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CRITERION FOR THE EXPONENTIAL STABILITY OF DISCRETE EVOLUTION FAMILY OVER BANACH SPACES

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Abstract. Let $\mathcal{T}(1)$ is the algebraic generator of the discrete semigroup $\boldsymbol{T} = \{\mathcal{T}(n)\}_{n\geq 0}$. We prove that the system $\xi_{n+1} = \mathcal{T}(1)\xi_n$ is uniformly exponentially stable if and only if for any θ, a real number and any p-periodic sequence $z(n)$ with $z(0) = 0$ the unique solution of the Cauchy Problem

$$\begin{cases} \xi_{n+1} = \mathcal{T}(1)\xi_n + e^{i\theta(n+1)}z(n+1), \\ \quad \xi_0 = 0 \end{cases} \qquad (\mathcal{T}(1), \theta, 0)$$

is bounded. We also extend the above result to p-periodic system $\zeta_{n+1} = \mathcal{A}_n\zeta_n$, i.e., we proved that the system $\zeta_{n+1} = \mathcal{A}_n\zeta_n$ is uniformly exponentially stable if and only if for $\theta \in \mathbb{R}$ and any p-periodic sequence $z(n)$, with $z(0) = 0$ the unique solution of the Cauchy Problem

$$\begin{cases} \zeta_{n+1} = \mathcal{A}_n\zeta_n + e^{i\theta(n+1)}z(n+1), \\ \quad \zeta_0 = 0 \end{cases} \qquad (\mathcal{A}_n, \theta, 0)$$

is bounded. Here, $\mathcal{A}_n$ is a sequence of bounded linear operators on Banach space $\mathcal{X}$.

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1. Introduction

In 1821, A. L. Cauchy addressed, in the Chapter V of his *Cours d'Analyse* [9], the following problem:

Déterminer la fonction $\psi(x)$ de manière qu'elle reste continue entre deux limites réelles quelconques de la variable x, et que l'on ait pour toutes les valeurs réelles des variables x et y

$$\psi(x+y) = \psi(x)\psi(y). \tag{1.1}$$

This means the question of determining a function $\psi(x)$ in such a way that it remains continuous between two arbitrary real limits of the variable x, and that, for all real values of the variables x and y, one has (1.1) is satisfied.

In modern notations the Cauchy question can be restated as: Find all the maps $\mathcal{T}(\cdot) : \mathbb{R}_+ \to \mathbb{C}$ which satisfy the following functional equation

$$\begin{cases} \mathcal{T}(t+s) = \mathcal{T}(t)\mathcal{T}(s), \\ \quad \mathcal{T}(0) = 1. \end{cases} \tag{1.2}$$

It was easy to check that the function $\mathcal{T}(t) = e^{at}$, for all $t \in \mathbb{R}_+$ satisfies (1.2). Also, it was found that the function $\mathcal{T}(t) = e^{at}$ satisfies the differential equation

$$\begin{cases} \dfrac{d\mathcal{T}(t)}{dt} = a\mathcal{T}(t), \\ \mathcal{T}(0) = 1. \end{cases}$$

The next question was to extend the same idea to functions of the form $\mathcal{T}(\cdot) : \mathbb{R}_+ \to \mathcal{M}_n(\mathbb{C})$ where $\mathcal{M}_n(\mathbb{C})$ is the space of all square matrices of order n. In this case the solution of the differential equation

$$\begin{cases} \dfrac{d\mathcal{T}(t)}{dt} = \mathcal{A}\mathcal{T}(t), \\ \mathcal{T}(0) = I \end{cases}$$

is $\mathcal{T}(t) = e^{tA}$ where, in 1888 the exponential of the matrix $\mathcal{A}$ were defined by G. Peano [15, 16]. Later on, the family $\mathcal{T}(t) = (e^{tA})_{t\geq 0}$, $t \geq 0$ was called semigroup of operators generated by the matrix $\mathcal{A}$.

In 1892, Liapunov gives his classical Liapunov stability theorem which is stated as

Theorem 1.1. ([12]) *Let $(e^{t\mathcal{A}})_{t\geq 0}$ be the family of operators generated by $\mathcal{A} \in \mathcal{M}_n(\mathbb{C})$. Then the following assertions are equivalent.*

(a) *The semigroup is stable, i.e.,* $\lim\limits_{t\to\infty} \|e^{t\mathcal{A}}\| = 0$.

(b) *All eigenvalues of $\mathcal{A}$ have negative real part, i.e., $Re(\lambda) < 0$ for all the eigenvalues λ of $\mathcal{A}$.*

In 1970, one of the remarkable results in the stability of strongly continuous semigroup $\boldsymbol{T} = \{\mathcal{T}(t)\}_{t\geq 0}$ was obtained by R. Datko [10]. This result states that a strongly continuous semigroup of bounded linear operators acting on complex or real Banach space is uniformly exponentially stable if and only if

$$\int_0^\infty \|\mathcal{T}(t)x\|dt < \infty.$$

In 1972, Pazy [15] extended the result of Datko to more stronger form which stated that a strongly continuous semigroup of bounded linear operators acting on real or complex Banach space is uniformily exponentially stable if and only if

$$\int_0^\infty \|\mathcal{T}(t)x\|^p dt < \infty, \quad \text{for any } p \geq 1.$$

In the last few decades, the theory of exponential stability of semigroups of operators is well developed. For further results on this topic, we recommend [1]-[7], [13, 14, 18].

