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MORE ON CERTAIN OPIAL–TYPE INEQUALITY FOR FRACTIONAL DERIVATIVES AND EXPONENTIALLY CONVEX FUNCTIONS

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Abstract. In this paper we consider a certain class of convex functions in Opial–type integral inequalities. Cauchy type mean value theorems are proved and used in studying Stolarsky type means defined by the observed integral inequalities. A method of producing n-exponentially convex and exponentially convex functions is applied. Also, some new Opial–type equalities are given involving fractional integrals and fractional derivatives.

1. Introduction

We consider a particular class of convex functions in Opial–type integral inequalities from which we construct functionals. Our main object is to give Cauchy type mean value theorems and use them for Stolarsky type means,

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all defined by the observed integral inequalities, and also, to prove the n-exponential convexity for the functionals. Also, we produce some new Opial–type inequalities for Riemann–Liouville fractional integrals and three main types of fractional derivatives: Riemann–Liouville, Canavati and Caputo type.

2. Preliminaries

We start with following inequality established in 1960 by Opial [13]:

Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then

$$\int_0^h |x(t)x'(t)|dt \leq \frac{h}{4}\int_0^h \left(x'(t)\right)^2 dt, \tag{2.1}$$

where constant $\frac{h}{4}$ is the best possible.

Over the last 50 years, Opial's inequality (2.1) is studied by many mathematicians and extended, generalized in different ways. It is recognized as fundamental result in the theory of differential equations (see the monograph [1]). Following theorems include such generalizations of Opial's inequality given in [2] and for them we need next characterization:

We say that a function $u : [a,b] \longrightarrow \mathbb{R}$ belongs to the class $U_1(v,K)$ if it admits the representation

$$u(x) = \int_a^x K(x,t)v(t)\,dt$$

where v is a continuous function and K is an arbitrary non-negative kernel such that $v(x) > 0$ implies $u(x) > 0$ for every $x \in [a,b]$. We also assume that all integrals under consideration exist and are finite.

Theorem 2.1. *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $u \in U_1(v,K)$ where $\left(\int_a^x (K(x,t))^p\,dt\right)^{\frac{1}{p}} \leq M$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned}&\int_a^b |u(x)|^{1-q}\,\phi'(|u(x)|)|v(x)|^q\,dx\\ \leq\;& \frac{q}{M^q\,(b-a)}\int_a^b \phi\left((b-a)^{\frac{1}{q}}M|v(x)|\right)dx.\end{aligned} \tag{2.2}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then the reverse inequality holds.

A similar result follows by using another class $U_2(v,K)$ of functions $u:[a,b]\longrightarrow\mathbb{R}$ which admits representation

$$u(x)=\int_x^b K(x,t)v(t)dt.$$

Theorem 2.2. *Let $\phi:[0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function such that for $q>1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0)=0$. Let $u\in U_2(v,K)$ where $\left(\int_x^b (K(x,t))^p\,dt\right)^{\frac{1}{p}}\le N$ and $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\begin{aligned}&\int_a^b |u(x)|^{1-q}\,\phi'(|u(x)|)|v(x)|^q dx\\ &\quad\le\ \frac{q}{N^q(b-a)}\int_a^b \phi\left((b-a)^{\frac{1}{q}}N|v(x)|\right)dx\,.\end{aligned}\tag{2.3}$$

If the function $\phi(x^{\frac{1}{q}})$ is concave, then reverse inequality holds.

This paper is continuation of our work on Opial–type integral inequalities (see [2, 9, 10]). In the following section we construct functionals and prove Cauchy type mean value theorems. Next, in Section 3, we prove some new Opial–type equalities for fractional integrals and fractional derivatives as an application of our main results. Improvements of composition identities for the fractional derivatives, given in papers [3, 4, 5], are applied in these results. In Section 4 we produce the n-exponentially convex functions by applying an elegant method of exponential convexity. At the end of the paper, we use Cauchy mean value theorems for Stolarsky type means defined by the observed functionals to give the related examples (see Section 5).

Recall, $C^n[a,b]$ denotes the space of all functions on $[a,b]$ which have continuous derivatives up to order n, and $AC[a,b]$ is the space of all absolutely continuous functions on $[a,b]$. By $AC^n[a,b]$ we denote the space of all functions $f\in C^{n-1}[a,b]$ with $f^{(n-1)}\in AC[a,b]$. By $L_p[a,b]$, $1\le p<\infty$, we denote the space of all Lebesgue measurable functions f for which $|f^p|$ is Lebesgue integrable on $[a,b]$.

3. Main Results

Motivated by the inequalities (2.2) and (2.3), we define next functional:

$$\begin{aligned}\Psi_\phi(u,v)\quad=\quad&\frac{q}{M^q\,(b-a)}\int_a^b \phi\left((b-a)^{\frac{1}{q}}M|v(x)|\right)dx\\ &-\int_a^b |u(x)|^{1-q}\,\phi'(|u(x)|)|v(x)|^q\,dx\,,\end{aligned}\tag{3.1}$$

where functions ϕ, u and v are as in Theorem 2.1.
If $\phi(x^{\frac{1}{q}})$ is a convex function ($q > 1$), then, by Theorem 2.1, $\Psi_\phi(u, v) \geq 0$.

For our results we need following definition given in [15, p.7] and lemma from [9].

Definition 3.1. If g is strictly monotonic, then f is said to be (strictly) convex with respect to g if $f \circ g^{-1}$ is (strictly) convex.

Lemma 3.2. *Let $I \subseteq (0, \infty)$, $\phi \in C^2(I)$, $g(x) = x^q$, $q > 1$ and let*

$$m_1 \leq \frac{\xi \phi''(\xi) - (q-1)\phi'(\xi)}{q^2 \xi^{2q-1}} \leq M_1 \,,$$

for all $\xi \in I$. Then the functions ϕ_1, ϕ_2 defined as

$$\phi_1(x) = \frac{M_1\, x^{2q}}{2} - \phi(x), \tag{3.2}$$

$$\phi_2(x) = \phi(x) - \frac{m_1\, x^{2q}}{2} \tag{3.3}$$

are convex functions with respect to $g(x) = x^q$, that is $\phi_i(x^{\frac{1}{q}})$ $(i = 1, 2)$ are convex.

Next two theorems are our main results, and they follow methods used in [9, 10].

