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COMMON FIXED POINTS OF STRICT PSEUDOCONTRACTIONS BY ITERATIVE ALGORITHMS IN HILBERT SPACES

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Abstract. In this paper, we present iteration schemes to weakly and strongly approximate common fixed points of a finite family of a class of strict pseudocontractions in Hilbert spaces. It is proved that the sequence generated by the iterative scheme converges strongly to a common point of the set of fixed points, which solves the variational inequality $\langle(\mu F - \gamma\phi)\tilde{x}, \tilde{x} - p\rangle \leq 0$, for $p \in \bigcap_{i=1}^{N} Fix(T_i)$. Our results improve and extend corresponding ones announced by many others.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be λ-strictly pseudo-contractive if there exists a constant $\lambda \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \lambda\|(I - T)x - (I - T)y\|^2, \ x, y \in C. \tag{1.1}$$

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It is clear that (1.1) is equivalent to the following:

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-\lambda}{2}\|(I - T)x - (I - T)y\|^2,$$

and $Fix(T)$ denotes the set of fixed points of the mapping T; that is, $Fix(T) = \{x \in C : Tx = x\}$.

Note that the class of λ-strictly pseudo-contractive mappings includes the class of nonexpansive mappings T on C (that is, $\|Tx-Ty\| \leq \|x-y\|, x, y \in C$) as a subclass. That is, T is nonexpansive if and only if T is 0-strictly pseudo-contractive.

Theorem 1.1. ([1]) *Let (X, d) be a complete metric space and let f be a contraction on X, that is, there exists $r \in (0, 1)$ such that $d(f(x), f(y)) \leq rd(x, y)$ for all $x, y \in X$. Then f has a unique fixed point.*

Theorem 1.2. ([2]) *Let (X, d) be a complete metric space and let ϕ be a Meir-Keeler contraction (MKC, for short) on X, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \varepsilon + \delta$ implies $d(\phi(x), \phi(y)) < \varepsilon$ for all $x, y \in X$. Then ϕ has a unique fixed point.*

This theorem is one of generalizations of Theorem 1.1, because contractions are Meir-Keeler contractions.

A mapping $F : C \to C$ is called k-Lipschitzian if there exists a positive constant k such that

$$\|Fx - Fy\| \leq k\|x - y\|, \quad \forall\, x, y \in C. \tag{1.2}$$

F is said to be η-strongly monotone if there exists a positive constant η such that

$$\langle Fx - Fy, x - y\rangle \geq \eta\|x - y\|^2, \quad \forall\, x, y \in C. \tag{1.3}$$

Let A be a strongly positive bounded linear operator on H, that is, there exists a constant $\tilde{\gamma} > 0$ such that

$$\langle Ax, x\rangle \geq \tilde{\gamma}\|x\|^2, \quad \forall\, x \in H.$$

A typical problem is that of minimizing a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in Fix(T)} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle,$$

where b is a given point in H.

Remark 1.3. ([4]) From the definition of A, we note that a strongly positive bounded linear operator A is a $\|A\|$-Lipschizian and $\tilde{\gamma}$-strongly monotone operator.

In 2010, Tian [3] introduced the following iterative method: for a nonexpansive mapping $T : H \to H$ with $Fix(T) \neq \emptyset$,

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)Tx_n, \ \forall n \geq 1, \tag{1.4}$$

where F is a k-Lipschitzian and η-strongly monotone operator. He obtained that the sequence $\{x_n\}$ generated by (1.4) converges strongly to a point q in $Fix(T)$, which is the unique solution of the variational inequality $\langle(\gamma f - \mu F)q, p - q\rangle \leq 0$, $p \in Fix(T)$.

Recently, Wang [4] considered the following iterative method: for $x_1 = x \in C$,

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)W_n x_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu\alpha_n F)y_n, \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

where W_n is a family of nonexpansive mappings, and F is a k-Lipschitzian and η-strongly monotone operator with $0 < \mu < \frac{2\eta}{k^2}$. She proved that if the parameters satisfy appropriate conditions, then $\{x_n\}$ defined by (1.5) converges strongly to a common element of the fixed points of an infinite family of λ_i-strictly pseudo-contractive mappings, which is a unique solution of the variational inequality $\langle(\gamma f - \mu F)q, p - q\rangle \leq 0$, $p \in \bigcap_{i=1}^{\infty} Fix(T_i)$.

Very recently, Colao and Marino [5] introduced the following iterative method:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)V_n x_n, \ \forall n \geq 1, \tag{1.6}$$

where V_n are mappings defined by (2.2), and $f : C \to C$ is a ρ-contraction. He given a new method to prove that the sequence $\{x_n\}$ generated by (1.6) converges strongly to the unique point $q \in F^* = \bigcap_{i=1}^{N} Fix(T_i)$, which satisfies the variational inequality $\langle q - f(q), j(q - p)\rangle \leq 0$, $p \in F^*$.