In 2008, the classical Lyapunov Theorem 1.1 was extended by the first author of this note [19], in the following manner:

Theorem 1.2. *The system $\dot{\xi}(t) = \mathcal{A}\xi(t)$ is exponentially stable if and only if for any $\theta \in \mathbb{R}$ and any $b \in \mathbb{C}^n$ the solution of the Cauchy Problem*

$$\begin{cases} \dot{\zeta}(t) = \mathcal{A}\zeta(t) + e^{i\theta t}b, \quad t \geq 0, \\ \zeta(0) = 0 \end{cases}$$

is bounded, where $\mathcal{A} \in \mathcal{M}_n(\mathbb{C})$.

In 2009, Buse and Zada [8] proved the following similar result for discrete systems:

Theorem 1.3. *The system $\xi_{n+1} = \mathcal{A}\xi_n$ is exponentially stable if and only if for any $\theta \in \mathbb{R}$ and any $b \in \mathbb{C}^m$ the solution of the discrete Cauchy Problem*

$$\begin{cases} \zeta_{n+1} = \mathcal{A}\zeta_n + e^{i\theta n}b, \quad n \in \mathbb{N}, \\ \quad \zeta_0 = 0 \end{cases}$$

is bounded.

In this article, we aim to study a similar result as above for the systems $\xi_{n+1} = \mathcal{T}(1)\xi_n$ and $\zeta_{n+1} = \mathcal{A}_n\zeta_n$, where $\mathcal{T}(1)$ is the algebric generator of the discrete semigroup $\{\mathcal{T}(n)\}_{n\geq 0}$ and $\mathcal{A}_n$ is the q-periodic sequence of bounded linear operators acting on X.

Indeed, we establish a result which states the following result as a corollary:

Theorem 1.4. *The system $\xi_{n+1} = \mathcal{T}(1)\xi_n$ is uniformly exponentially stable if and only if for each real number θ and each p-periodic sequence $z(n)$ with $z(0) = 0$ the unique solution of the Cauchy Problem*

$$\begin{cases} \xi_{n+1} = \mathcal{T}(1)\xi_n + e^{i\theta(n+1)}z(n+1), \\ \quad \xi_0 = 0 \end{cases} \tag{$\mathcal{T}(1), \theta, 0$}$$

is bounded.

Moreover, we prove the following theorem:

Theorem 1.5. *The system $\zeta_{n+1} = \mathcal{A}_n\zeta_n$ is uniformly exponentially stable if and only if for any real number θ and any p-periodic sequence $z(n)$ with $z(0) = 0$ the unique solution of the Cauchy Problem*

$$\begin{cases} \zeta_{n+1} = \mathcal{A}_n\zeta_n + e^{i\theta(n+1)}z(n+1), \\ \quad \zeta_0 = 0 \end{cases} \tag{$\mathcal{A}_n, \theta, 0$}$$

is bounded.

2. Notations and preliminaries

Let $\mathcal{X}$ be a real or complex Banach space and $\mathcal{B}(\mathcal{X})$ the Banach algebra of all linear and bounded operators acting on $\mathcal{X}$.

We denote by $\|\cdot\|$ the norms of operators and vectors. Denote by $\mathbb{R}_+$ the set of real numbers and by $\mathbb{N}$ the set of all non-negative integers.

Let $B(\mathbb{N}, \mathcal{X})$ be the space of $\mathcal{X}$-valued bounded sequences with the supremum norm, and $P_0^p(\mathbb{N}, \mathcal{X})$ be the space of p-periodic (with $p \geq 2$) sequences z with $z(0) = 0$. Then clearly $P_0^p(\mathbb{N}, \mathcal{X})$ is a closed subspace of $B(\mathbb{N}, \mathcal{X})$.

Throughout this paper $\mathcal{A} \in \mathcal{B}(\mathcal{X})$, $\sigma(\mathcal{A})$ denotes the spectrum of A and $r(\mathcal{A})$ denotes the spectral radius of $\mathcal{A}$, and is defined as $r(\mathcal{A}) = \sup\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$. It is well known that $r(\mathcal{A}) := \lim\limits_{n\to\infty} \|\mathcal{A}^n\|^{\frac{1}{n}}$. The resolvent set of $\mathcal{A}$ is defined as $\rho(\mathcal{A}) := \mathbb{C}\backslash\sigma(\mathcal{A})$, i.e the set of all $\lambda \in \mathbb{C}$ for which $\mathcal{A} - \lambda I$ is an invertible operator in $\mathcal{B}(\mathcal{X})$.