Theorem 3.3. *Let $\phi : [0, \infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Let $u \in U_1(v, K)$ where $\left(\int_a^x (K(x,t))^p \, dt\right)^{\frac{1}{p}} \leq M$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\phi \in C^2(I)$, where $I \subseteq (0, \infty)$ is compact interval, then there exists $\xi \in I$ such that the following equality holds*

$$\Psi_\phi(u, v) = \frac{\xi\, \phi''(\xi) - (q-1)\, \phi'(\xi)}{2\, q\, \xi^{2q-1}} \left((b-a) M^q \int_a^b |v(x)|^{2q} \, dx \right.$$
$$\left. -2 \int_a^b |u(x)|^q \, |v(x)|^q \, dx \right). \tag{3.4}$$

Proof. Suppose that $\psi(y)$ is bounded and $\min(\psi(y)) = m_1$, $\max(\psi(y)) = M_1$ where

$$\psi(y) = \frac{y\, \phi''(y) - (q-1)\phi'(y)}{q^2 y^{2q-1}}.$$

If we apply Theorem 2.1 for ϕ_1 defined by (3.2), then inequality (2.2) becomes

$$\frac{q}{M^q\,(b-a)}\int_a^b \phi\left((b-a)^{\frac{1}{q}}M|v(x)|\right)dx-\int_a^b |u(x)|^{1-q}\,\phi'(|u(x)|)|v(x)|^q\,dx$$
$$\leq\ \frac{q\,M_1}{2}\left((b-a)M^q\int_a^b |v(x)|^{2q}\,dx-2\int_a^b |u(x)|^q\,|v(x)|^q\,dx\right). \tag{3.5}$$

Similarly, if we apply Theorem 2.1 for ϕ_2 defined by (3.3), then inequality (2.2) becomes

$$\frac{q}{M^q\,(b-a)}\int_a^b \phi\left((b-a)^{\frac{1}{q}}M|v(x)|\right)dx-\int_a^b |u(x)|^{1-q}\,\phi'(|u(x)|)|v(x)|^q\,dx$$
$$\geq\ \frac{q\,m_1}{2}\left((b-a)M^q\int_a^b |v(x)|^{2q}\,dx-2\int_a^b |u(x)|^q\,|v(x)|^q\,dx\right). \tag{3.6}$$

By combining the above two inequalities with the fact

$$m_1\leq\frac{y\,\phi''(y)-(q-1)\phi'(y)}{q^2y^{2q-1}}\leq M_1,$$

there exists $\xi\in I$ such that (3.4) follows. □

Theorem 3.4. *Let $\phi_1,\phi_2:[0,\infty)\longrightarrow\mathbb{R}$ be differentiable functions such that for $q>1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0)=0$, $i=1,2$. Let $u\in U_1(v,K)$ where $\left(\int_a^x (K(x,t))^p\,dt\right)^{\frac{1}{p}}\leq M$ and $\frac{1}{p}+\frac{1}{q}=1$. If $\phi_1,\phi_2\in C^2(I)$, where $I\subseteq(0,\infty)$ is compact interval and*

$$(b-a)M^q\int_a^b |v(x)|^{2q}\,dx-2\int_a^b |u(x)|^q|v(x)|^q dx\neq 0\,,$$

then there exists an $\xi\in I$ such that we have

$$\frac{\Psi_{\phi_1}(u,v)}{\Psi_{\phi_2}(u,v)}=\frac{\xi\,\phi_1''(\xi)-(q-1)\,\phi_1'(\xi)}{\xi\,\phi_2''(\xi)-(q-1)\,\phi_2'(\xi)}\,, \tag{3.7}$$

provided the denominators are not equal to zero.

Proof. Let us consider $h\in C^2(I)$ defined by

$$h=\Psi_{\phi_2}(u,v)\,\phi_1-\Psi_{\phi_1}(u,v)\,\phi_2\,.$$

For this function, (3.1) gives us $\Psi_h(u,v)=0$. By Theorem 3.3 used on h follows that there exists $\xi\in I$ such that

$$\Psi_{\phi_2}(u,v)\frac{\xi\phi_1''(\xi)-(q-1)\phi_1'(\xi)}{2\,q\,\xi^{2q-1}}-\Psi_{\phi_1}(u,v)\frac{\xi\phi_2''(\xi)-(q-1)\phi_2'(\xi)}{2\,q\,\xi^{2q-1}}$$
$$\cdot\left((b-a)M^q\int_a^b |v(x)|^{2q}\,dx-2\int_a^b |u(x)|^q|v(x)|^q dx\right)=0.$$

From this we get (3.7). □

Remark 3.5. By considering nonnegative difference of inequality given in Theorem 2.2, similar results can be done analogously (for details see [10]).

4. Opial–type equalities for fractional integrals and fractional derivatives

Here we give applications of our main results for fractional integrals and fractional derivatives. We observe Riemann–Liouville fractional integrals and three main types of fractional derivatives: Riemann–Liouville, Canavati and Caputo type. For more details on the fractional calculus see the monograph [12].

Let $[a,b]$, $-\infty < a < b < \infty$, be a finite interval on the real axis $\mathbb{R}$. For $f \in L_1[a,b]$ the *left-sided and right-sided Riemann–Liouville fractional integrals* of order $\alpha > 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt, \quad x > a\,,$$

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)\, dt, \quad x < b\,.$$

Here Γ is the gamma function $\Gamma(\alpha) = \int_0^\infty e^{-t}\, t^{\alpha-1}\, dt$.

Theorem 4.1. *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\phi \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a compact interval. Let $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$ and $v \in L_1[a,b]$. Then there exists $\xi \in I$ such that the following equality holds*

$$\Psi_\phi(J_{a+}^{\alpha} v, v) = \frac{\xi\phi''(\xi) - (q-1)\phi'(\xi)}{2q\xi^{2q-1}} \left[\frac{(b-a)^{q\alpha}}{\Gamma^q(\alpha)\left[p(\alpha - \frac{1}{q})\right]^{\frac{q}{p}}} \int_a^b |v(x)|^{2q} dx - 2\int_a^b |J_{a+}^{\alpha} v(x)|^q |v(x)|^q dx \right]. \tag{4.1}$$

Proof. We follow the same idea as in [9, Theorem 6] and [2, Theorem 3.1]. For $x \in [a,b]$ let

$$K(x,t) = \begin{cases} \frac{1}{\Gamma(\alpha)}(x-t)^{\alpha-1}, & a \le t \le x\,; \\ 0, & x < t \le b\,, \end{cases}$$

$$u(x) = J_{a+}^{\alpha} v(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} v(t)\, dt\,, \tag{4.2}$$

$$P(x) = \left(\int_a^x (K(x,t))^p \, dt \right)^{\frac{1}{p}} = \frac{(x-a)^{\alpha - \frac{1}{q}}}{\Gamma(\alpha) \left[p \left(\alpha - \frac{1}{q} \right) \right]^{\frac{1}{p}}} .$$

It is easy to see that for $\alpha > \frac{1}{q}$ the function P is increasing on $[a,b]$, thus

$$\max_{x \in [a,b]} P(x) = \frac{(b-a)^{\alpha - \frac{1}{q}}}{\Gamma(\alpha) \, p^{\frac{1}{p}} \left(\alpha - \frac{1}{q} \right)^{\frac{1}{p}}} = M .$$

Hence $\left(\int_a^x K(x,t)^p dt \right)^{\frac{1}{p}} \leq M$, which with the function u defined by (4.2) and Theorem 3.3 gives us (4.1). □

Theorem 4.2. *Let $\phi_1, \phi_2 : [0, \infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0) = 0, i = 1, 2$. Further, let $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a compact interval. Let $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$ and $v \in L_1[a,b]$. Then there exists $\xi \in I$ such that the following equality holds*

$$\frac{\Psi_{\phi_1}(J^\alpha_{a+} v, v)}{\Psi_{\phi_2}(J^\alpha_{a+} v, v)} = \frac{\xi \, \phi_1''(\xi) - (q-1)\phi_1'(\xi)}{\xi \, \phi_2''(\xi) - (q-1)\phi_2'(\xi)},$$

provided that denominators are not equal to zero.