In this work, motivated and inspired by the above results, we consider the following iterative method: for $x_1 = x \in C$,

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)V_n x_n, \\ x_{n+1} = \alpha_n \gamma \phi(x_n) + (I - \mu\alpha_n F)y_n, \quad \forall n \geq 1, \end{cases} \tag{1.7}$$

where V_n are mappings defined by (2.2), ϕ is a Meir-Keeler contraction (MKC, for short) and F is a k-Lipschitzian and η-strongly monotone operator with $0 < \mu < \frac{2\eta}{k^2}$. We will prove that if the parameters satisfy appropriate conditions, then $\{x_n\}$ defined by (1.7) converges strongly to a common element of the fixed points of a finite family of λ_i-strictly pseudo-contractive mappings, which is a unique solution of the variational inequality $\langle(\mu F - \gamma\phi)(\tilde{x}), \tilde{x} - p\rangle \leq 0$, $p \in F^* = \bigcap_{i=1}^{N} Fix(T_i)$. Our results extend and improve the corresponding results of Wang [4], Colao and Marino [5] and many others.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. For the sequence $\{x_n\}$ in H, we write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. $x_n \to x$ means that $\{x_n\}$ converges strongly to x. In a real Hilbert space H, we have

$$\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle, \ \forall\, x, y \in H.$$

In order to prove our main results, we need the following lemmas.

Lemma 2.1. *In a Hilbert space H, the following inequality holds:*

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \quad x, y \in H. \tag{2.1}$$

Lemma 2.2. ([6, Lemma 2.3]) *Let ϕ be a MKC on a convex subset C of a Banach space E. Then for each $\varepsilon > 0$, there exists $r \in (0, 1)$ such that*

$$\|x - y\| \ge \varepsilon \quad \textit{implies} \quad \|\phi x - \phi y\| \le r\|x - y\|, \quad \forall\, x, y \in C.$$

Lemma 2.3. ([4, Lemma 2.2]) *Let F be a k-Lipschitzian and η-strongly monotone operator on a Hilbert space H with $k > 0, \eta > 0$, $0 < \mu < \frac{2\eta}{k^2}$ and $0 < t < 1$. Then $S = (I - t\mu F) : H \to H$ is a contraction with contractive coefficient $1 - t\tau$ and $\tau = \frac{1}{2}\mu(2\eta - \mu k^2)$.*

Lemma 2.4. ([7]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space E and $\{\gamma_n\}$ be a sequence in $[0, 1]$ which satisfies the following condition:*

$$0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1.$$

Suppose that

$$x_{n+1} = \gamma_n x_n + (1 - \gamma_n) z_n, \quad n \ge 0,$$

and

$$\limsup_{n\to\infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.5. ([8, 9]) *Let $\{s_n\}$ be a sequence of non-negative real numbers satisfying*

$$s_{n+1} \le (1 - \lambda_n)s_n + \lambda_n\delta_n + \gamma_n, \ n \ge 0,$$

where $\{\lambda_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\lambda_n\} \subset [0, 1]$ *and* $\sum_{n=0}^{\infty} \lambda_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \le 0$ *or* $\sum_{n=0}^{\infty} \lambda_n\delta_n < \infty$,
(iii) $\gamma_n \ge 0 (n \ge 0)$, $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.6. ([10]) *Let* $\{y_m\}$ *be a bounded sequence contained in a separable subset* K *of a Banach space* E. *Then there is a subsequence* $\{y_{m_k}\}$ *of* $\{y_m\}$ *such that* $\lim_k \|y_{m_k} - z\|$ *exists for all* $z \in K$.

Lemma 2.7. ([10]) *Let* C *be a closed convex subset of a Banach space* E *with a uniformly Gâteaux differentiable norm, and let* $\{y_m\}$ *be a sequence in* K *such that* $h(z) = \lim_m \|y_m - z\|$ *exists for all* $z \in C$. *If* h *attains its minimum over* C *at* u, *then*

$$\limsup_m \langle z - u, j(y_m - u)\rangle \leq 0,$$

for all $z \in C$.

Lemma 2.8. ([11]) *Let* E *be a reflexive Banach space and let* C *be a closed convex subset of* E. *Let* h *be a proper convex lower semicontinuous function of* C *into* $(-\infty, \infty]$ *and suppose that* $h(x_n) \to \infty$ *as* $\|x_n\| \to \infty$. *Then, there exists* $x_0 \in D(h)$ *such that*

$$h(x_0) = \inf\{h(x) : x \in C\}.$$

The following lemmas are obtained from the reference [5].

Lemma 2.9. ([5, Lemma 3]) *Let* C *be a nonempty closed convex subset of a real Hilbert space* H *and* $T : C \to C$ *be a* λ*-strictly pseudocontractive mapping. Define a mapping* $S : C \to C$ *by* $Sx = (1-\alpha)x + \alpha Tx$ *for all* $x \in C$ *and* $\alpha \in (0, 1-\lambda)$. *Then* S *is a nonexpansive mapping such that* $Fix(S) = Fix(T)$.