We give some results in the framework of general Banach space and spaces of sequences as defined above.

Recall that $\mathcal{A}$ is power bounded if there exists a positive constant M such that $\|\mathcal{A}^n\| \leq M$ for all $n \in \mathbb{N}$.

We introduce few Lemmas with their proofs from [5], for the sake of the self-containedness of the paper.

Lemma 2.1. ([5]) *Let $\mathcal{A} \in \mathcal{B}(\mathcal{X})$. If there exists $M > 0$ such that*

$$\sup_{n\in\mathbb{N}} \|I + \mathcal{A} + \cdots + \mathcal{A}^n\| = M < \infty, \tag{2.1}$$

then $\mathcal{A}$ is power bounded and $1 \in \rho(\mathcal{A})$.

Proof. The proof is given in [5], but for convenience we will prove this Lemma in the sequel.

Since we have the identity

$$\mathcal{A}^{n+1} = I + (\mathcal{A} - I)(I + \mathcal{A} + \cdots + \mathcal{A}^n),$$

by using the inequality (2.1) we get that A is power bounded.

Next, suppose that $1 \in \sigma(\mathcal{A})$. Then there exists a sequence $(\xi_m)_{m\in\mathbb{N}}$ with $\xi_m \in \mathcal{X}$, $\|\xi_m\| = 1$ and $(I - \mathcal{A})\xi_m \to 0$ as $m \to \infty$. Now $\mathcal{A}$ is power bounded, and hence $\mathcal{A}^k(I - \mathcal{A})x_m \to 0$ as $m \to \infty$ uniformly for $k \in \mathbb{N}$. Then, $N \in \mathbb{N}$, $N > 2M$ and $m \in \mathbb{N}$ such that

$$\|\mathcal{A}^k(I - \mathcal{A})x_m\| \le \frac{1}{2N},\ k = 0, 1, \cdots, N.$$

Therefore,

$$\begin{aligned} M &\ge \|\sum_{k=0}^{N} \mathcal{A}^k \xi_m\| = \|\xi_m + \sum_{k=1}^{N} \mathcal{A}^k \xi_m\| \\ &= \|\xi_m + \sum_{k=1}^{N}(\xi_m + \sum_{j=0}^{k-1} \mathcal{A}^j(\mathcal{A} - I)\xi_m)\| \\ &= \|(N+1)\xi_m + \sum_{k=1}^{N}\sum_{j=0}^{k-1} \mathcal{A}^j(\mathcal{A} - I)\xi_m\| \\ &\ge (N+1) - \frac{N(N+1)}{4N} > \frac{N}{2} > M, \end{aligned}$$

which is absurd and hence $1 \in \rho(\mathcal{A})$. □

Lemma 2.2. *Let $\mathcal{V} \in \mathcal{B}(\mathcal{X})$ and $\theta \in \mathbb{R}$. If*

$$\sup_{n\in\mathbb{N}} \|\sum_{k=0}^{n} e^{i\theta k} \mathcal{V}^k\| = M_\theta < \infty. \tag{2.2}$$

Then $\mathcal{V}$ is power bounded and $e^{-i\theta} \in \rho(\mathcal{V})$.

Proof. Let $\mathcal{A} = e^{i\theta}\mathcal{V}$, then by Lemma 2.1 we have $\mathcal{A}$ is power bounded but $\|\mathcal{A}\| = \|\mathcal{V}\|$, hence $\mathcal{V}$ is power bounded. Also again, by Lemma 2.1, we have $1 \in \rho(\mathcal{A}) = \rho(e^{i\theta}\mathcal{V})$, i.e., $e^{i\theta}\mathcal{V} - I$ is invertible, from this we get $\mathcal{V} - e^{-i\theta}I$ is invertible. Hence $e^{-i\theta} \in \rho(\mathcal{V})$. □

Lemma 2.3. *Let $\mathcal{V} \in \mathcal{B}(\mathcal{X})$. If the inequality (2.2) holds true for all $\theta \in \mathbb{R}$, then $r(\mathcal{V}) < 1$.*

Proof. From Lemma 2.2 we have $\mathcal{V}$ is power bounded, so there exists $M > 0$ such that $\|\mathcal{V}^n\| \leq M$ for all $n \in \mathbb{N}$, then clearly $r(\mathcal{V}) := \lim_{n\to\infty} \|\mathcal{V}^n\|^{\frac{1}{n}} \leq 1$. But $e^{i\theta} \in \rho(\mathcal{V})$ for all $\theta \in R$ and $\sigma(\mathcal{V})$ is compact, hence $r(\mathcal{V}) < 1$. □

3. Exponential stability of discrete semigroup

We recall that a discrete semigroup is a family $\boldsymbol{T} = \{\mathcal{T}(n) : n \in \mathbb{N}\}$ of bounded linear operators acting on $\mathcal{X}$ which satisfies the following conditions:

(1) $\mathcal{T}(0) = I$, the identity operator on $\mathcal{X}$,
(2) $\mathcal{T}(n + m) = \mathcal{T}(n)\mathcal{T}(m)$ for all $n, m \in \mathbb{N}$.

