Nonlinear Functional Analysis and Applications
Vol. 19, No. 4 (2014), pp. 627-632

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A NOTE ON D-OPERATOR PAIR

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Abstract. In this paper, we show that under contractive conditions proving existence of common fixed point by assuming D-operator pair [Some common fixed point theorems for D-operator pair with applications to nonlinear integral equations, Nonlinear Functional Analysis and Applications, Vol. 18, No. 2 (2013), 205-218] is equivalent to proving the existence of common fixed point by assuming the existence of common fixed point.

1. Introduction and Preliminaries

In a recent work, Rai and Pathak [11] defined D-operator pair of single valued mappings and obtain some common fixed point theorems for this class of maps under relaxed conditions. They also discussed the existence of solutions for some nonlinear integral equations that oftenly appear in nonlinear analysis.

In 1982, Sessa [7] initiated the study of existence of common fixed point of weakly commuting mappings, a weaker version of commutativity condition

[0]Received June 17, 2014. Revised September 30, 2014.
[0]2010 Mathematics Subject Classification: 54H25, 47H10.
[0]Keywords: Coincidence point, point of coincidence, common fixed point, commutativity conditions.
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[2]. In recent years several authors have considered several generalizations of commuting mappings or weaker notions of commutativity. A systmetic servey of various noncommuting conditions upto 2014 can be found in [12]. Now, it has been shown that weak compatibility is the minimal noncommuting condition for the existence of common fixed points of contractive type mapping pairs. More recently, several authors claimed to introduce some weaker noncommuting notions and pretended to show, weak compatibility as a proper subclass of their weaker notions. This is, however, not true. In view of the results of Alghamdi et al. ([1] see also, [9], [10], [14]) most of the generalized commutativity notions fall in the subclass of weak compatibility in the setting of a unique common fixed point (or unique point of coincidence). If there are just two maps involved, and they only have one coincidence point (which turns out to be the unique fixed point), then, of course D-operator pair and all of the generalizations of commutativity coincide. If there are no coincidence points, then there cannot be any fixed points, and no D-operators points either. Those generalizaitions of commuting mappings are noval but for their actual applications one should go beyond contractive conditions. In fact, under contractive conditions, proving the existence of common fixed points by assuming several weaker noncommuting notions is equivalent to proving the existence of common fixed points by assuming the existence of common fixed points ([10, 14]).

Following Pathak and Rai [8] (see also [1]), we assume that (X, d) is a metric space. For $x \in X$ and $A \subset X$,

$$d(x, A) = \inf\{d(x, y) : y \in A\}.$$

Let f and g be self-maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence (POC) of f and g. The set of coincidence points of f and g will be denoted as $C(f, g)$. Let $PC(f, g)$ represent the set of points of coincidence of f and g. A point $x \in X$ is a common fixed point of f and g if $x = fx = gx$. The set of all common fixed points of f and g is denoted by $F(f, g)$.

Definition 1.1. Let X be a non-empty set and d be a function $d : X \times X \to [0, \infty)$ such that

$$d(x, y) = 0 \quad \text{if and only if} \quad x = y, \quad \text{for each} \quad x, y \in X. \tag{1.1}$$

For a space (X, d) satisfying (1.1) and $A \subset X$, the diameter of A is defined by

$$diam(A) = \sup\{\max\{d(x, y), d(y, x)\} : x, y \in A\}.$$

Definition 1.2. A pair of self-maps (f, g) of a metric space (X, d) is said to be

(i) commuting [2] if $fgx = gfx$ for all x in X.

(ii) weakly commuting [7] if

$$d(fgx, gfx) \leq d(fx, gx)$$

for all x in X.

(iii) $R-$weakly commuting [6] if

$$d(fgx, gfx) \leq Rd(fx, gx)$$

for all x in X and $R > 0$.

(iv) compatible [3] if and only if

$$\lim_n d(fgx_n, gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_n fx_n = \lim_n gx_n = t$ for some t in X.

(v) weakly compatible (WC) [4] if the pair commutes on the set of coincidence points, i.e., $fgx = gfx$ whenever $fx = gx$ for some $x \in X$.

(vi) occasionally weakly compatible (OWC) [5] if $fx = gx$ and $fgx = gfx$ for some $x \in X$.

(vii) a PD-operator pair [8], if there is a point $x \in X$ such that $x \in C(f, g)$ and

$$d(fgx, gfx) \leq \ diam(PC(f, g)),$$

for some $x \in C(f, g)$.

(viii) a D-operator pair [11], if there is a point $x \in X$ such that $x \in C(f, g)$ and

$$d(fgx, gfx) \leq R \ diam(PC(f, g)),$$

for all $R > 0$.

Definition 1.3. A symmetric on a set X is a mapping $d : X \times X \longrightarrow [0, \infty)$ such that

(1) $d(x, y) = 0$ if and only if $x = y$, and

(2) $d(x, y) = d(y, x)$.

A set X, together with a symmetric d is called a symmetric space.