Proof. It follows directly for the function u defined by (4.2) and Theorem 3.4. □

Using Theorems 2.2, 3.3 and 3.4, analogous results follows for the right-sided Riemann–Liouville fractional integrals. The proofs are similar and omitted.

Theorem 4.3. *Let $\phi : [0, \infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\phi \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a compact interval. Let $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$ and $v \in L_1[a,b]$. Then there exists $\xi \in I$ such that the following equality holds*

$$\begin{aligned} \Psi_\phi(J^\alpha_{b-} v, v) \quad = \quad & \frac{\xi \phi''(\xi) - (q-1)\phi'(\xi)}{2q \xi^{2q-1}} \left[\frac{(b-a)^{q\alpha}}{\Gamma^q(\alpha) \left[p(\alpha - \frac{1}{q}) \right]^{\frac{q}{p}}} \int_a^b |v(x)|^{2q} dx \right. \\ & \left. -2 \int_a^b |J^\alpha_{b-} v(x)|^q |v(x)|^q dx \right] . \end{aligned} \tag{4.3}$$

Theorem 4.4. *Let $\phi_1, \phi_2 : [0, \infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0) = 0, i = 1, 2$. Further, let*

$\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a compact interval. Let $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha > \frac{1}{q}$ and $v \in L_1[a,b]$. Then there exists $\xi \in I$ such that the following equality holds

$$\frac{\Psi_{\phi_1}(J^{\alpha}_{b-}v, v)}{\Psi_{\phi_2}(J^{\alpha}_{b-}v, v)} = \frac{\xi\, \phi_1''(\xi) - (q-1)\phi_1'(\xi)}{\xi\, \phi_2''(\xi) - (q-1)\phi_2'(\xi)},$$

provided that denominators are not equal to zero.

Next, we observe the **Caputo fractional derivatives** (for details see [12, Section 2.4]): for $\alpha \geq 0$ define n as

$$n = [\alpha] + 1, \text{ for } \alpha \notin \mathbb{N}_0\,; \quad n = [\alpha], \text{ for } \alpha \in \mathbb{N}_0\,, \tag{4.4}$$

where $[\cdot]$ is the integral part. For $f \in AC^n[a,b]$ the *left-sided and right-sided Caputo fractional derivatives* of order α are defined by

$${}^{C}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1}\, f^{(n)}(t)\, dt = J^{n-\alpha}_{a+} f^{(n)}(x)\,,$$

$${}^{C}D^{\alpha}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (t-x)^{n-\alpha-1}\, f^{(n)}(t)\, dt = (-1)^n J^{n-\alpha}_{b-} f^{(n)}(x)\,.$$

Theorem 4.5. *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\phi \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a compact interval. Let $\alpha \geq 0$, n given by (4.4) and $v \in AC^n[a,b]$. If $n - \alpha > \frac{1}{q}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then there exists $\xi \in I$ such that the following equality holds*

$$\begin{aligned} &\Psi_{\phi}({}^{C}D^{\alpha}_{a+}v, v^{(n)}) \\ = \quad & \frac{\xi\, \phi''(\xi) - (q-1)\phi'(\xi)}{2q\xi^{2q-1}} \left[\frac{(b-a)^{q(n-\alpha)}}{\Gamma^q(n-\alpha)\left[p\left(n-\alpha-\frac{1}{q}\right)\right]^{\frac{q}{p}}} \int_a^b |v^{(n)}(x)|^{2q} dx \right. \\ & \left. -2\int_a^b |{}^{C}D^{\alpha}_{a+}v(x)|^q |v^{(n)}(x)|^q dx \right]. \end{aligned} \tag{4.5}$$

Proof. For $x \in [a,b]$, let

$$K(x,t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)}(x-t)^{n-\alpha-1}, & a \leq t \leq x\,; \\ 0, & x < t \leq b\,, \end{cases}$$

$$u(x) = {}^{C}D^{\alpha}_{a+}v(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} v^{(n)}(t)\, dt\,, \tag{4.6}$$

$$Q(x) = \left(\int_a^x (K(x,t))^p dt \right)^{\frac{1}{p}} = \frac{(x-a)^{n-\alpha-\frac{1}{q}}}{\Gamma(n-\alpha)\left[p\left(n-\alpha-\frac{1}{q}\right)\right]^{\frac{1}{p}}}\,.$$

For $n-\alpha > \frac{1}{q}$ the function Q is increasing on $[a,b]$, thus

$$\max_{x\in[a,b]} Q(x) = \frac{(b-a)^{n-\alpha-\frac{1}{q}}}{\Gamma(n-\alpha)\, p^{\frac{1}{p}} \left(n-\alpha-\frac{1}{q}\right)^{\frac{1}{p}}} = M\,.$$

Hence $\left(\int_a^x K(x,t)^p dt\right)^{\frac{1}{p}} \leq M$, which with $v = v^{(n)}$, u as in (4.6) and Theorem 3.3 gives us (4.5). □

Theorem 4.6. *Let $\phi_1, \phi_2 : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0) = 0, i = 1, 2$. Further, let $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a compact interval. Let $\alpha \geq 0$, n given by (4.4) and $v \in AC^n[a,b]$. If $n-\alpha > \frac{1}{q}$ and $\frac{1}{p}+\frac{1}{q} = 1$, then there exists $\xi \in I$ such that the following equality holds*

$$\frac{\Psi_{\phi_1}({}^C D^{\alpha}_{a+} v, v^{(n)})}{\Psi_{\phi_2}({}^C D^{\alpha}_{a+} v, v^{(n)})} = \frac{\xi\phi_1''(\xi) - (q-1)\phi_1'(\xi)}{\xi\phi_2''(\xi) - (q-1)\phi_2'(\xi)},$$

provided that denominators are not equal to zero.

Proof. It follows directly for the function u defined by (4.6) and Theorem 3.4. □

The proofs for the equalities involving the right-sided Caputo fractional derivatives are similar and omitted.