It is clear that $\mathcal{T}(n) = \mathcal{T}^n(1)$ for all $n \in \mathbb{N}$, $\mathcal{T}(1)$ is called the algebraic generator of the semigroup $\boldsymbol{T}$.

The growth bound of $\boldsymbol{T}$ is denoted by $\omega_0(\boldsymbol{T})$ and is defined as

$$\omega_0(\boldsymbol{T}) := \inf\Big\{\omega \in \mathbb{R} : \text{ there exists } M_\omega \geq 1 \text{ such that } \forall n \in \mathbb{N},\ \|\mathcal{T}(n)\| \leq M_\omega e^{\omega n}\Big\}.$$

The family $\boldsymbol{T}$ is uniformly exponentially stable if $\omega_0(\boldsymbol{T})$ is negative, or equivalently, if there exists $M \geq 1$ and $\omega > 0$ such that $\|\mathcal{T}(n)\| \leq Me^{-\omega n}$ for all $n \in \mathbb{N}$.

Let us divide n by p, i.e., $n = lp + r$ for some $l \in \mathbb{N}$, where $r \in \{0, 1, \cdots, p-1\}$. We consider the following sets which will be useful along this work

$$\mathcal{A}_j := \{1 + jp, 2 + jp, \cdots, (j + 1)p - 1\}, \quad \text{for all } \ j \in \mathbb{N}.$$

If $r \geq 1$ then

$$B_l := \{lp + 1, lp + 2, \cdots, lq + r\}$$

and

$$C := \{0, p, 2p, \cdots, lp\}.$$

It is clear that

$$\bigcup_{j=0}^{l-1} A_j \bigcup B_l \bigcup C = \{0, 1, 2, \cdots, n\}. \tag{3.1}$$

We denote by $\mathcal{W}$ the class of all sequences from the space $\mathbb{P}_0^p(\mathbb{N}, X)$ in the form of $\{z(n) : z(n) = n(p - n)T(n)\}$, that is,

$$\mathcal{W} = \Big\{z(n) \in \mathbb{P}_0^p(\mathbb{N}, X) : z(n) = n(p - n)\mathcal{T}(n)\Big\}. \tag{3.2}$$

It is not difficult to see that $\mathcal{W}$ is the subspace of $\mathbb{P}_0^p(\mathbb{N}, X)$.

Our first result is stated as

Theorem 3.1. *Let $\mathcal{T}(1)$ is the algebraic generator of the discrete semigroup $\mathbf{T} = \{\mathcal{T}(n) : n \in \mathbb{N}\}$ on X and $\theta \in \mathbb{R}$. Then the following statements hold true.*

- **(1)** *If the system $\xi_{n+1} = \mathcal{T}(1)\xi_n$ is uniformly exponentially stable then for each real number θ and for each p-periodic sequence $z(n)$ with $z(0) = 0$ the unique solution solution of the Cauchy Problem $(\mathcal{T}(1), \theta, 0)$ is bounded.*
- **(2)** *If for each real number θ and for each p-periodic sequence $z(n)$ in $\mathcal{W}$ the unique solution solution of the Cauchy Problem $(\mathcal{T}(1), \theta, 0)$ is bounded, then $T(n)$ is uniformly exponentially stable.*

Proof. **(1)** First we will show that if $\mathcal{T}(n)$ is uniformly exponentially stable then the unique solution ζ_n of $(\mathcal{T}(1), \theta, 0)$ is bounded.

As $\mathcal{T}(n)$ is uniformly exponentially stable; there exist two positive constants M and β such that $\|\mathcal{T}(n)\| \leq Me^{-\beta n}$, for all $n \in \mathbb{N}$. The unique solution of the Cauchy Problem $(\mathcal{T}(1), \theta, 0)$ is given by

$$\xi_n = \sum_{k=0}^{n} e^{i\theta k}\mathcal{T}(n-k)z(k).$$

Taking the norm of both sides, we get

$$\begin{aligned}
\|\xi_n\| &= \|\sum_{k=0}^{n} e^{i\theta k}\mathcal{T}(n-k)z(k)\| \\
&\leq \sum_{k=0}^{n} \|e^{i\theta k}\mathcal{T}(n-k)z(k)\| = \sum_{k=0}^{n} \|e^{i\theta k}\|\|\mathcal{T}(n-k)\|\|z(k)\| \\
&\leq \sum_{k=0}^{n} \|\mathcal{T}(n-k)\|\|z(k)\| \\
&\leq \sum_{k=0}^{n} Me^{-\beta(n-k)}M' = M''e^{-\beta n}\sum_{k=0}^{n} e^{\beta k} \\
&\leq M'''.
\end{aligned}$$

Thus, the solution of the Cauchy Problem $(\mathcal{T}(1), \theta, 0)$ is bounded.