Lemma 2.10. ([5, Lemma 9]) *Let* C *be a closed convex and nonempty subset of a Hilbert space* H. *Let* $\{T_i\}_{i=1}^N$ *be a family of mappings from* C *into itself, such that* $F^* = \bigcap_{i=1}^N Fix(T_i) \neq \emptyset$ *and for each* $i \in \{1, \ldots, N\}$, T_i *is* λ_i*-strictly pseudocontractive. Moreover, let* $\gamma_1, \ldots, \gamma_N \in (0, \min_{i=1,\ldots,N}\{1 - \lambda_i\})$ *and define the mapping* V *of* C *into itself as follows:*

$$\begin{cases} U_1 = \gamma_1 T_1 + (1-\gamma_1) I, \\ U_2 = \gamma_2 T_2 U_1 + (1-\gamma_2) U_1, \\ \vdots \\ U_{N-1} = \gamma_{N-1} T_{N-1} U_{N-2} + (1-\gamma_{N-1}) U_{N-2}, \\ V \equiv U_N = \gamma_N T_N U_{N-1} + (1-\gamma_N) U_{N-1}. \end{cases} \tag{2.2}$$

Then $U_1, \ldots, U_{N-1}$ *and* V *are nonexpansive. Moreover,* $Fix(V) = F^*$.

Lemma 2.11. ([5, Lemma 10]) *Let C, E and the family $\{T_i\}_{i=1}^N$ be as in Lemma 2.10. Moreover let the maps $\tilde{V}$ and V be generated following the scheme (2.2) by the family $\{T_i\}_{i=1}^N$ and coefficients $\tilde{\gamma_1}, \ldots, \tilde{\gamma_N}$ and $\gamma_1, \ldots, \gamma_N$ respectively. Fix $w \in F^*$, then for any $x \in C$ the following holds*

$$\|\tilde{V}x - Vx\| \leq \sum_{i=1}^{N} \mid \tilde{\gamma_i} - \gamma_i \mid M_i \|x - w\|,$$

where $M_i = \frac{2(2-\lambda_i)}{1-\lambda_i}$.

Lemma 2.12. ([12]) *Let H be a Hilbert space, C a closed convex subset of H and $T : C \to C$ a nonexpansive mapping with $Fix(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y, then $(I - T)x = y$.*

Let C be a closed subspace of H. Let F be a k-Lipschitzian and η-strongly monotone operator on C with $k > 0$, $\eta > 0$ and $V_n : C \to C$ be a family of nonexpansive mappings. Now given $\phi : C \to C$ be a MKC, let us have $\alpha_n \in (0, 1)$, $0 < \mu < \frac{2\eta}{k^2}$, $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2}) = \tau$, $\tau < 1$, and consider a mapping S_n on H defined by

$$S_n x = \alpha_n \gamma \phi(x) + (I - \alpha_n \mu F)V_n x, \quad x \in C.$$

It is easy to see that S_n is a contraction. Indeed, from Lemma 2.3, we have

$$\begin{aligned}\|S_n x - S_n y\| &\leq \alpha_n \gamma \|\phi(x) - \phi(y)\| + \|(I - \alpha_n \mu F)V_n x - (I - \alpha_n \mu F)V_n y\| \\ &\leq \alpha_n \gamma \|x - y\| + (1 - \alpha_n \tau)\|x - y\| \\ &= [1 - \alpha_n(\tau - \gamma)]\|x - y\|,\end{aligned}$$

for all $x, y \in H$. Hence it has a unique point, denoted as y_n, which uniquely solves the fixed point equation

$$y_n = \alpha_n \gamma \phi(y_n) + (I - \alpha_n \mu F)V_n y_n, \quad y_n \in C. \tag{2.3}$$

3. Main Results

Theorem 3.1. *Let H be a separable Hilbert space, and C be a closed subspace of H. Let $\{T_i\}_{i=1}^N$ be a family of mappings from C into itself, such that $F^* = \bigcap_{i=1}^N Fix(T_i)$ is nonempty and for each $i \in \{1, \ldots, N\}$, T_i is a λ_i-strictly pseudocontractive and $\{\gamma_{i,n}\}_{i=1}^N \subset [a, b] \subset (0, \min_{i=1,\ldots,N}\{1 - \lambda_i\})$. Define the mappings V_n as in (2.2). Let $F : C \to C$ be a k-Lipschitzian and η-strongly monotone operator with $0 < \mu < \frac{2\eta}{k^2}$, $\phi : C \to C$ be a MKC with*

$0 < \gamma < \mu(\eta - \frac{\mu k^2}{2}) = \tau$, $\tau < 1$ *and* $\{\alpha_n\} \subset (0,1)$. *Define the sequence* $\{y_n\}$ *by*

$$y_n = \alpha_n \gamma \phi(y_n) + (I - \alpha_n \mu F) V_n y_n, \quad n \in \mathbb{N}. \tag{3.1}$$

