suppose there exists another subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges strongly to z^* (say). Since $\lim \|x_n - T(t)x_n\| = 0$ for each $t \in \mathbb{R}^+$, we have that z^* is a fixed point of $\mathcal{T}$. Thus from (3.14), we have

$$\langle x^* - fx^*, j(x^* - z^*) \rangle \leq 0\,. \tag{3.15}$$

Now, since $x^* \in F$, using (3.8) again, we get

$$\langle x_{n_k} - fx_{n_k}, j(x_{n_k} - x^*) \rangle \leq \left(\frac{k_{n_k} - 1}{1 - \alpha_{n_k}} \right) \|x_{n_k} - x^*\|^2\,, \tag{3.16}$$

taking limit in (3.16), we get

$$\langle z^* - fz^*, j(z^* - x^*) \rangle \leq 0\,. \tag{3.17}$$

Adding (3.15) and (3.17), we get

$$\langle x^* - z^* + fz^* - fx^*, j(x^* - z^*) \rangle \leq 0\,.$$

This gives

$$\|x^* - z^*\|^2 \leq \langle fx^* - fz^*, j(x^* - z^*) \rangle \leq \alpha\,\|x^* - z^*\|^2\,.$$

Since $\alpha \in (0,1)$, we have, $x^* = z^*$. This proves that $\{x_n\}$ converges strongly to $x^* \in F$, which is the unique solution to the variational inequality (3.3). This completes the proof. □

Remark 3.2. Theorem 3.1 includes as special case the corresponding results in [3, 5, 6, 11, 14, 16, 17].

Remark 3.3. In Theorem 3.1, viscosity iteration method involves strong pseudocontractive mapping, and therefore $x^* \in F$ is the solution of larger class of variational inequality (3.3).

References

[1] R.P. Agarwal, D. O'Regan and D.R. Sahu, *Fixed point theory for Lipschitzian-type mapping with applications*, Springer, New York, 2009.

[2] F.E. Browder, *Convergence of approximates to fixed points of nonexpamsive nonlinear mappings in Banach spaces*, Arch. Ration. Mech. Anal. **24** (1967), 82-90.

[3] L.C. Ceng, H.K. Xu and J.C. Yao, *The viscosity approximation method for asymptotically nonexpansive mappings in Banach spaces*, Nonlinear Anal. **69** (2008), 1402–1412.

[4] S.S. Chang, *Some problems and results in the study of nonlinear analysis*, Nonlinear Anal. **30** (7) (1997), 4197–4208.

[5] C.E. Chidume, *Strong convergence theorems for fixed points of asymptotically pseudocontractive semi-groups*, J. Math. Anal. Appl. **296** (2004), 410–421.

[6] S.Y. Cho and S.M. Kang, *Approximation of fixed points of pseudocontraction semigroups based on viscosity iteration process*, Appl. Math. Lett. **24** (2011), 224–228.

[7] K. Deimling, *Zeros of accretive operators*, Manuscripta Math. **13** (1974), 365–374.

[8] A.L. Dontchev and T. Zolezzi, *Well-posed optimization problems*, Lecture Notes in Mathematics, Vol. 1543 Springer-Verlag, 1993.

[9] K. Goebel and W.A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. **35** (1972), 171–174.

[10] B. Halpern, *Fixed points of nonexpansive maps*, Bull. Amer. Math. Soc. **73** (1967), 957–961.

[11] X.L. Li and N.J. Huang, *Strong convergence theorems for fixed points of pseudocontractive semigroups*, Bull. Aust. Math. Soc. **76** (2007), 441–452.

[12] A. Moudafi, *Viscosity approximation methods for fixed-points problems*, J. Math. Anal. Appl. **241** (2000), 46–55.

[13] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, J. Math. Anal. Appl. **158** (1991), 407–413.

[14] T. Suzuki, *On strong convergence to common fixed points of nonexpansive semigroups in Hilbert spaces*, Proc. Amer. Math. Soc. **131** (2002), 2133–2136.

[15] A. Tychonov, *Solution of incorrectly formulated problems and the regularization method*, Soviet Math. Dokl. **4** (1963), 1035–1038.

[16] H.K. Xu, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298** (2004), 279–291.

[17] H.K. Xu, *A strong convergence theorem for contraction semigroups in Banach spaces*, Bull. Aust. Math. Soc. **72** (2005), 371–379.