(2) The proof of the second part is not so easy.

As the unique solution of the Cauchy Problem

$$\begin{cases} \xi_{n+1} = \mathcal{T}(1)x_n + e^{i\theta(n+1)}z(n+1), \\ \quad \xi_0 = 0, \end{cases}$$

is

$$\xi_n = \sum_{k=0}^{n} e^{i\theta k}\mathcal{T}(n-k)z(k)$$

where $z(k) \in \mathcal{W}$, i.e., $z(k) = k(p-k)\mathcal{T}(k)$, according to the partition (3.1), i.e., $\{0, 1, 2, \cdots, n\} = \cup_{j=0}^{N-1} A_j \cup B_N \cup C$, let us replace k by $k - jp$ in $z(k)$ for $\mathcal{A}_j$ and define it as

$$z(k) = \begin{cases} (k-jp)[(1+j)p-k]\mathcal{T}(k-jp), & \text{if } \ k \in \mathcal{A}_j, \\ k(p-k)\mathcal{T}(k), & \text{if } \ k \in B_l, \\ 0, & \text{if } \ k \in C. \end{cases}$$

Then clearly $z(k) \in \mathcal{W}$. This implies that

$$\begin{aligned} \xi_n &= \sum_{k=0}^{n} e^{i\theta k}\mathcal{T}(n-k)z(k) \\ &= \sum_{k\in\bigcup_{j=0}^{l-1}\mathcal{A}_j\cup B_l\cup C} e^{i\theta k}\mathcal{T}(lp+r-k)z(k) \\ &= \sum_{k\in\bigcup_{j=0}^{l-1}\mathcal{A}_j} e^{i\theta k}\mathcal{T}(lp+r-k)z(k) + \sum_{k\in B_l} e^{i\theta k}\mathcal{T}(lp+r-k)z(k) \\ &\quad + \sum_{k\in C} e^{i\theta k}\mathcal{T}(lp+r-k)z(k) \\ &= \sum_{j=0}^{l-1}\sum_{k=1+jp}^{(p-1+jp)} e^{i\theta k}\mathcal{T}(lp+r-k)z(k) + \sum_{k=lp+1}^{lp+r} e^{i\theta k}\mathcal{T}(lp+r-k)z(k) + 0 \\ &= J_1 + J_2, \end{aligned}$$

where

$$\begin{aligned} J_1 &= \sum_{j=0}^{l-1}\sum_{k=1+jp}^{(p-1+jp)} e^{i\theta k}\mathcal{T}(lp+r-k)(k-jq)[p-(k-jp)]\mathcal{T}(k-jp)x \\ &= \sum_{j=0}^{l-1}\mathcal{T}(lp+r-jp)\sum_{k=1+jp}^{(p-1+jp)} e^{i\theta k}(k-jp)[p-(k-jp)]x \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{l-1} \mathcal{T}(lp+r-jp)e^{i\theta jp} \sum_{\nu=1}^{p-1} e^{i\theta\nu}\nu(p-\nu)x \\
&= \sum_{j=0}^{l-1} e^{-i\theta(lp+r-jp)}\mathcal{T}(lp+r-jp)e^{i\theta(lp+r)} \sum_{\nu=1}^{p-1} e^{i\theta\nu}\nu(p-\nu)x \\
&= \sum_{\omega=r+p}^{r+lp} e^{-i\theta\omega}\mathcal{T}^{\omega}(1)e^{i\theta n} \sum_{\nu=1}^{p-1} e^{i\theta\nu}\nu(p-\nu)x \\
&= \sum_{\omega=r+p}^{n} e^{-i\theta\omega}\mathcal{T}^{\omega}(1)S(x)
\end{aligned}$$

with $S(x) = e^{i\theta n}\sum_{\nu=1}^{p-1} e^{i\theta\nu}\nu(p-\nu)x$. And

$$\begin{aligned}
J_2 &= \sum_{\rho=0}^{r-1} e^{i\theta(lp+r-\rho)}\mathcal{T}(\rho)z(lp+r-\rho)x \\
&= \sum_{\rho=0}^{r-1} e^{i\theta(lp+r-\rho)}\mathcal{T}(\rho)z(r-\rho)x.
\end{aligned}$$

Hence,

$$\xi_n = \sum_{k=0}^{n} e^{i\theta k}\mathcal{T}(n-k)z(k) = \sum_{\omega=r+p}^{n} e^{-i\theta\omega}T^{\omega}(1)S(x) + \sum_{\rho=0}^{r-1} e^{i\theta(n-\rho)}\mathcal{T}(\rho)z(r-\rho)x.$$

Now by our assumption x_n is bounded, i.e.,

$$\|\xi_n\| = \sup_{n\geq 0}\Big\| \sum_{k=0}^{n} e^{i\theta k}\mathcal{T}(n-k)z(k)\Big\| < \infty.$$

