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BOUNDARY AND POINT CONTROLS FOR SEMILINEAR STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we consider a class of semilinear stochastic partial differential equations with nonhomogeneous boundary conditions including noise and (boundary) control. The system is formulated as an abstract evolution equation in a suitable Hilbert space. We prove existence and regularity of mild solutions. We consider Bolza problem and prove existence of optimal controls for two classes of admissible controls, one being the class of $\mathcal{G}_t$-adapted measurable stochastic processes with values in a weakly compact subset of a suitable Hilbert space, and the other being the class weak star measurable $\mathcal{G}_t$-adapted signed Borel measures containing point controls (or Dirac measures) as a special case.

1. Introduction

In recent years intensive research has been carried out in the area of necessary conditions of optimality for stochastic systems of finite and some for infinite dimensions along the line of the Pontryagin minimum principle [1-2], [9,11,13,14,18]. See also the extensive references given therein. In [2] we consider semi-linear neutral stochastic evolution equations with controls in the drift and the diffusion operators and present necessary conditions of optimality. In [11] Duncan and Pasic-Duncan consider linear stochastic differential equations on Hilbert spaces with exponential-quadratic cost functionals giving differential operator Ricatti equations. They present also several interesting examples from initial boundary value problems. Fuhrman, Hu and Tessitore [13] present maximum principle for a class of stochastic partial differential

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equations subject to finite dimensional Brownian motion with controls appearing in the drift and the diffusion coefficients. The cost functional is of Bolza type. Fuhrman et. al. consider regular controls and they develop second order necessary conditions. In [14] Hu and Peng develop some fundamental results on the question of existence and uniqueness of solutions for a large class of backward stochastic evolution equations (BSDE) on Hilbert spaces. In [18] Zhou develops necessary conditions of optimality (maximum principle) for a very general class of linear non-degenerate (strictly elliptic) second order partial differential equations on a d-dimensional space with all the coefficients containing control. Except [11], the above mentioned papers do not consider boundary value problems which arise so naturally in all physical problems. For example, see [9] where Clason, Kaltenbacher, and Veljovic use boundary controls for Westervelt-Kuznetsov equation arising in acoustic and vibration problems. In [7] boundary controls were considered and left point controls as an open problem. Here we consider both boundary controls and point controls.

Most of the papers mentioned above develop necessary conditions of optimality. Very often the question of existence of optimal policies are either ignored or taken for granted. Necessary conditions without existence may turn out to be vacuous. Here in this paper we concentrate on this question using regular as well as relaxed controls. We consider a general class of stochastic semilinear initial boundary value problems with noise appearing both in the interior of the spatial domain as well as on some part of its boundary. The control is exercised on the remaining part of the boundary. The nonlinear drift and the diffusion coefficients (operators) are allowed to contain differential expressions. In other words, these operators map from a smaller space to a larger one and therefore not bounded in the state space. Using the theory of analytic semigroups, we reformulate the stochastic partial differential equation (SPDE) as a stochastic evolution equation on a Hilbert space with the boundary noise term including the boundary control containing the same unbounded operator. The author is not aware of any work in the literature where noise appears both in the interior and a part of the boundary with controls appearing in the complementary part of the boundary. This is one of the motivations of this paper. The second motivation comes from the necessity of allowing nonlinearities which are unbounded on the state space. Further, we consider partially observed regular controls (measurable stochastic processes adapted to a subsigma algebra) and prove existence of optimal controls. We also consider weak star adapted Borel measures as controls (Borel measure valued random processes) and again prove existence of optimal controls.

The paper is organized as follows. In section 2, we present the mathematical model of the system and reformulate this as an abstract stochastic evolution equation on Hilbert space. In section 3, after basic assumptions are introduced, we prove the existence and regularity of mild solutions. Existence of optimal control is proved in section 4 giving two results, one for regular controls, and the other for measure valued controls containing point controls (Dirac measures). For illustration, the paper is concluded with some examples and certain comments on open problems.

2. System model with distributed and boundary forces

Large class of dynamic systems arising in physical sciences and engineering can be described by the following class of partial differential equations:

$$\begin{aligned} &\partial\varphi/\partial t + \mathcal{A}\varphi = f(t,\xi,\varphi) + \sigma(t,\xi,\varphi)V_d(t,\xi), \quad (t,\xi) \in I \times \Sigma, \\ &(\mathcal{B}\varphi)(t,\xi) = V_b(t,\xi), \quad (t,\xi) \in I \times \partial\Sigma_n, \\ &(\mathcal{B}\varphi)(t,\xi) = u_b(t,\xi), \quad (t,\xi) \in I \times \partial\Sigma_c, \\ &\varphi(0,\xi) = \varphi_0(\xi), \quad \xi \in \Sigma, \end{aligned} \tag{2.1}$$