Theorem 4.7. *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\phi \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a compact interval. Let $\alpha \geq 0$, n given by (4.4) and $v \in AC^n[a,b]$. If $n-\alpha > \frac{1}{q}$ and $\frac{1}{p}+\frac{1}{q} = 1$, then there exists $\xi \in I$ such that the following equality holds*

$$\begin{aligned}
&\Psi_{\phi}({}^C D^{\alpha}_{b-} v, v^{(n)}) \\
&= \frac{\xi\,\phi''(\xi) - (q-1)\phi'(\xi)}{2q\xi^{2q-1}} \left[\frac{(b-a)^{q(n-\alpha)}}{\Gamma^q(n-\alpha)\left[p\left(n-\alpha-\frac{1}{q}\right)\right]^{\frac{q}{p}}} \int_a^b |v^{(n)}(x)|^{2q} dx \right. \\
&\quad \left. -2\int_a^b |{}^C D^{\alpha}_{b-} v(x)|^q |v^{(n)}(x)|^q dx \right]. \qquad (4.7)
\end{aligned}$$

Theorem 4.8. *Let $\phi_1, \phi_2 : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0) = 0, i = 1, 2$. Further, let $\phi_1, \phi_2 \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a compact interval. Let $\alpha \geq 0$, n given by*

(4.4) *and* $v \in AC^n[a,b]$. *If* $n-\alpha > \frac{1}{q}$ *and* $\frac{1}{p}+\frac{1}{q}=1$, *then there exists* $\xi \in I$ *such that the following equality holds*

$$\frac{\Psi_{\phi_1}({}^CD^{\alpha}_{b-}v, v^{(n)})}{\Psi_{\phi_2}({}^CD^{\alpha}_{b-}v, v^{(n)})} = \frac{\xi\phi_1''(\xi)-(q-1)\phi_1'(\xi)}{\xi\phi_2''(\xi)-(q-1)\phi_2'(\xi)},$$

provided that denominators are not equal to zero.

We continue with extensions that require the composition identity for the left-sided Caputo fractional derivatives, given in [5]:

Lemma 4.9. *Let* $\beta > \alpha \geq 0$, m *and* n *given by* (4.4) *for* β *and* α *respectively. Let* $f \in AC^m[a,b]$ *be such that* $f^{(i)}(a) = 0$ *for* $i = n, n+1, ..., m-1$. *Let* ${}^CD^{\beta}_{a+}f, {}^CD^{\alpha}_{a+}f \in L_1[a,b]$. *Then*

$${}^CD^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_a^x (x-t)^{\beta-\alpha-1}\, {}^CD^{\beta}_{a+}f(t)\,dt\,, \qquad x \in [a,b]\,.$$

Theorem 4.10. *Let* $\phi : [0,\infty) \longrightarrow \mathbb{R}$ *be a differentiable function such that for* $q > 1$ *the function* $\phi(x^{\frac{1}{q}})$ *is convex and* $\phi(0) = 0$. *Further, let* $\phi \in C^2(I)$, *where* $I \subseteq \mathbb{R}_+$ *is a compact interval. Let* $\frac{1}{p}+\frac{1}{q}=1$, $\beta-\alpha > \frac{1}{q}$, $\alpha \geq 0$, m *and* n *given by* (4.4) *for* β *and* α *respectively. Let* $v \in AC^m[a,b]$ *be such that* $v^{(i)}(a) = 0$ *for* $i = n, n+1, ..., m-1$. *Let* ${}^CD^{\beta}_{a+}v \in L_q[a,b]$ *and* ${}^CD^{\alpha}_{a+}v \in L_1[a,b]$. *Then there exists* $\xi \in I$ *such that the following equality holds*

$$\begin{aligned}&\Psi_{\phi}({}^CD^{\beta}_{a+}v, {}^CD^{\alpha}_{a+}v)\\ &= \frac{\xi\phi''(\xi)-(q-1)\phi'(\xi)}{2q\xi^{2q-1}}\left[\frac{(b-a)^{q(\beta-\alpha)}}{\Gamma^q(\beta-\alpha)\left[p\left(\beta-\alpha-\frac{1}{q}\right)\right]^{\frac{q}{p}}}\int_a^b |{}^CD^{\alpha}_{a+}v(x)|^{2q}dx\right.\\ &\qquad \left.-2\int_a^b |{}^CD^{\beta}_{a+}v(x)|^q|{}^CD^{\alpha}_{a+}v(x)|^q dx\right]. \qquad (4.8)\end{aligned}$$

Proof. For $x \in [a,b]$, let

$$K(x,t) = \begin{cases} \frac{1}{\Gamma(\beta-\alpha)}(x-t)^{\beta-\alpha-1}, & a \leq t \leq x\,;\\ 0, & x < t \leq b\,,\end{cases}$$

$$u(x) = {}^CD^{\alpha}_{a+}v(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_a^x (x-t)^{\beta-\alpha-1}\, {}^CD^{\beta}_{a+}v(t)\,dt\,, \qquad (4.9)$$

$$R(x) = \left(\int_a^x (K(x,t))^p dt\right)^{\frac{1}{p}} = \frac{(x-a)^{\beta-\alpha-\frac{1}{q}}}{\Gamma(\beta-\alpha)\left[p\left(\beta-\alpha-\frac{1}{q}\right)\right]^{\frac{1}{p}}}\,.$$

For $\beta-\alpha>\frac{1}{q}$ the function R is increasing on $[a,b]$, thus

$$\max_{x\in[a,b]} R(x)=\frac{(b-a)^{\beta-\alpha-\frac{1}{q}}}{\Gamma(\beta-\alpha)\,p^{\frac{1}{p}}\left(\beta-\alpha-\frac{1}{q}\right)^{\frac{1}{p}}}=M\,.$$

Hence $\left(\int_a^x K(x,t)^p dt\right)^{\frac{1}{p}}\leq M$, which with $v={}^CD^{\beta}_{a+}v$, u as in (4.9) and Theorem 3.3 gives us (4.8). □

Theorem 4.11. *Let $\phi_1,\phi_2:[0,\infty)\longrightarrow\mathbb{R}$ be a differentiable function such that for $q>1$ the function $\phi_i(x^{\frac{1}{q}})$ is convex and $\phi_i(0)=0, i=1,2$. Further, let $\phi_1,\phi_2\in C^2(I)$, where $I\subseteq\mathbb{R}_+$ is a compact interval. Let $\frac{1}{p}+\frac{1}{q}=1$, $\beta-\alpha>\frac{1}{q}$, $\alpha\geq 0$, m and n given by (4.4) for β and α respectively. Let $v\in AC^m[a,b]$ be such that $v^{(i)}(a)=0$ for $i=n,n+1,...,m-1$. Let ${}^CD^{\beta}_{a+}v\in L_q[a,b]$ and ${}^CD^{\alpha}_{a+}v\in L_1[a,b]$. Then there exists $\xi\in I$ such that the following equality holds*

$$\frac{\Psi_{\phi_1}({}^CD^{\beta}_{a+}v,{}^CD^{\alpha}_{a+}v)}{\Psi_{\phi_2}({}^CD^{\beta}_{a+}v,{}^CD^{\alpha}_{a+}v)}=\frac{\xi\phi_1''(\xi)-(q-1)\phi_1'(\xi)}{\xi\phi_2''(\xi)-(q-1)\phi_2'(\xi)},$$

provided that denominators are not equal to zero.

Proof. It follows directly for $v={}^CD^{\beta}_{a+}v$, u defined by (4.9) and Theorem 3.4. □

Remark 4.12. Using Theorem 2.2, 3.3, 3.4 and composition identities for the right-sided Caputo fractional derivatives given in [5, Theorem 2.2], similar results can be stated and proved for the right-sided Caputo fractional derivatives (for details see [2, 10]).