Thus

$$\sup_{n\geq 0}\| \sum_{\omega=r+p}^{n} e^{-i\theta\omega}\mathcal{T}^{\omega}(1)S(x) + \sum_{\rho=0}^{r-1} e^{i\theta(n-\rho)}\mathcal{T}(\rho)z(r-\rho)x\| < \infty,$$

which implies that

$$\sup_{n\geq 0}\| \sum_{\omega=r+p}^{n} e^{-i\theta\omega}\mathcal{T}^{\omega}(1)S(x)\| < \infty,$$

i.e.,

$$\sup_{n\geq 0}\| \sum_{\omega=r+p}^{n} e^{-i\theta\omega}\mathcal{T}^{\omega}(1)\| < \infty.$$

Thus by Lemma 3.4, $\mathbf{T} = \{\mathcal{T}(n)\}_{n\geq 0}$ is uniformly exponentially stable. □

As a consequence of this theorem, we state the following Corollary.

Corollary 3.2. *The system $\xi_{n+1} = \mathcal{T}(1)\xi_n$ is uniformly exponentially stable if and only if for any real number θ and any p-periodic sequence $z(n)$ with $z(0) = 0$ the unique solution of the Cauchy Problem $(\mathcal{T}(1), \theta, 0)$ is bounded.*

4. Uniform exponential stability of discrete evolution family

The family $\mathcal{U} = \{\mathbb{U}(m, n) : m, n \in \mathbb{Z}_+, m \geq n\}$ of bounded linear operators is called p-periodic discrete evolution family, for a fixed integer $p \geq 2$, if it satisfies the following properties

- $\mathbb{U}(m, m) = I$, for all $m \in \mathbb{Z}_+$.
- $\mathbb{U}(m, n)\mathbb{U}(n, r) = \mathbb{U}(m, r)$, for all $m \geq n \geq r$, $m, n, r \in \mathbb{Z}_+$.
- $\mathbb{U}(m + p, n + p) = \mathbb{U}(m, n)$, for all $m \geq n$, $m, n \in \mathbb{Z}_+$.
- The map $(m, n) \mapsto \mathbb{U}(m, n)x : \{(m, n) : m, n \in \mathbb{Z}_+ \ : m \geq n\} \to \mathcal{X}$ is continuous for all $m \geq n$.

It is well known that $\mathcal{U}$ is exponentially bounded, that is there exist $\omega \in \mathbb{R}$ and $M_\omega \geq 0$ such that

$$\|\mathbb{U}(m, n)\| \leq M_\omega e^{\omega(m-n)}, \quad \text{for all} \quad m \geq n. \tag{4.1}$$

The growth bound of exponentially bounded evolution family $\mathcal{U}$ is defined by

$$\omega_0(\mathcal{U}) := \inf\Big\{\omega \in \mathbb{R} : \text{ there is } M_\omega \geq 0 \text{ such that (4.1) holds}\Big\}.$$

Let us consider the following discrete Cauchy Problem:

$$\begin{cases} \zeta_{n+1} = \mathcal{A}_n\zeta_n + e^{i\theta(n+1)}z(n+1), & n \in \mathbb{Z}_+, \\ \quad \zeta_0 = 0, \end{cases}$$

where the sequence $(\mathcal{A}_n)$ is q-periodic, i.e., $\mathcal{A}(n + p) = \mathcal{A}_n$ for all $n \in \mathbb{Z}_+$ and a fixed $p \geq 2$.

Let

$$\mathbb{U}(n, k) = \begin{cases} \mathcal{A}_{n-1}\mathcal{A}_{n-2}\cdots\mathcal{A}_k, & if \quad k \leq n - 1, \\ I, & if \quad k = n. \end{cases}$$

Then, the family $\{\mathbb{U}(n, k)\}_{n\geq k\geq 0}$ is a discrete p-periodic evolution family and the solution ζ_n of the Cauchy Problem $(\mathcal{A}_n, \theta, 0)$ in terms of the discrete evolution family $\mathbb{U}(n, k)$ is given by:

$$\zeta_n = \sum_{k=1}^{n} e^{i\theta k}\mathbb{U}(n, k)z(k).$$

We denote by $\mathcal{B}$ the class of all sequences from the space $\mathbb{P}_0^p(\mathbb{N}, X)$ in the form of $\{z(n) : z(n) = n(p-n)\mathbb{U}(n,0)\}$, i.e.,

$$\mathcal{W} = \Big\{ z(n) \in \mathbb{P}_0^p(\mathbb{N}, X) : z(n) = n(p-n)\mathbb{U}(n,0) \Big\}. \tag{4.2}$$

Clearly $\mathcal{W}$ is the subspace of $\mathbb{P}_0^p(\mathbb{N}, X)$.

Now we are in position to state and prove our main result.