If the control sequence $\{\alpha_n\}$ *satisfies:*

(*A*1) $\qquad \lim_n \alpha_n = 0,$

then the sequence $\{y_n\}$ *converges to the unique point* $\tilde{z} \in F^* = \bigcap_{i=1}^N Fix(T_i)$ *which satisfies the inequality:*

$$\langle (\mu F - \gamma\phi)\tilde{z}, \tilde{z} - w\rangle \leq 0, \quad \forall\, w \in F^*. \tag{3.2}$$

Proof. Fix $n \in \mathbb{N}$. By (2.3) we have that the contractive map $S_n x = \alpha_n \gamma \phi(x) + (I - \alpha_n \mu F) V_n x$ maps C into itself. Then its unique fixed point must lie in C. That is

$$y_n = \alpha_n \gamma \phi(y_n) + (I - \alpha_n \mu F) V_n y_n \in C.$$

Boundedness of the sequence $\{y_n\}$ follows directly from nonexpansivity of V_n and from $Fix(V_n) = F^* \neq \emptyset$. In fact, for a fixed $w \in F^*$, we have from Lemma 2.1 and Lemma 2.3

$$\begin{aligned}
\|y_n - w\|^2 &= \|\alpha_n \gamma \phi(y_n) + (I - \alpha_n \mu F) V_n y_n - w\|^2 \\
&= \|\alpha_n \gamma \phi(y_n) - \alpha_n \mu F w + \alpha_n \mu F w + (I - \alpha_n \mu F) V_n y_n - w\|^2 \\
&= \|\alpha_n (\gamma \phi(y_n) - \mu F w) + (I - \alpha_n \mu F) V_n y_n - (I - \alpha_n \mu F) w\|^2 \\
&\leq [(1 - \alpha_n \tau)\|y_n - w\| + \alpha_n \gamma \|y_n - w\|]^2 \\
&\quad + 2\alpha_n \langle \gamma \phi(w) - \mu F w, y_n - w\rangle \\
&\leq [1 - \alpha_n(\tau - \gamma)]\|y_n - w\|^2 + 2\alpha_n \langle \gamma \phi(w) - \mu F w, y_n - w\rangle \qquad (3.3) \\
&\leq [1 - \alpha_n(\tau - \gamma)]\|y_n - w\|^2 + 2\alpha_n \|\gamma \phi(w) - \mu F w\| \|y_n - w\|.
\end{aligned}$$

Thus

$$\|y_n - w\| \leq \frac{2}{\tau - \gamma}\|\gamma \phi(w) - \mu F w\|.$$

Now, our purpose is to prove that

$$\limsup_n \langle \gamma \phi(\tilde{z}) - \mu F \tilde{z}, y_n - \tilde{z}\rangle \leq 0.$$

Set

$$\Gamma := \limsup_n \langle \gamma \phi(\tilde{z}) - \mu F \tilde{z}, y_n - \tilde{z}\rangle.$$

Since $\{y_n\}$ is bounded and is contained in a separable set C, by Lemma 2.6, we can choose a sequence $\{n_v\} \subset \mathbb{N}$ with the properties that

(Pr.1) $\lim_v \langle \gamma \phi(\tilde{z}) - \mu F \tilde{z}, y_{n_v} - \tilde{z}\rangle = \Gamma$,
(Pr.2) $\gamma_{i,n_v} \to \gamma_i \in [a,b] \subset (0, \min_{i=1,\dots,N}\{1 - \lambda_i\})(i = 1, \dots, N)$,
(Pr.3) $\lim_v \|y_{n_v} - z\|$ exists for all $z \in C$.

Denoted by V the map generated by the finite family $\{T_i\}_{i=1}^N$ and coefficient $\gamma_1, \ldots, \gamma_N$ following the scheme (2.2), it results from Lemma 2.10 that V is nonexpansive and $Fix(V) = F^*$. Moreover in view of Lemma 2.11, for every fixed $x \in C$ we have

$$\lim_v \|V_{n_v} x - V x\| = 0. \tag{3.4}$$

Define $h : C \to \mathbb{R}$ by $h(x) := \lim_v \|y_{n_v} - x\|$. The map h is well defined by (Pr.3) and h is continuous, convex and $h(x) \to \infty$ as $\|x\| \to \infty$. By Lemma 2.8 h attains a minimum in C. Thus

$$A := \{x \in C : \; h(x) = \inf_{y \in C} h(y)\}$$

is nonempty and bounded. For any fixed $x \in A$, we have

$$\begin{aligned} \|y_{n_v} - Vx\| &= \|\alpha_{n_v} \gamma \phi(y_{n_v}) + (I - \alpha_{n_v} \mu F) V_{n_v} y_{n_v} - Vx\| \\ &= \|\alpha_{n_v} \gamma \phi(y_{n_v}) - \alpha_{n_v} \mu F V_{n_v} y_{n_v} + V_{n_v} y_{n_v} - V_{n_v} x + V_{n_v} x - Vx\| \\ &\leq \alpha_{n_v} \|\gamma \phi(y_{n_v}) - \mu F V_{n_v} y_{n_v}\| + \|y_{n_v} - x\| + \|V_{n_v} x - Vx\|. \end{aligned}$$

