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REMARKS ON THE PAPER "EXTENSION OF CARISTI'S FIXED POINT THEOREM TO VECTOR VALUED METRIC SPACES"

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Abstract. In this article, we state that all theorems in the paper [1] are the same Caristi's Theorem in ordinary case.

1. Introduction and preliminary

In this work we have mentioned to an extension of Caristi's fixed point theorem for vector valued metric spaces stated by Agarwal and Khamsi [1], which is the same case of Caristi's fixed point theorem in ordinary case.
Recall that this theorem states that any map $T : M \to M$ has a fixed point provided that M is complete and there exists a lower semi-continuous map ϕ mapping M into the nonnegative numbers such that

$$d(x, Tx) \leq \phi(x) - \phi(Tx),$$

for every $x \in M$.

Also, we take look at to cone metric spaces were introduced by Huang and Zhang in 2007. They replaced the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mapping satisfying different contractions [9]. The study of fixed point theorems in such spaces followed by some other mathematicians, see [2, 3, 4, 10].

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Now consider $(V, \preceq)$ be an ordered Banach space. The cone $V_+ = \{v \in V : \theta \preceq v\}$, where θ is the zero-vector of V, satisfies the usual properties:
(i) $V_+ \cap -V_+ = \{\theta\}$,
(ii) $V_+ + V_+ \subset V_+$,
(iii) $aV_+ \subseteq V_+$ for all $a \geq 0$.

The space V can be partially ordered by the cone $V_+ \subset V$; that is,

$$x \preceq y \quad \Leftrightarrow \quad y - x \in V_+, \quad \text{for each } \; x, y \in V.$$

Also we write

$$x \prec\prec y \quad \text{if} \quad y - x \in {V_+}^o,$$

where ${V_+}^o$ denotes the interior of V_+. A cone V_+ is called normal if there exists a constant $K > 0$ such that $0 \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. Also you can consider equivalent definition

$$\inf\{\|x + y\| : x, y \in V, \; \|x\| = \|y\| = 1\} > 0.$$

Throughout the remainder of this work we will assume that V is a complete Banach lattice which is order continuous.

The concept of vector valued metric spaces relies on the following definition.

Definition 1.1. ([1]) Let X be a nonempty set. Let M be a set. A map $d : M \times M \to V$ defines a distance if:
(i) $d(x, y) = \theta \iff x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in M$,
(iii) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in M$.
The pair (M, d) is called a vector valued metric space (vvms for short).

For some examples please refer to [3, 4, 10] to find vvms as special case of the cone metric spaces, normal cones with constant $K = 1$, $K > 1$ and non-normal cone.

In the next section we need to following results.

Proposition 1.2. ([4] and [7, Proposition 1.7.59, pp. 117]) *If E is an ordered Banach space with positive cone P, then P is a normal cone if and only if there exists an equivalent norm $|\cdot|$ on E which is monotone.*

So by renorming the E we can suppose P is a normal cone with constant one.

Lemma 1.3. ([4]) *Let (M, d) be a cone metric space, P a normal cone with constant one and $T : M \to C(M)$ be a set-valued map, then*

$$\|d(x, Tx)\| = \| \inf_{y \in Tx} d(x, y)\| = \inf_{y \in Tx} \|d(x, y)\|.$$

2. Main Results

We want to discuss, are the all of theorems in the paper [1], the same Caristi's Theorem?

Let (M, d) be a Vector valued metric space(vvms for short) and $F : M \to V_+$, where $(V, \preceq)$ is an order Banach space, $V_+ := \{v \in V : \theta \preceq v\}$ be a cone and θ is zero-vector of V. For any map $T : M \to M$ which satisfies

$$d(x, T(x)) \preceq F(x) - F(T(x)),$$

for any $x \in M$, has a fixed point and it will be fix any minimal point. Therefore the fixed point problem shifts to the existence of minimal points of the order $\preceq$ in M. We will approach this question through Bronsted order. Indeed the map F defines an order on M as follows:

$$x \preceq_M y \iff d(x, y) \preceq_{V_+} F(y) - F(x),$$

for any $x, y \in M$. Using this order, any map $T : M \to M$ which satisfies

$$d(x, T(x)) \preceq_{V_+} F(x) - F(T(x)),$$

for any $x \in M$ will fix any minimal point.