Theorem 4.1. *Let $\mathcal{U} = \{\mathbb{U}(m,n) : m,n \in \mathbb{N}\}$ be a discrete evolution family on $\mathcal{X}$ and θ is any real number. The following statements hold true.*

(1) *If the system $\xi_{n+1} = \mathcal{A}_n\xi_n$ is uniformly exponentially stable, equivalently $\mathcal{U}$ is uniformly exponentially stable then for each real number θ and each p-periodic sequence z_n with $z_0 = 0$ the unique solution of the Cauchy Problem $(\mathcal{A}_n, \theta, 0)$ is bounded.*

(2) *If for each real number θ and each p-periodic sequence z_n in $\mathcal{W}$ the unique solution of the Cauchy Problem $(\mathcal{A}_n, \theta, 0)$ is bounded, then $\mathcal{U}$ is uniformly exponentially stable.*

Proof. **(1)** Here we will show that if $\mathbb{U}(m,n)$ is uniformly exponentially stable then the unique solution ζ_n of $(\mathcal{A}_n, \theta, 0)$ is bounded. As $\mathbb{U}(m,n)$ is uniformly exponentially stable thus there exist two positive constants M and ν such that $\|\mathbb{U}(m,n)\| \leq Me^{-\nu(m-n)}$, for all $m,n \in \mathbb{N}$.

The unique solution of the Cauchy Problem $(\mathcal{A}_n, \theta, 0)$ is given by

$$\zeta_n = \sum_{k=1}^{n} e^{i\theta k}\mathbb{U}(n,k)z_k.$$

Taking the norm of both sides, we obtain

$$\begin{aligned}
\|\zeta_n\| &= \|\sum_{k=1}^{n} e^{i\theta k}\mathbb{U}(n,k)z_k\| \\
&\leq \sum_{k=1}^{n} \|e^{i\theta k}\mathbb{U}(n,k)z_k\| = \sum_{k=1}^{n} \|e^{i\theta k}\|\|\mathbb{U}(n,k)\|\|z_k\| \\
&\leq \sum_{k=1}^{n} \|\mathbb{U}(n,k)\|\|z_k\| \\
&\leq \sum_{k=1}^{n} Me^{-\nu(n-k)}M' = M''e^{-\nu n}\sum_{k=1}^{n} e^{\nu k} \\
&\leq M'''.
\end{aligned}$$

Thus, the solution of the Cauchy Problem $(\mathcal{A}_n, \theta, 0)$ is bounded.

(2) The proof of the second part is not so easy. As the unique solution of the Cauchy Problem

$$\begin{cases} \zeta_{n+1} = \mathcal{A}_n\zeta_n + e^{i\theta(n+1)}z(n+1), \\ \quad \zeta_0 = 0, \end{cases}$$

in terms of evolution family $\mathbb{U}(n,k)$ is

$$\zeta_n = \sum_{k=0}^{n} e^{i\theta k}\mathbb{U}(n,k)z(k)$$

where $z(k) \in \mathcal{B}$, i.e., $z(k) = k(q-k)\mathbb{U}(k,0)$. According to the partition (3.1), i.e., $\{0,1,2,\cdots,n\} = \bigcup_{j=0}^{l-1}\mathcal{A}_j \bigcup B_l \bigcup C$, let us replace k by $k - jp$ in $z(k)$ for $\mathcal{A}_j$ and define it as

$$z(k) = \begin{cases} (k-jp)[(1+j)p-k]\mathbb{U}(k-jp,0), & \text{if} \quad k \in \mathcal{A}_j, \\ k(p-k)\mathbb{U}(k,0), & \text{if} \quad k \in B_l, \\ 0, & \text{if} \quad k \in C. \end{cases}$$

Then clearly $z(k) \in \mathcal{B}$. Thus

$$\begin{aligned}
\zeta_n &= \sum_{k=1}^{n} e^{i\theta k}\mathbb{U}(n,k)z(k) = \sum_{k=1}^{lp+r} e^{i\theta k}\mathbb{U}(lp+r,k)z(k) \\
&= \sum_{k\in\cup_{j=0}^{l-1}A_j\cup B_l\cup C} e^{i\theta k}\mathbb{U}(lp+r,k)z(k) \\
&= \sum_{k\in\cup_{j=0}^{l-1}A_j} e^{i\theta k}\mathbb{U}(lp+r,k)z(k) + \sum_{k\in B_l} e^{i\theta k}\mathbb{U}(lp+r,k)z(k) \\
&\quad + \sum_{k\in C} e^{i\theta k}\mathbb{U}(lp+r,k)z(k) \\
&= \sum_{j=0}^{l-1}\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}\mathbb{U}(lp+r,k)z(k) + \sum_{k=lp+1}^{lp+r} e^{i\theta k}\mathbb{U}(lp+r,k)z(k) \\
&\quad + \sum_{k\in C} e^{i\theta k}\mathbb{U}(lp+r,k)z(k) \\
&= \sum_{j=0}^{l-1}\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}\mathbb{U}(lp+r,k)(k-jp)[(1+j)p-k]\mathbb{U}(k-jp,0)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=lp+1}^{lp+r} e^{i\theta k}\mathbb{U}(lp+r,k)k(p-k)\mathbb{U}(k,0)\\
&+\sum_{k\in C} e^{i\theta k}\mathbb{U}(lp+r,k)0\\
=\;& \sum_{j=0}^{l-1}\sum_{k=1+jp}^{p-1+jp} e^{i\mu k}\mathbb{U}(lp+r,k)(k-jp)[(1+j)p-k]\mathbb{U}(k-jp,0)\\
&+\sum_{k=lp+1}^{lp+r} e^{i\theta k}\mathbb{U}(lp+r,k)k(p-k)\mathbb{U}(k,0)\\
=\;& I_1+I_2,
\end{aligned}
$$