Hence, by (3.4) we obtain

$$\lim_v \|y_{n_v} - Vx\| \leq \lim_v \|y_{n_v} - x\|,$$

that is $V : A \to A$. Since H is uniformly smooth, A is closed, convex and bounded and V is nonexpansive then V has a fixed point $\tilde{y} \in A$, that is $\tilde{y} \in Fix(V) \bigcap A = F^* \bigcap A$. Furthermore, $\tilde{y}$ minimizes h over C. Thus, from Lemma 2.7, it follows

$$\limsup_v \langle x - \tilde{y}, y_{n_v} - \tilde{y} \rangle \leq 0, \quad \forall \, x \in C.$$

In particular, for $x = \gamma \phi(\tilde{y}) - \mu F \tilde{y} + \tilde{y}$, we obtain

$$\limsup_v \langle \gamma \phi(\tilde{y}) - \mu F \tilde{y}, y_{n_v} - \tilde{y} \rangle \leq 0.$$

Since $\tilde{y}$ also belongs to F^*, from (3.3) we derive

$$\|y_{n_v} - \tilde{y}\|^2 \leq \frac{2}{\tau - \gamma} \langle \gamma \phi(\tilde{y}) - \mu F \tilde{y}, y_{n_v} - \tilde{y} \rangle.$$

Passing the last inequality to $\limsup_v$ we obtain

$$\limsup_v \|y_{n_v} - \tilde{y}\|^2 \leq \frac{2}{\tau - \gamma} \limsup_v \langle \gamma \phi(\tilde{y}) - \mu F \tilde{y}, y_{n_v} - \tilde{y} \rangle \leq 0,$$

hence $y_{n_v} \to \tilde{y}$. Note that for any fixed $n \in \mathbb{N}$, we have

$$(\mu F - \gamma \phi) y_n = -\frac{1}{\alpha_n} [(I - V_n) y_n - \alpha_n \mu F y_n + \alpha_n \mu F V_n y_n].$$

Notice

$$\begin{aligned}\langle (I-V_n)y_n-(I-V_n)w, y_n-w\rangle &= \|y_n-w\|^2-\langle V_ny_n-V_nw, y_n-w\rangle\\ &\ge \|y_n-w\|^2-\|V_ny_n-V_nw\|\|y_n-w\|\\ &\ge \|y_n-w\|^2-\|y_n-w\|^2\\ &\ge 0. \end{aligned}\tag{3.5}$$

It follows that, for $w \in F^* = Fix(V_n)$,

$$\begin{aligned}\langle (\mu F-\gamma\phi)y_n, y_n-w\rangle &= -\frac{1}{\alpha_n}\langle (I-V_n)y_n-\alpha_n\mu Fy_n+\alpha_n\mu FV_ny_n, y_n-w\rangle\\ &= -\frac{1}{\alpha_n}\langle (I-V_n)y_n-(I-V_n)w, y_n-w\rangle\\ &\quad+\langle(\mu F-\mu FV_n)y_n, y_n-w\rangle\\ &\le \langle(\mu F-\mu FV_n)y_n, y_n-w\rangle. \end{aligned}\tag{3.6}$$

Now replacing y_n in (3.6) with y_{n_v} and letting $v \to \infty$, noticing $(\mu F - \mu FV_n)y_{n_v} \to (\mu F-\mu FV_n)\tilde{y} = 0$ for $\tilde{y} \in F^* = Fix(V_n)$, we obtain

$$\langle (\mu F-\gamma\phi)\tilde{y}, \tilde{y}-w\rangle \le 0,\ \forall w \in F^*,$$

which means that $\tilde{y} \in F^*$ is the unique solution of (3.2), i.e., $\tilde{y} = \tilde{z}$. From (Pr.1) we have then

$$\Gamma = \lim_v \langle (\mu F-\gamma\phi)\tilde{z}, y_{n_v}-\tilde{z}\rangle \le 0.$$

Passing to $\limsup_n$ in (3.3) with $w = \tilde{z}$, we derive

$$\lim_n \|y_n-\tilde{z}\| \le \frac{2\Gamma}{\tau-\gamma} \le 0$$

and the proof is complete. □

Theorem 3.2. *Let H, C and $\{T_i\}_{i=1}^N$, $\{\gamma_{i,n}\}_{i=1}^N$ be as in Theorem 3.1. Construct the mappings $V_n (n \in \mathbb{N})$ as in (2.2). Let $F : C \to C$ be a k-Lipschitzian and η-strongly monotone operator with $0 < \mu < \frac{2\eta}{k^2}$, $\phi : C \to C$ be a MKC with $0 < \gamma < \mu(\eta - \frac{\mu k^2}{2}) = \tau$, $\tau < 1$ and let $\{\alpha_n\}$ and $\{\beta_n\} \subset (0,1)$. If the control sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_{i,n}\}_{i=1}^N$ do satisfy:*

- (B1) $\lim_n \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
- (B2) $\limsup_n \frac{1}{\alpha_n}\sum_{i=1}^N |\gamma_{i,n+1}-\gamma_{i,n}| = 0$;
- (B3) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n \le a < 1$ *for some constant* $a \in (0,1)$.