Lemma 2.1. *Let (M, d) be a vvms and $F : M \to V_+$, be a lsc map and $(V, \preceq)$ is an order complete and order continuous Banach lattice V and V_+, a normal cone with constant one. Then*

$$\sup_{n \geq 1} \|F(x_n)\| = \| \sup_{n \geq 1} F(x_n)\|,$$
$$\inf_{n \geq 1} \|F(x_n)\| = \| \inf_{n \geq 1} F(x_n)\|.$$

We note that $V_+ = \mathbb{R}^n$ is normal, minihedral and strongly minihedral with $V_+^o \neq \emptyset$. Since V_+ called minihedral cone if $\sup\{x, y\}$ exists for all $x, y \in V$, and strongly minihedral if every subset of V which is bounded from above has a supremum. Let (M, d) a cone metric space. Since cone V_+ is strongly minihedral, hence every subset of V_+ has infimum(see [4]).

Proof. Let $b = \sup_{n \geq 1} F(x_n)$ and $a = \sup_{n \geq 1} \|F(x_n)\|$. We show that $a = \|b\|$. For every $k \geq 1$, we have $F(x_k) \preceq b$. Since V_+ is normal cone with constant one so

$$\|F(x_k)\| \leq \| \sup_{n \geq 1} F(x_n)\|, \quad \forall\, k \geq 1,$$

or

$$\sup_{n\geq 1}\|F(x_n)\| \leq \|\sup_{n\geq 1}F(x_n)\|,$$

therefore $a \leq \|b\|$.
For the inverse, let for all $0 \leq a \leq r$. Then, for every $n \geq 1$, $\|F(x_n)\| \leq r$. Since $b = \sup_{n\geq 1} F(x_n)$, for every $c \succ\succ \theta$ there exists $k \geq 1$ such that $b - c \prec F(x_k)$, thus

$$|\|b\| - \|c\|| \leq \|b - c\| \leq \|F(x_k)\| \leq r, \quad \forall\, c \succ\succ \theta.$$

Therefore $\|b\| \leq r$. □

Theorem 2.2. *Let (M, d) be a vvms and $F : M \to V_+$, where $(V, \preceq)$ is an order complete and order continuous Banach lattice V. Let $F : M \to V_+$ be a lsc map. Then $T : M \to M$ such that*

$$d(x, T(x)) \preceq F(x) - F(T(x)),$$

for any $x \in M$, has a fixed point.

Proof. To prove we state this theorem is ordinary case of Caristi's Theorem.

If we consider $V_+ := \mathbb{R}^N_+$, then V_+ is normal cone with constant $K = 1$. And we have the following partial order on $\mathbb{R}_+$

$$\|d(x, y)\| \leq \|F(y)\| - \|F(x)\|,$$

where according to lower semi-continuous(lsc for short) of F; $\|F(x)\|$ is lsc, and $D(x, y) = \|d(x, y)\|$ is its equivalent metric, in this case we have exactly Caristi's Theorem.

When V_+ be a normal cone with constant $K > 1$ by Proposition 1.2 with renorming Banach space we can convert the normal constant to one.

If we have non-normal cone V_+, it is enough we consider its equivalent metric

$$D(x, y) = \inf\{\|u\| : u \in V_+,\ d(x, y) \preceq u\},$$

which has found by Feng and Mao in [8] and Asadi and Vaezpour in [5], so we have

$$D(x, y) \leq \|d(x, y)\| \leq \|F(y)\| - \|F(x)\|,$$

now it is enough that we say $\|F(x)\|$ is lsc, we reach in all cases to Caristi's Theorem. For lsc of $\|F\|$, we note that when $x_n \to x$, we have

$$F(x) \preceq \liminf_{n\to\infty} F(x_n).$$

Since we have non-normal cone V_+ by [6], we can convert every non-normal cone to normal cone with constant one by the renorming Banach space. Therefore in those cases we come back to first step, i.e., normal cone with constant one. Therefore $\|F(x)\|$ is lsc by new norm. □

Example 2.3. ([6]) Let $E := C^1_{\mathbb{R}}([0,1])$ with

$$\|x\| := \|x\|_\infty + \|x'\|_\infty.$$

Hence $P := \{x \in E : x \geq 0\}$ is a non-normal cone. Now if we consider $n(\cdot)$ by $n(x) := \|x\|_\infty$, then P is normal cone with $K = 1$.

This is an example which show us that every non-normal cone to normal cone could be converted to normal cone with constant one by the renorming Banach space.

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