Results given for the Caputo fractional derivatives can be analogously done for two other types of fractional derivative that we observe: Canavati type and Riemann–Liouville type. Here, as an example equality for each type of fractional derivatives, we give equality analogous to the (4.8) obtain with composition identity, for the left-sided fractional derivatives. Proofs are omitted.

For more details on the **Canavati fractional derivatives**(see [7]): we consider subspace $C^{\alpha}_{a+}[a,b]$ defined by

$$C^{\alpha}_{a+}[a,b]=\left\{f\in C^{n-1}[a,b]\colon J^{n-\alpha}_{a+}f^{(n-1)}\in C^1[a,b]\right\}.$$

For $f\in C^{\alpha}_{a+}[a,b]$ the *left-sided Canavati fractional derivative* of order α is defined by

$${}^{\bar{C}}D^{\alpha}_{a+}f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d}{dx}\int_a^x(x-t)^{n-\alpha-1}\,f^{(n-1)}(t)\,dt=\frac{d}{dx}J^{n-\alpha}_{a+}f^{(n-1)}(x)\,.$$

The composition identity for the left-sided Canavati fractional derivatives is given in [3]:

Lemma 4.13. *Let $\beta > \alpha > 0$, $m = [\beta] + 1$, $n = [\alpha] + 1$. Let $f \in C^{\beta}_{a+}[a,b]$ be such that $f^{(i)}(a) = 0$ for $i = n-1, n, \ldots, m-2$. Then $f \in C^{\alpha}_{a+}[a,b]$ and*

$$^{\bar{C}}D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_a^x (x-t)^{\beta-\alpha-1}\, ^{\bar{C}}D^{\beta}_{a+}f(t)\,dt\,,\ \ x \in [a,b]\,.$$

Theorem 4.14. *Let $\phi : [0,\infty) \longrightarrow \mathbb{R}$ be a differentiable function such that for $q > 1$ the function $\phi(x^{\frac{1}{q}})$ is convex and $\phi(0) = 0$. Further, let $\phi \in C^2(I)$, where $I \subseteq \mathbb{R}_+$ is a compact interval. Let $\frac{1}{p} + \frac{1}{q} = 1$, $\beta - \alpha > \frac{1}{q}$, $\alpha \geq 0$, $m = [\beta] + 1$ and $n = [\alpha] + 1$. Let $v \in C^{\beta}_{\alpha+}[a,b]$ be such that $v^{(i)}(a) = 0$ for $i = n-1, n, ..., m-2$. Let $^{C}D^{\beta}_{a+}v \in L_q[a,b]$. Then there exists $\xi \in I$ such that the following equality holds*

$$\begin{aligned}
&\Psi_{\phi}(^{\bar{C}}D^{\beta}_{a+}v, {}^{\bar{C}}D^{\alpha}_{a+}v) \\
&= \frac{\xi\phi''(\xi) - (q-1)\phi'(\xi)}{2q\xi^{2q-1}}\left[\frac{(b-a)^{q(\beta-\alpha)}}{\Gamma^q(\beta-\alpha)\left[p\left(\beta-\alpha-\frac{1}{q}\right)\right]^{\frac{q}{p}}}\int_a^b |^{\bar{C}}D^{\beta}_{a+}v(x)|^{2q}dx\right. \\
&\quad \left. -2\int_a^b |^{\bar{C}}D^{\alpha}_{a+}v(x)|^q|^{\bar{C}}D^{\beta}_{a+}v(x)|^q dx\right].
\end{aligned}$$

For more details on **Riemann–Liouville fractional derivatives**(see [12, Section 2.1]): for $f : [a,b] \to \mathbb{R}$ the *left-sided Riemann–Liouville fractional derivative* of order α is defined by

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}\, f(t)\,dt = \frac{d^n}{dx^n}\, J^{n-\alpha}_{a+}f(x)\,.$$

The following lemma summarizes conditions in the composition identity for the left-sided Riemann–Liouville fractional derivatives (for details see [4]):

Lemma 4.15. *Let $\beta > \alpha \geq 0$, $m = [\beta] + 1$, $n = [\alpha] + 1$. The composition identity*

$$D^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\beta-\alpha)}\int_a^x (x-t)^{\beta-\alpha-1}\, D^{\beta}_{a+}f(t)\,dt\,, \quad x \in [a,b], \tag{4.10}$$

is valid if one of the following conditions hold:

(i) $f \in J^{\beta}_{a+}(L_1[a,b]) = \{f : f = J^{\beta}_{a+}\varphi, \varphi \in L_1[a,b]\}$.
(ii) $J^{m-\beta}_{a+}f \in AC^m[a,b]$ *and* $D^{\beta-k}_{a+}f(a) = 0$ *for* $k = 1, \ldots m$.

(iii) $D_{a+}^{\beta-1}f \in AC[a,b]$, $D_{a+}^{\beta-k}f \in C[a,b]$ *and* $D_{a+}^{\beta-k}f(a) = 0$ *for* $k = 1,\dots m$.

(iv) $f \in AC^m[a,b]$, $D_{a+}^{\beta}f, D_{a+}^{\alpha}f \in L_1[a,b]$, $\beta-\alpha \notin \mathbb{N}$, $D_{a+}^{\beta-k}f(a) = 0$ *for* $k = 1,\dots,m$ *and* $D_{a+}^{\alpha-k}f(a) = 0$ *for* $k = 1,\dots,n$.

(v) $f \in AC^m[a,b]$, $D_{a+}^{\beta}f, D_{a+}^{\alpha}f \in L_1[a,b]$, $\beta-\alpha = l \in \mathbb{N}$, $D_{a+}^{\beta-k}f(a) = 0$ *for* $k = 1,\dots,l$.

(vi) $f \in AC^m[a,b]$, $D_{a+}^{\beta}f, D_{a+}^{\alpha}f \in L_1[a,b]$ *and* $f^{(k)}(a) = 0$ *for* $k = 0,\dots,m-2$.

(vii) $f \in AC^m[a,b]$, $D_{a+}^{\beta}f, D_{a+}^{\alpha}f \in L_1[a,b]$, $\beta \notin \mathbb{N}$ *and* $D_{a+}^{\beta-1}f$ *is bounded in a neighborhood of* $m = a$.