Then the sequences defined by (1.7) *converges to the unique point* $\tilde{x} \in F^* = \bigcap_{i=1}^N Fix(T_i)$*, which satisfies the variational inequality:*

$$\langle \mu F\tilde{x} - \gamma\phi(\tilde{x}), \tilde{x} - p\rangle \leq 0, \quad \forall\, p \in F^*.$$

Proof. We proceed with the following steps.

Step 1. We claim that $\{x_n\}$ is bounded. In fact, for a fixed $w \in F^* = Fix(V_n)$, we have

$$\begin{aligned} \|y_n - w\| &= \|\beta_n(x_n - w) + (1-\beta_n)(V_n x_n - w)\| \\ &\leq \beta_n\|x_n - w\| + (1-\beta_n)\|V_n x_n - w\| \\ &\leq \|x_n - w\|. \end{aligned} \tag{3.7}$$

Then from (1.7) and (3.7) and Lemma 2.3, we obtain

$$\begin{aligned} \|x_{n+1} - w\| &= \|\alpha_n\gamma\phi(x_n) + (I - \mu\alpha_n F)y_n - w\| \\ &= \|\alpha_n\gamma\phi(x_n) - \mu\alpha_n Fw + \mu\alpha_n Fw + (I - \mu\alpha_n F)y_n - w\| \\ &= \|\alpha_n(\gamma\phi(x_n) - \mu Fw) + (I - \mu\alpha_n F)y_n - (I - \mu\alpha_n F)w\| \\ &\leq (1 - \alpha_n\tau)\|y_n - w\| + \alpha_n[\|\gamma\phi(x_n) - \gamma\phi(w)\| \\ &\quad + \|\gamma\phi(w) - \mu Fw\|] \\ &\leq (1 - \alpha_n\tau)\|x_n - w\| + \alpha_n\gamma\|x_n - w\| + \alpha_n\|\gamma\phi(w) - \mu Fw\| \\ &\leq [1 - \alpha_n(\tau - \gamma)]\|x_n - w\| + \alpha_n\|\gamma\phi(w) - \mu Fw\| \\ &\leq [1 - \alpha_n(\tau - \gamma)]\|x_n - w\| + \alpha_n(\tau - \gamma)\frac{\|\gamma\phi(w) - \mu Fw\|}{\tau - \gamma} \\ &\leq \max\left\{\|x_n - w\|, \frac{\|\gamma\phi(w) - \mu Fw\|}{\tau - \gamma}\right\}, \quad n \geq 1. \end{aligned}$$

By induction, we have

$$\|x_n - w\| \leq \max\left\{\|x_1 - w\|, \frac{\|\gamma\phi(w) - \mu Fw\|}{\tau - \gamma}\right\} =: M, \quad n \geq 1. \tag{3.8}$$

Thus $\{y_n\}, \{\mu F y_n\}, \{\phi(x_n)\}$ and $\{V_n x_n\}$ are bounded too.

Step 2. We claim that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. To this end, define a sequence $\{z_n\}$ by $z_n = (x_{n+1} - \beta_n x_n)/(1 - \beta_n)$, such that $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n$.

We now observe that

$$
\begin{aligned}
z_{n+1}-z_n &= \frac{x_{n+2}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} - \frac{x_{n+1}-\beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}\gamma\phi(x_{n+1}) + (I-\mu\alpha_{n+1}F)y_{n+1} - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n\gamma\phi(x_n) + (I-\mu\alpha_n F)y_n - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma\phi(x_{n+1}) - \mu F y_{n+1}) + \frac{y_{n+1}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n}{1-\beta_n}(\gamma\phi(x_n) - \mu F y_n) - \frac{y_n - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma\phi(x_{n+1}) - \mu F y_{n+1}) \\
&\quad + \frac{[\beta_{n+1}x_{n+1} + (1-\beta_{n+1})V_{n+1}x_{n+1}] - \beta_{n+1}x_{n+1}}{1-\beta_{n+1}} \\
&\quad - \frac{\alpha_n}{1-\beta_n}(\gamma\phi(x_n) - \mu F y_n) - \frac{[\beta_n x_n + (1-\beta_n)V_n x_n] - \beta_n x_n}{1-\beta_n} \\
&= \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\gamma\phi(x_{n+1}) - \mu F y_{n+1}) - \frac{\alpha_n}{1-\beta_n}(\gamma\phi(x_n) - \mu F y_n) \\
&\quad + V_{n+1}x_{n+1} - V_n x_n. \qquad (3.9)
\end{aligned}
$$