subject to distributed and boundary noise $\{V_d, V_b\}$ defined on the domain $\Sigma \subset R^n$ and part of its boundary $\partial\Sigma_n$ respectively. The control is applied on the boundary $\partial\Sigma_c$ while the full boundary $\partial\Sigma$ is given by the union $\partial\Sigma_n \cup \partial\Sigma_c$ of the two disjoint parts $\partial\Sigma_n$ and $\partial\Sigma_c$. The domain Σ is an open bounded set with smooth boundary and $I = [0,T]$ is an interval. The operator $\mathcal{A}$ is

$$(\mathcal{A}\varphi)(\xi) \equiv \sum_{|\theta| \le 2m} a_\theta(\xi) D^\theta \varphi, \tag{2.2}$$

on Σ, with multi index $\theta = \{\theta_i\}_{i=1}^n$, $|\theta| \equiv \sum_{i=1}^n \theta_i$, $\theta_i \in N_0 \equiv \{0,1,2,\cdots\}$. The boundary operator $\mathcal{B}$ is also a partial differential operator of order at most $2m-1$, given by

$$\begin{aligned} &\mathcal{B}\varphi = \{\mathcal{B}_j, j = 1,2,\cdots,m\}, \\ &(\mathcal{B}_j\varphi)(\xi) \equiv \sum_{|\vartheta| \le m_j \le 2m-1} b_\vartheta^j(\xi) D^\vartheta \varphi, \quad \xi \in \partial\Sigma, \end{aligned} \tag{2.3}$$

where $\vartheta = \{\vartheta_i\}_{i=1}^n$, $|\vartheta| \equiv \sum \vartheta_i$, $\vartheta_i \in N_0$, $i = 1,2,\cdots,n$. The nonlinear operators $\{f,\sigma\}$ are defined shortly. For nonhomogeneous boundary conditions one needs the trace theorem which states that under sufficient smoothness conditions on the boundary $\partial\Sigma$ and the coefficients $\{b_\vartheta, |\vartheta| \le 2m-1\}$, the boundary operator $\mathcal{B}|_{Ker\mathcal{A}}$ is an isomorphism of $W_2^{2m}(\Sigma)/Ker\mathcal{B}$ on to $Y(\partial\Sigma) \equiv \Pi_{j=1}^m W_2^{2m-m_j-1/2}(\partial\Sigma)$ called the trace space. Thus it has a bounded (right) inverse denoted by $\mathcal{R} \equiv (\mathcal{B}|_{Ker\mathcal{A}})^{-1}$. For details on this topic the reader is referred to [4, pp.59-63] and [15,Vol.1, Theorem 5.4, p.165, Theorem 6.6, p.177]. We denote by $\mathcal{R}_n$ the restriction of the operator $\mathcal{R}$ to the space $Y(\partial\Sigma_n)$ and $\mathcal{R}_c$ the restriction of $\mathcal{R}$ to the space $Y(\partial\Sigma_c)$. Then, under fairly general assumptions on the coefficients $\{a_\theta, b_\vartheta\}$ and the principal part of $\mathcal{A}$ and smoothness of the boundary $\partial\Sigma$, one can prove that the negative of the operator $A \equiv \mathcal{A}|_{Ker(\mathcal{B})}$ with domain given by

$$D(A) \equiv \{\psi \in L_2(\Sigma) : \mathcal{A}\psi \in L_2(\Sigma) \ \& \ \mathcal{B}\psi = 0\}$$

generates an analytic semigroup $\{S(t), t \geq 0\}$ on the Hilbert space $E \equiv L_2(\Sigma)$. For details see [4, p.60] and [5, p.85, Theorems 3.2.8 A-D]. Using this semigroup the (nonhomogeneous boundary value problem) system (2.1) can be formulated as an integral equation on the Hilbert space E,

$$\begin{aligned} \varphi(t) &= S(t)\varphi_0 + \int_0^t S(t-\tau) f(\tau, \varphi(\tau)) d\tau + \int_0^t S(t-\tau)\sigma(\tau, \varphi(\tau)) V_d(\tau) d\tau \\ &\quad + \int_0^t AS(t-\tau)\mathcal{R}_n V_b(\tau) d\tau + \int_0^t AS(t-\tau)\mathcal{R}_c u_b(\tau) d\tau, \end{aligned} \tag{2.4}$$