Theorem 4.16. *Let* $\phi : [0,\infty) \longrightarrow \mathbb{R}$ *be a differentiable function such that for* $q > 1$ *the function* $\phi(x^{\frac{1}{q}})$ *is convex and* $\phi(0) = 0$. *Further, let* $\phi \in C^2(I)$, *where* $I \subseteq \mathbb{R}_+$ *is a compact interval. Let* $\frac{1}{p}+\frac{1}{q} = 1$, $\beta-\alpha > \frac{1}{q}$, $\alpha \geq 0$. *Suppose that one of conditions in* (i) – (vii) *in Lemma* 4.15 *holds for* $\{\beta,\alpha,v\}$ *and let* ${}^{C}D_{a+}^{\beta}v \in L_q[a,b]$. *Then there exists* $\xi \in I$ *such that the following equality holds*

$$\begin{aligned}&\Psi_{\phi}(D_{a+}^{\beta}v, D_{a+}^{\alpha}v)\\ &= \frac{\xi\phi''(\xi)-(q-1)\phi'(\xi)}{2q\xi^{2q-1}}\left[\frac{(b-a)^{q(\beta-\alpha)}}{\Gamma^q(\beta-\alpha)\left[p\left(\beta-\alpha-\frac{1}{q}\right)\right]^{\frac{q}{p}}}\int_a^b |D_{a+}^{\beta}v(x)|^{2q}dx\right.\\ &\quad\left. -2\int_a^b |D_{a+}^{\alpha}v(x)|^q|D_{a+}^{\beta}v(x)|^q dx\right].\end{aligned}$$

5. Exponential convexity method

Following definitions and properties of exponentially convex functions comes from [11], also [6], [14].

Let I be an interval in $\mathbb{R}$.

Definition 5.1. A function $\psi\colon I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} \xi_i\,\xi_j\,\psi\left(\frac{x_i+x_j}{2}\right) \geq 0$$

holds for all choices $\xi_i \in \mathbb{R}$ and $x_i \in I$, $i = 1,\dots,n$.

A function $\psi\colon I \to \mathbb{R}$ is n-exponentially convex if it is n-exponentially convex in the Jensen sense and continuous on I.

Remark 5.2. It is clear from the definition that 1-exponentially convex functions in the Jensen sense are in fact nonnegative functions. Also, n-exponentially convex functions in the Jensen sense are k-exponentially convex in the Jensen sense for every $k \in \mathbb{N}$, $k \leq n$.

By definition of positive semi-definite matrices and some basic linear algebra we have the following proposition.

Proposition 5.3. *If ψ is an n-exponentially convex in the Jensen sense, then the matrix $\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k}$ is a positive semi-definite matrix for all $k \in \mathbb{N}, k \leq n$. Particularly,* $\det\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{k} \geq 0$ *for all $k \in \mathbb{N}$, $k \leq n$.*

Definition 5.4. A function $\psi\colon I \to \mathbb{R}$ is exponentially convex in the Jensen sense on I if it is n-exponentially convex in the Jensen sense for all $n \in \mathbb{N}$.

A function $\psi\colon I \to \mathbb{R}$ is exponentially convex if it is exponentially convex in the Jensen sense and continuous.

Remark 5.5. It is known (and easy to show) that $\psi : I \to (0, \infty)$ is log-convex in the Jensen sense if and only if

$$\alpha^2\psi(x) + 2\alpha\beta\psi\left(\frac{x+y}{2}\right) + \beta^2\psi(y) \geq 0$$

holds for every $\alpha, \beta \in \mathbb{R}$ and $x, y \in I$. It follows that a function is log-convex in the Jensen sense if and only if it is 2-exponentially convex in the Jensen sense.

Also, using basic convexity theory it follows that a function is log-convex if and only if it is 2-exponentially convex.

We will also need following results (see for example [15]).

Proposition 5.6. *If $x_1, x_2, x_3 \in I$ are such that $x_1 < x_2 < x_3$, then the function $f : I \to \mathbb{R}$ is convex if and only if the inequality*

$$(x_3 - x_2)\, f(x_1) + (x_1 - x_3)\, f(x_2) + (x_2 - x_1)\, f(x_3) \;\geq\; 0$$

holds.

Proposition 5.7. *If f is a convex function on an interval I and if $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid*

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \;\leq\; \frac{f(y_2) - f(y_1)}{y_2 - y_1}\,.$$

If the function f is concave, then the inequality reverses.

Proposition 5.8. *Let f be a log-convex function and assume that $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$. Then the following inequality is valid*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{\frac{1}{x_2-x_1}} \leq \left(\frac{f(y_2)}{f(y_1)}\right)^{\frac{1}{y_2-y_1}}. \tag{5.1}$$

Next we need divided differences, commonly used when dealing with functions that have different degree of smoothness.

Definition 5.9. The second order divided difference of a function $f\colon I \to \mathbb{R}$ at mutually different points $y_0, y_1, y_2 \in I$ is defined recursively by

$$\begin{aligned} [y_i; f] &= f(y_i), \quad i = 0,1,2, \\ [y_i, y_{i+1}; f] &= \frac{f(y_{i+1}) - f(y_i)}{y_{i+1} - y_i}, \quad i = 0,1, \\ [y_0, y_1, y_2; f] &= \frac{[y_1, y_2; f] - [y_0, y_1; f]}{y_2 - y_0}. \end{aligned} \tag{5.2}$$

Remark 5.10. The value $[y_0, y_1, y_2; f]$ is independent of the order of the points y_0, y_1 and y_2. This definition may be extended to include the case in which some or all the points coincide. Namely, taking the limit $y_1 \to y_0$ in (5.2), we get

$$\begin{aligned} &\lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_2; f] \\ &= \frac{f(y_2) - f(y_0) - f'(y_0)(y_2 - y_0)}{(y_2 - y_0)^2}, \quad y_2 \neq y_0 \end{aligned}$$

provided that f' exists, and furthermore, taking the limits $y_i \to y_0$, $i = 1, 2$, in (5.2), we get

$$\lim_{y_2 \to y_0} \lim_{y_1 \to y_0} [y_0, y_1, y_2; f] = [y_0, y_0, y_0; f] = \frac{f''(y_0)}{2}$$

provided that f'' exists.

We use a method of producing n-exponentially convex and exponentially convex functions given in [11], to prove the n-exponential convexity for the functional $\Psi_\phi(u, v)$ defined by (3.1).

Theorem 5.11. *Let J be an interval in $\mathbb{R}$ and $\Upsilon = \{\phi_s : s \in J\}$ be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is n-exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$, where $F_{\phi_s}(y) = \phi_s(y^{\frac{1}{q}})$. Let*

$\Psi_{\phi_s}(u,v)$ be a linear functional defined by (3.1)*. Then $s \mapsto \Psi_{\phi_s}(u,v)$ is n-exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Psi_{\phi_s}(u,v)$ is also continuous on J, then it is n-exponentially convex on J.*

Proof. For $\xi_i \in \mathbb{R}$, $s_i \in J$, $i = 1, \dots, n$, we define the function

$$h(y) = \sum_{i,j=1}^{n} \xi_i \xi_j \phi_{\frac{s_i+s_j}{2}}(y)\,.$$

Set

$$H(y) = h(y^{\frac{1}{q}})\,.$$

Using the assumption that the function $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is n-exponentially convex in the Jensen sense, we have

$$[y_0, y_1, y_2; H] = \sum_{i,j=1}^{n} \xi_i \xi_j [y_0, y_1, y_2; F_{\phi_{\frac{s_i+s_j}{2}}}] \geq 0\,,$$

which in turn implies that H is a convex function on I. Therefore we have $\Psi_h(u,v) \geq 0$. Hence

$$\sum_{i,j=1}^{n} \xi_i \xi_j \Psi_{\phi_{\frac{s_i+s_j}{2}}}(u,v) \geq 0\,.$$

We conclude that the function $s \mapsto \Psi_{\phi_s}(u,v)$ is n-exponentially convex on J in the Jensen sense. If the function $s \mapsto \Psi_{\phi_s}(u,v)$ is also continuous on J, then it is n-exponentially convex by definition. □