It follows from (3.9) that

$$
\begin{aligned}
\|z_{n+1}-z_n\| &\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}(\|\gamma\phi(x_{n+1})\| + \|\mu F y_{n+1}\|) \\
&\quad + \frac{\alpha_n}{1-\beta_n}(\|\gamma\phi(x_n)\| + \|\mu F y_n\|) + \|V_{n+1}x_{n+1} - V_n x_n\|. \qquad (3.10)
\end{aligned}
$$

Using Lemma 2.11 and for M and w as in (3.8), it follows

$$
\begin{aligned}
\|V_{n+1}x_{n+1} - V_n x_n\| &\le \|V_n x_{n+1} - V_n x_n\| + \|V_{n+1}x_{n+1} - V_n x_{n+1}\| \\
&\le \|x_{n+1}-x_n\| + \sum_{i=1}^{N} \mid \gamma_{i,n+1} - \gamma_{i,n} \mid M_i M. \qquad (3.11)
\end{aligned}
$$

Substituting (3.11) into (3.10), we obtain

$$
\begin{aligned}
\|z_{n+1}-z_n\| &\le \bar{M}\left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}\right) + \|x_{n+1}-x_n\| \\
&\quad + \sum_{i=1}^{N} \mid \gamma_{i,n+1} - \gamma_{i,n} \mid M_i M, \qquad (3.12)
\end{aligned}
$$

where $\bar{M} = \sup\{\|\gamma\phi(x_n)\| + \|\mu F y_n\|, n \geq 1\}$. It follows from (3.12) that

$$\begin{aligned} &\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\| \\ &\leq \bar{M}(\frac{\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n}) + \sum_{i=1}^{N} \mid \gamma_{i,n+1} - \gamma_{i,n} \mid M_i M. \end{aligned} \tag{3.13}$$

Observing condition $(B1), (B2), (B3)$ and (3.13), it follows that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Hence, by Lemma 2.4, we can obtain

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.14}$$

It follows from $(B3)$ and (3.14) that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1-\beta_n)\|z_n - x_n\| = 0. \tag{3.15}$$

Step 3. We claim that $\lim_{n\to\infty} \|x_n - V_n x_n\| = 0$. As a direct consequence of (3.15), we note that

$$\begin{aligned} \|x_n - V_n x_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \|y_n - V_n x_n\| \\ &= \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| + \beta_n\|x_n - V_n x_n\|. \end{aligned}$$

From $(B1), (B3)$ and using Step 2, we have

$$\begin{aligned} (1-a)\|x_n - V_n x_n\| &\leq (1-\beta_n)\|x_n - V_n x_n\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|\gamma\phi(x_n) - \mu F y_n\| \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

This implies that

$$\|x_n - V_n x_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.16}$$

Step 4. We claim that $\limsup_n \langle \gamma\phi(\tilde{x}) - \mu F\tilde{x}, x_n - \tilde{x}\rangle \leq 0$, where $\tilde{x} = \lim_m y_m$ with $y_m = \alpha_m \gamma\phi(y_m) + (I - \mu\alpha_m F)V y_m$.

For this purpose, let $\{x_{n_k}\}$ be a subsequence chosen in such a way that $\lim_{k\to\infty}\langle \gamma\phi(\tilde{x}) - \mu F\tilde{x}, x_{n_k} - \tilde{x}\rangle = \limsup_{n\to\infty}\langle \gamma\phi(\tilde{x}) - \mu F\tilde{x}, x_n - \tilde{x}\rangle$, $x_{n_k} \rightharpoonup z$ and $\gamma_{i,n_k} \to \gamma_i (i = 1, \ldots, N)$. Let V be the maps generated by the finite $\{T_i\}_{i=1}^N$ and coefficient $\gamma_1, \ldots \gamma_N$ following the scheme (2.2). From (3.16) and Lemma 2.11, we obtain

$$\|x_{n_k} - V x_{n_k}\| \leq \|x_{n_k} - V_{n_k} x_{n_k}\| + \|V_{n_k} x_{n_k} - V x_{n_k}\| \to 0,$$

we have $Vx_{n_k} \rightharpoonup z$. From Lemma 2.12, we know $z \in F^*$. Hence, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \gamma\phi(\tilde{x}) - \mu F\tilde{x}, x_n - \tilde{x}\rangle &= \lim_{k\to\infty}\langle \gamma\phi(\tilde{x}) - \mu F\tilde{x}, x_{n_k} - \tilde{x}\rangle \\ &= \langle \gamma\phi(\tilde{x}) - \mu F\tilde{x}, z - \tilde{x}\rangle \\ &\leq 0.\end{aligned}$$