2. Main Results

Proposition 2.1. ([9]) *Let a pair of mappings (f, g) has a unique POC. Then the pair(f, g) is WC if and only if it is OWC.*

Proposition 2.2. *Let $d : XxX \to [0, \infty)$ be a mapping such that $d(x, y) = 0$ if and only if $x = y$. Let a pair of mappings (f, g) has a unique POC. If (f, g) is a pair of $D-$operators, then it is WC.*

Proof. First we have that $C(f, g) \neq \phi$ because $PC(f, g) \neq \phi$ ($PC(f, g)$ is a singleton). Since (f, g) is a $D-$operator, there exists some $x \in C(f, g)$ such that

$$d(fgx, gfx) \leq R\ diam(PC(f, g)) = 0,$$

for some $R > 0$. Hence $d(fgx, gfx) = 0$, i.e., there exists $fx = gx$ with $fgx = gfx$. Therefore the pair (f, g) is OWC. According to [9] (f, g) is WC. □

Let $\phi : R_+ \to R_+$ be a nondecreasing function satisfying the condition $\phi(t) < t$, for each $t > 0$.

Proposition 2.3. *Let $d : XxX \to [0, \infty)$ be a mapping such that $d(x, y) = 0$ if and only if $x = y$. Suppose (f, g) is $D-$operator pair and satisfy the condition:*

$$d(fx, fy) \leq \phi\big(\max\{d(gx, gy), d(gx, fy), d(fx, gy), d(gy, fy)\}\big), \tag{2.1}$$

for each $x, y \in X$. Then f and g are WC.

Proof. By hypothesis, there exists some $x \in X$ such that $w = fx = gx$. It remains to show that (f, g) has a unique POC. Suppose there exists another point $w_1 = fy = gy$ with $w \neq w_1$. Then, we have

$$\begin{aligned} d(w, w_1) &= d(fx, fy) \\ &\leq \phi(\max\{d(gx, gy), d(gx, fy), d(fx, gy), d(gy, fy)\}) \\ &< d(w, w_1), \end{aligned}$$

a contradiction. Thus (f, g) has a unique POC. By Proposition 2.2, The pair (f, g) is WC. □

In a recent work, Rai and Pathak [11] proved the following theorem:

Theorem 2.4. ([11]) *Let f and g be two self-maps of a symmetric space X. Suppose that (f, g) is a $D-$operator pair and satisfy the condition* (2.1)*. Then f and g have a unique common fixed point.*

Now we are in a position to prove our main result.

Theorem 2.5. *Let the pair (f, g) has the contractive condition* (2.1)*. Then the condition of D-operators and the existence of a unique common fixed point are equivalent.*

Proof. We first observe that under the contractive condition (2.1), the assumption of $D-$operators and the existence of a unique common fixed point are equivalent conditions. To see this, first suppose that f and g satisfy the contractive condition (2.1). If f and g have a common fixed point, say z, then $z = fz = gz$, $fgz = gfz = z$. Thus, f and g are $D-$operators, since contractive condition (2.1) exclude the existence of two point of coincidences or common fixed points.

On the other hand, suppose that f and g are $D-$operators such that $fx = gx$ and

$$d(fgx, gfx) \leq Rdiam(PC(f,g))$$

for some $x \in C(f,g)$. Now, in view of condition (2.1), we get

$$diam(PC(f,g)) = 0,$$

(since contractive condition (2.1) exclude the existence of two point of coincidences). Hence $fgx = gfx$. If $fx \neq ffx$, using (2.1) we get

$$\begin{aligned} d(ffx, fx) &\leq \phi(\max(d(gfx, gx), d(gfx, fx), d(ffx, gx), d(gx, fx))) \\ &< d(ffx, fx), \end{aligned}$$

which is a contradiction. Hence $fx = ffx$ and $fx = ffx = gfx$. This means that fx is a common fixed point of f and g. Uniqueness of the fixed point follows from contractive condition (2.1). $\square$

Remark 2.6. Let us remark that the same result as in Theorem 2.5 will also be true for many contractive conditions assumed in the paper [11], e.g.,

$$d(fx, fy) < \max(d(gx, gy), d(gx, fy), d(gy, fx), d(gy, fy)), \tag{2.2}$$

$$d(fx, fy) \leq h \max(d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)), \tag{2.3}$$

for $0 \leq h < 1$,

$$d(fx, fy) \leq h \max\left\{d(gx, gy), \frac{d(fx, gx) + d(fy, gy)}{2}, \frac{d(fx, gy) + d(fy, gx)}{2}\right\}, \tag{2.4}$$

for $0 \leq h < 1$,

$$\begin{aligned} d(fx, fy) \leq\ & a\, d(gx, gy) + b \max\big(d(fx, gx), d(fy, gy)\big) \\ &+ c \max\big(d(gx, gy), d(gx, fx), d(gy, fy)\big), \end{aligned} \tag{2.5}$$

where $a, b, c > 0, a + b + c = 1$ and $a + c < 1$.

Therefore, under contractive conditions (2.2)-(2.5), the existence of a common fixed point and D-operators are equivalent conditions. In order to find actual applications of D-operators, we should go beyond contractive conditions, since contractive conditions do not allow more than one point of coincidence or fixed point.

Remark 2.7. When $R = 1$, D-operators reduce into PD-operators, therefore the same conclusion can be drawn from the PD-operators as well, i.e., under contractive conditions, proving the existence of common fixed point by assuming the notion of PD-operator is equivalent to prov the existence of common fixed point by assuming the existence of common fixed point [13].

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