Corollary 5.12. *Let J be an interval in $\mathbb{R}$ and $\Upsilon = \{\phi_s : s \in J\}$ be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is exponentially convex in the Jensen sense on J for every three mutually different points $y_0, y_1, y_2 \in I$, where $F_{\phi_s}(y) = \phi_s(y^{\frac{1}{q}})$. Let $\Psi_{\phi_s}(u,v)$ be a linear functional defined by* (3.1)*. Then $s \mapsto \Psi_{\phi_s}(u,v)$ is exponentially convex function in the Jensen sense on J. If the function $s \mapsto \Psi_{\phi_s}(u,v)$ is continuous on J, then it is exponentially convex on J.*

Let us denote means for $\phi_s, \phi_p \in \Omega$ by

$$\mu_{s,p}(\Psi, \Omega) = \begin{cases} \left(\frac{\Psi_{\phi_s}(u,v)}{\Phi_{\phi_p}(u,v)} \right)^{\frac{1}{s-p}}, & s \neq p\,, \\ \exp\left(\frac{\frac{d}{ds}\Psi_{\phi_s}(u,v)}{\Psi_{\phi_s}(u,v)} \right), & s = p\,. \end{cases} \tag{5.3}$$

Theorem 5.13. *Let J be an interval in $\mathbb{R}$ and $\Omega = \{\phi_s : s \in J\}$ be a family of functions defined on an interval I in $\mathbb{R}$, such that the function $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is 2-exponentially convex in the Jensen sense on J for every* *three mutually different points* $y_0, y_1, y_2 \in I$*, where* $F_{\phi_s}(y) = \phi_s(y^{\frac{1}{q}})$*. Let* $\Psi_{\phi_s}(u,v)$ *be a linear functional defined by* (3.1)*. Then the following statements hold:*

(i) *If the function* $s \mapsto \Psi_{\phi_s}(u,v)$ *is continuous on* J*, then it is* 2*-exponentially convex function on* J*. If the function* $s \mapsto \Psi_{\phi_s}(u,v)$ *is additionally positive, then it is also log-convex on* J*, and for* $r, s, t \in J$ *such that* $r < s < t$*, we have*

$$(\Psi_{\phi_s}(u,v))^{t-r} \leq (\Psi_{\phi_r}(u,v))^{t-s} (\Psi_{\phi_t}(u,v))^{s-r} . \tag{5.4}$$

(ii) *If the function* $s \mapsto \Psi_{\phi_s}(u,v)$ *is positive and differentiable on* J*, then for every* $s, p, r, t \in J$*, such that* $s \leq r$ *and* $p \leq t$*, we have*

$$\mu_{s,p}(\Psi, \Omega) \ \leq \ \mu_{r,t}(\Psi, \Omega) . \tag{5.5}$$

Proof. (i) The first part is an immediate consequence of Theorem 5.11 and in second part log-convexity on J follows from Remark 5.5. Since $s \mapsto \Psi_{\phi_s}(u,v)$ is positive, for $r, s, t \in J$ such that $r < s < t$, with $f(s) = \log \Psi_{\phi_s}(u,v)$ in Proposition 5.6, we have

$$(t-s)\log \Psi_{\phi_r}(u,v) + (r-t)\log \Psi_{\phi_s}(u,v) + (s-r)\log \Psi_{\phi_t}(u,v) \ \geq \ 0 .$$

This is equivalent to inequality (5.4).
(ii) The function $s \mapsto \Psi_{\phi_s}(u,v)$ is log-convex on J by (i), that is, the function $s \mapsto \log \Psi_{\phi_s}(u,v)$ is convex on J. Applying Proposition 5.7 we get

$$\frac{\log \Psi_{\phi_s}(u,v) - \log \Psi_{\phi_p}(u,v)}{s-p} \ \leq \ \frac{\log \Psi_{\phi_r}(u,v) - \log \Psi_{\phi_t}(u,v)}{r-t} \tag{5.6}$$

for $s \leq r$, $p \leq t$, $s \neq p$, $r \neq t$, and therefore we have

$$\mu_{s,p}(\Psi, \Omega) \ \leq \ \mu_{r,t}(\Psi, \Omega) .$$

Cases $s = p$ and $r = t$ follows from (5.6) as limit cases. □

Remark 5.14. The results from Theorem 5.11, Corollary 5.12 and Theorem 5.13 still hold when two of the points $y_0, y_1, y_2 \in I$ coincide, for a family of differentiable functions ϕ_s such that the function $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is n-exponentially convex in the Jensen sense (exponentially convex in the Jensen sense, log-convex in the Jensen sense). Furthermore, they still hold when all three points coincide for a family of twice differentiable functions with the same property. The proofs can be obtained by recalling Remark 5.10 and suitable characterization of convexity.

6. Applications to Stolarsky type means

In this section, we use Cauchy type mean value Theorem 3.3 and Theorem 3.4 for Stolarsky type means and functional $\Psi_\phi(u,v)$. Several families of functions which fulfil the conditions of Theorem 5.11, Corollary 5.12 and Theorem 5.13 (and Remark 5.14) that we present here, enable us to construct large families of functions which are exponentially convex.

Example 6.1. Consider a family of functions

$$\Omega_1 = \{\phi_s : [0,\infty) \to \mathbb{R} : s > 0\}$$

defined for $q > 1$ by

$$\phi_s(x) = \begin{cases} \frac{q^2}{s(s-q)}\, x^s, & s \neq q; \\ q\, x^q \log x, & s = q. \end{cases}$$

Then $\left[\phi_s(x^{\frac{1}{q}})\right]'' = x^{\frac{s-2q}{q}} = e^{\frac{s-2q}{q}\ln x} > 0$ which show that ϕ_s is convex function with respect to $g(x) = x^q$ for $x > 0$, and $s \mapsto \left[\phi_s(x^{\frac{1}{q}})\right]''$ is exponentially convex by definition. Notice $\phi_s(0) = 0$, with the convention $0 \log 0 = 0$.