Step 5. We claim that $\{x_n\}$ converges strongly to $\tilde{x}$. From (1.7), Lemma 2.1 and Lemma 2.3, we have

$$\begin{aligned}&\|x_{n+1} - \tilde{x}\|^2 \\ &= \|\alpha_n\gamma\phi(x_n) + (I - \mu\alpha_n F)y_n - \tilde{x}\|^2 \\ &= \|(I - \mu\alpha_n F)y_n - (I - \mu\alpha_n F)\tilde{x} + \alpha_n(\gamma\phi(x_n) - \mu F\tilde{x})\|^2 \\ &\leq \|(I - \mu\alpha_n F)y_n - (I - \mu\alpha_n F)\tilde{x}\|^2 + 2\alpha_n\langle \gamma\phi(x_n) - \mu F\tilde{x}, x_{n+1} - \tilde{x}\rangle \\ &\leq (1 - \alpha_n\tau)^2\|y_n - \tilde{x}\|^2 + 2\alpha_n\langle \gamma\phi(x_n) - \gamma\phi(\tilde{x}), x_{n+1} - \tilde{x}\rangle \\ &\quad + 2\alpha_n\langle \gamma\phi(\tilde{x}) - \mu F\tilde{x}, x_{n+1} - \tilde{x}\rangle \\ &\leq (1 - \alpha_n\tau)^2\|x_n - \tilde{x}\|^2 + \alpha_n\gamma(\|x_n - \tilde{x}\|^2 + \|x_{n+1} - \tilde{x}\|^2) \\ &\quad + 2\alpha_n\langle \gamma\phi(\tilde{x}) - \mu F\tilde{x}, x_{n+1} - \tilde{x}\rangle.\end{aligned}$$

It then follows that

$$\begin{aligned}&\|x_{n+1} - \tilde{x}\|^2 \\ &\leq \frac{(1 - \alpha_n\tau)^2 + \alpha_n\gamma}{1 - \alpha_n\gamma}\|x_n - \tilde{x}\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma}\langle \gamma\phi(\tilde{x}) - \mu F\tilde{x}, x_{n+1} - \tilde{x}\rangle \\ &\leq (1 - \frac{2\alpha_n(\tau - \gamma)}{1 - \alpha_n\gamma})\|x_n - \tilde{x}\|^2 \\ &\quad + \frac{2\alpha_n(\tau - \gamma)}{1 - \alpha_n\gamma}\left[\frac{1}{\tau - \gamma}\langle \gamma\phi(\tilde{x}) - \mu F\tilde{x}, x_{n+1} - \tilde{x}\rangle + \frac{\alpha_n\tau^2}{2(\tau - \gamma)}\tilde{M}\right],\end{aligned}$$

where $\tilde{M} = \sup_{n\geq 1}\|x_n - \tilde{x}\|^2$. From (B1) and Step 4, it follows that

$$\sum_{n=1}^{\infty}\frac{2\alpha_n(\tau - \gamma)}{1 - \alpha_n\gamma} = \infty$$

and

$$\limsup_{n\to\infty}\frac{1}{\tau - \gamma}\langle \gamma\phi(\tilde{x}) - \mu F\tilde{x}, x_{n+1} - \tilde{x}\rangle + \frac{\alpha_n\tau^2}{2(\tau - \gamma)}\tilde{M} \leq 0.$$

Hence, by Lemma 2.5, the sequence $\{x_n\}$ converges strongly to the unique point $\tilde{x} \in F^* = \bigcap_{i=1}^N Fix(T_i)$, which satisfies the variational inequality:

$$\langle \mu F\tilde{x} - \gamma\phi(\tilde{x}), \tilde{x} - p\rangle \leq 0, \quad \forall\, p \in F^*.$$

□

Remark 3.3. Our results improve and extend the results of Wang [4] in the following aspects:

(i) a family of nonexpansive mappings W_n is replaced by V_n;
(ii) contractive mapping is replaced by a MKC.

Remark 3.4. If $\beta_n = 0, \gamma = 1, \mu = 1$, F be an identity operator and ϕ is replaced by a contractive mapping f in Theorem 3.2, we can obtain Theorem 14 of Colao and Marino [5] in Hilbert spaces.

Corollary 3.5. *Let H, C and $\{T_i\}_{i=1}^N$, $\{\gamma_{i,n}\}_{i=1}^N$ be as in Theorem* 3.2. *Construct the mappings $V_n (n \in \mathbb{N})$ as in* (2.2). *Let A be a strongly positive bounded linear operator on C with coefficient $0 < \tilde{\gamma} < \|A\|$, $\phi : C \to C$ be a MKC with $0 < \gamma < \mu(\bar{\gamma} - \frac{\mu\|A\|^2}{2}) = \tau$, $\tau < 1$ and let $\{\alpha_n\}$ and $\{\beta_n\} \subset (0, 1)$. If the control sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_{i,n}\}_{i=1}^N$ do satisfy the conditions* (B1), (B2) *and* (B3). *Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$:*

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) V_n x_n, \\ x_{n+1} = \alpha_n \gamma \phi(x_n) + (I - \alpha_n \mu A) y_n, \quad \forall\, n \geq 1. \end{cases}$$

Then $\{x_n\}$ converges to the unique point $\tilde{x} \in F^* = \bigcap_{i=1}^N Fix(T_i)$, which satisfies the variational inequality:

$$\langle \mu A\tilde{x} - \gamma\phi(\tilde{x}), \tilde{x} - p \rangle \leq 0, \quad \forall p \ \in F^*.$$

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