where

$$
I_1=\sum_{j=0}^{l-1}\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}\mathbb{U}(lp+r,k)(k-jp)[(1+j)p-k]\mathbb{U}(k-jp,0)
$$

and

$$
I_2=\sum_{k=lp+1}^{lp+r} e^{i\theta k}\mathbb{U}(lp+r,k)k(p-k)\mathbb{U}(k,0).
$$

Now to further simplify I_1

$$
\begin{aligned}
I_1 &= \sum_{j=0}^{l-1}\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}\mathbb{U}(lp+r,k)(k-jp)[(1+j)p-k)]\mathbb{U}(k-jp,0)\\
&= \sum_{j=0}^{l-1}\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}\mathbb{U}(lp+r,k)(k-jp)[(1+j)p-k)]\mathbb{U}(k,jp)\\
&= \sum_{j=0}^{l-1}\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}\mathbb{U}(lp+r,jp)(k-jp)[(1+j)p-k)]\\
&= \sum_{j=0}^{l-1}\mathbb{U}(lp+r,jp)\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}(k-jp)[(1+j)p-k)]\\
&= \sum_{j=0}^{-1}\mathbb{U}(lp+r,jp)\sum_{k=1+jp}^{p-1+jp} e^{i\theta k}(k-jp)[p-(k-jp)]\\
&= \sum_{j=0}^{l-1}\mathbb{U}(r,0)\mathbb{U}^{l-j}(p,0)e^{i\theta jp}\sum_{v=1}^{p-1} e^{i\theta v}v(p-v)
\end{aligned}
$$

$$\begin{aligned}
&= \mathbb{U}(r,0)\sum_{v=1}^{p-1} e^{i\theta v} v(p-v)\sum_{j=0}^{l-1} e^{i\theta jp}\mathbb{U}^{l-j}(p,0)\\
&= \mathbb{U}(r,0)\sum_{v=1}^{p-1} e^{i\theta v} v(p-v)\sum_{\alpha=1}^{l} e^{i\theta p(\alpha)}\mathbb{U}^{\alpha}(p,0)\\
&= \mathbb{U}(r,0)\sum_{v=1}^{p-1} e^{i\theta v} v(p-v)e^{i\theta pl}\sum_{\alpha=1}^{l} e^{i\theta p\alpha}\mathbb{U}^{\alpha}(p,0)\\
&= G(\mu,p)\sum_{\alpha=1}^{l} e^{i\theta p\alpha}\mathbb{U}^{\alpha}(p,0),
\end{aligned}$$

where $G(\mu,p) = \mathbb{U}(r,0)\sum_{v=1}^{p-1} e^{i\theta v} v(p-v)e^{-i\theta pl} \neq 0$. Also

$$\begin{aligned}
I_2 &= \sum_{k=lp+1}^{lp+r} e^{i\theta k}\mathbb{U}(lp+r,k)k(p-k)\mathbb{U}(k,0)\\
&= \sum_{k=lp+1}^{lp+r} e^{i\theta k}\mathbb{U}(lp+r,0)k(p-k)\\
&= \mathbb{U}(lp+r,0)\sum_{k=lp+1}^{lp+r} e^{i\theta k}k(p-k).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{k=0}^{n} e^{i\theta k}\mathbb{U}(n,k)z(k)\\
&= G(\theta,p)\sum_{\alpha=1}^{l} e^{i\theta p\alpha}\mathbb{U}^{\alpha}(p,0)x + \mathbb{U}(lp+r,0)\sum_{k=lp+1}^{lp+r} e^{i\theta k}k(p-k).
\end{aligned}$$

As ζ_n is bounded, we have I_1 is bounded, i.e.,

$$\sup_{l\geq 0}\Big\|\sum_{\alpha=0}^{l} e^{i\mu p\alpha}\mathbb{U}^{\alpha}(p,0)\Big\| < \infty.$$

Now applying Lemma (2.2), we obtain that $\mathbb{U}(p,0)$ is power bounded and $e^{-i\mu p} \in \rho(\mathbb{U}(p,0))$. Therefore, $\mathcal{U}$ is uniformly exponentially stable and hence the proof is completed. □

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