Analogously as in the proof of Theorem 5.11 we conclude that $s \mapsto [y_0, y_1, y_2; F_{\phi_s}]$ is exponentially convex (and so exponentially convex in the Jensen sense), where $F_{\phi_s}(y) = \phi_s(y^{\frac{1}{q}})$. By Corollary 5.12 we have that $s \mapsto \Psi_{\phi_s}(u,v)$ is exponentially convex in the Jensen sense. It is easy to verify that this mappings are continuous, so they are exponentially convex. Hence, we have

$$\Psi_{\phi_s}(u,v) = \begin{cases} \frac{q^3 (b-a)^{\frac{s}{q}-1} M^{s-q}}{s(s-q)} \int_a^b |v(x)|^s\, dx - \frac{q^2}{s-q} \int_a^b |u(x)|^{s-q}\, |v(x)|^q\, dx\,, \\ s \neq q\,, \\ \\ q^2 \int_a^b |v(x)|^q \log\left[(b-a)^{\frac{1}{q}}\, M|v(x)|\right]\, dx - q\int_a^b |v(x)|^q\, [q \log|u(x)| + 1]\, dx\,, \\ s = q\,. \end{cases}$$

For this family of functions, $\mu_{s,t}(\Psi, \Omega_1)$ from (5.3) becomes

$$\mu_{s,t}(\Psi,\Omega_1) = \begin{cases} \left(\frac{\Psi_{\phi_s}(u,v)}{\Psi_{\phi_t}(u,v)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \\ \exp\left(\frac{q-2s}{s(s-q)} + \frac{\Psi_{\phi_s \cdot \log}(u,v)}{\Psi_{\phi_s}(u,v)}\right), & s = t \neq q, \\ \\ \exp\left(-\frac{1}{q} + \frac{\Psi_{\phi_q \cdot \log}(u,v)}{2\Psi_{\phi_q}(u,v)}\right), & s = t = q \end{cases}$$

and by (5.5) it is monotonous in parameters s and t.
For the functional $\Psi_\phi(u,v)$ we get

$$\mu_{s,t}(\Psi,\Omega_1) = \begin{cases} \left(\dfrac{qt(t-q)(b-a)^{\frac{s}{q}-1} M^{s-q} \int_a^b |v(x)|^s\,dx - s\int_a^b |u(x)|^{s-q}|v(x)|^q\,dx}{qs(s-q)(b-a)^{\frac{t}{q}-1} M^{t-q} \int_a^b |v(x)|^t\,dx - t\int_a^b |u(x)|^{t-q}|v(x)|^q\,dx} \right)^{\frac{1}{s-t}}, & s \neq t\,, \\ \exp\left(\dfrac{q-2s}{s(s-q)} + \dfrac{q(b-a)^{\frac{s}{q}-1} M^{s-q} \int_a^b |v(x)|^s \log\left[(b-a)^{\frac{1}{q}} M|v(x)|\right] dx - \int_a^b [s\log|u(x)|+1]|u(x)|^{s-q}|v(x)|^q\,dx}{q(b-a)^{\frac{s}{q}-1} M^{s-q} \int_a^b |v(x)|^s\,dx - s\int_a^b |u(x)|^{s-q}|v(x)|^q\,dx} \right), & s = t \neq q\,, \\ \exp\left(-\dfrac{1}{q} + \dfrac{q\int_a^b |v(x)|^q \log^2\left[(b-a)^{\frac{1}{q}} M|v(x)|\right] dx - \int_a^b [q\log|u(x)|+2]|v(x)|^q \log|u(x)|\,dx}{2q\int_a^b |v(x)|^q \log\left[(b-a)^{\frac{1}{q}} M|v(x)|\right] dx - 2\int_a^b [q\log|u(x)|+1]|v(x)|^q\,dx} \right), & s = t = q\,. \end{cases}$$

Example 6.2. Consider a family of functions

$$\Omega_2 = \{\varphi_s : [0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$$

defined for $q > 1$ by

$$\varphi_s(x) = \begin{cases} \frac{e^{sx^q}-1}{s^2}, & s \neq 0\,, \\ \frac{x^{2q}}{2}\,, & s = 0\,. \end{cases}$$

Since $\left[\varphi_s(x^{\frac{1}{q}})\right]'' = e^{sx} > 0$, then φ_s is convex function with respect to $g(x) = x^q$ for $x > 0$, and $s \mapsto \left[\varphi_s(x^{\frac{1}{q}})\right]''$ is exponentially convex by definition. Notice that $\varphi_s(0) = 0$. Arguing as in the previous example, we get that the mapping $s \mapsto \Phi_{\varphi_s}(u,v)$ is exponentially convex.

We have

$$\Psi_{\varphi_s}(u,v) = \begin{cases} \frac{q}{s^2 M^q (b-a)} \int_a^b \{\exp[s(b-a)M^q|v(x)|^q] - 1\}\,dx \\ -\frac{q}{s}\int_a^b |v(x)|^q \exp[s|u(x)|^q]dx\,, & s \neq 0\,, \\ \frac{q(b-a)M^q}{2}\int_a^b |v(x)|^{2q}dx - q\int_a^b |u(x)|^q|v(x)|^q\,dx\,, & s = 0\,. \end{cases}$$

For this family of functions, $\mu_{s,t}(\Phi, \Omega_2)$ from (5.3) becomes

$$\mu_{s,t}(\Psi, \Omega_2) = \begin{cases} \left(\frac{\Psi_{\varphi_s}(u,v)}{\Psi_{\varphi_t}(u,v)}\right)^{\frac{1}{s-t}}, & s \neq t, \\ \exp\left(-\frac{2}{s} + \frac{\Psi_{x^q \cdot \varphi_s}(u,v)}{\Psi_{\varphi_s}(u,v)}\right), & s = t \neq 0, \\ \exp\left(\frac{\Psi_{x^q \cdot \varphi_0}(u,v)}{3\Psi_{\varphi_0}(u,v)}\right), & s = t = 0, \end{cases}$$

and by (5.5) it is monotonous in parameters s and t.
For the functional $\Psi_\varphi(u, v)$ we get

$$\mu_{s,t}(\Psi, \Omega_2)$$
$$= \begin{cases} \left(\frac{s^{-2}\int_a^b \{\exp[s(b-a)M^q|v(x)|^q]-1\}dx - s^{-1}M^q(b-a)\int_a^b \exp(s|u(x)|^q)|v(x)|^q\,dx}{t^{-2}\int_a^b \{\exp[t(b-a)M^q|v(x)|^q]-1\}dx - t^{-1}M^q(b-a)\int_a^b \exp(t|u(x)|^q)|v(x)|^q\,dx}\right)^{\frac{1}{s-t}}, \\ s \neq t, \\ \exp\left(-\frac{2}{s} + \frac{\int_a^b \{\exp[s(b-a)M^q|v(x)|^q] - \exp[s\,|u(x)|^q] - s|u(x)|^q\ \exp[s|u(x)|^q]\}|v(x)|^q\,dx}{\frac{1}{M^q(b-a)}\int_a^b \{\exp[s(b-a)M^q|v(x)|^q]-1\}dx - s\int_a^b \exp(s|u(x)|^q)|v(x)|^q dx}\right), \\ s = t \neq 0, \\ \exp\left(\frac{(b-a)^2 M^{2q}\int_a^b |v(x)|^{3q}dx - 3\int_a^b |u(x)|^{2q}|v(x)|^q dx}{3(b-a)M^q\int_a^b |v(x)|^{2q}dx - 6\int_a^b |u(x)|^q|v(x)|^q dx}\right), \\ s = t = 0. \end{cases}$$

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