for $t \geq 0$. Note that the unbounded operator $-A$, that generates the semigroup $S(t), t \geq 0$, also appears in the integral equation (2.4). This integral equation is our starting point and using this equation we construct a rigorous model for the stochastic initial boundary value problem. First, we define the admissible controls. For $\alpha \in (0, 1]$ introduce the family of trace spaces $Y_\alpha(\partial\Sigma_c) = \prod_{j=1}^m W_2^{2\alpha m - m_j - 1/2}(\partial\Sigma_c)$ and let U be a closed bounded convex subset of the trace space $Y_\alpha(\partial\Sigma_c)$. For admissible controls we choose $\mathcal{U}_{ad} \equiv L_2(I, U) \subset L_2(I, Y_\alpha(\partial\Sigma_c))$. For a rigorous mathematical model of the noise processes we consider the complete filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ where $\mathcal{F}_{t\geq 0} \subset \mathcal{F}$ is a nondecreasing family of subsigma algebras of the sigma algebra $\mathcal{F}$ and P is the probability measure on Ω. Let Σ_0 be any open subset of the set Σ and denote by H the Hilbert space $L_2(\Sigma_0)$. Let $W_d \equiv \{W_d(t), t \geq 0\}$ with $P\{W_d(0) = 0\} = 1$, denote an $\mathcal{F}_t$-adapted H-valued Brownian motion with V_d being its distributional derivative. Similarly, let $W_b \equiv \{W_b(t), t \geq 0\}$ with $P\{W_b(0) = 0\} = 1$, denote an $\mathcal{F}_t$-adapted Brownian motion taking values from the vector space $Y_\alpha(\partial\Sigma_n) \equiv \prod_{j=1}^m W_2^{2\alpha m - m_j - 1/2}(\partial\Sigma_n)$ with distributional derivative V_b. Using this formalism we can write the stochastic integral equation (2.4) in the canonical form as follows

$$\begin{aligned} x(t) &= S(t)x_0 + \int_0^t S(t-\tau) f(\tau, x(\tau)) d\tau + \int_0^t S(t-\tau)\sigma(\tau, x(\tau)) dW_d(\tau) \\ &\quad + \int_0^t AS(t-\tau)\mathcal{R}_n dW_b(\tau) + \int_0^t AS(t-\tau)\mathcal{R}_c u_b(\tau) d\tau, \end{aligned} \tag{2.5}$$

for $t \geq 0$, where $\varphi_0 = x_0$. For detailed proof leading to the above results and more on nonhomogeneous nonlinear boundary value problems see [4, Chapter 3, p.59] and [5, Example 3.2.8, p.85].

Before we can proceed with the analysis of the integral equation (2.5) we must introduce a family of suitable vector spaces. We have already noted that the operator

$$\mathcal{B}/Ker(\mathcal{A}) \in iso\big(W_2^{2m}(\Sigma)/Ker(\mathcal{B}), Y(\partial\Sigma)\big).$$

For $\alpha \in [0,1]$, let us introduce the following interpolation spaces

$$X_\alpha \equiv W_2^{2\alpha m}(\Sigma)/Ker(\mathcal{B}) \text{ and } Y_\alpha \equiv \Pi_{j=1}^m W_2^{2\alpha m - m_j - 1/2}(\partial\Sigma).$$

Note that for $0 \leq \alpha < \beta \leq 1$,

$$W_2^{2m}(\Sigma)/Ker(\mathcal{B}) = X_1 \hookrightarrow X_\beta \hookrightarrow X_\alpha \hookrightarrow X_0 \subset E$$

and

$$\Pi_{j=1}^m W_2^{2m-m_j-1/2}(\partial\Sigma) = Y_1 \hookrightarrow Y_\beta \hookrightarrow Y_\alpha \hookrightarrow Y_0 = \prod_{j=1}^m W_2^{-m_j-1/2}(\partial\Sigma).$$

Clearly, for each $\alpha \in (0,1]$, the operator $\mathcal{B}/Ker(\mathcal{A}) \in iso\big(X_\alpha, Y_\alpha\big)$. Hence it has a continuous inverse, $(\mathcal{B}/Ker(\mathcal{A}))^{-1} \equiv \mathcal{R} \in \mathcal{L}(Y_\alpha, X_\alpha)$. The state space for the Brownian motion W_b can then be chosen as any of the interpolation spaces Y_α with $\alpha \in (0,1]$ in the sense that, for any $t \geq 0$ and $y^* \in Y_\alpha^*$ (the dual of Y_α), we have

$$P\{|(W_b(t), y^*)_{Y_\alpha, Y_\alpha^*}| < \infty\} = 1$$

and $(W_b(t), y^*)_{Y_\alpha, Y_\alpha^*}$ is an $\mathcal{F}_t$-adapted real valued Gaussian random process with mean zero and variance $t(Q_b y^*, y^*)$ with Q_b being a positive nuclear operator from Y_α^* to Y_α.

3. Basic assumptions and system analysis

Now we are prepared to introduce the basic assumptions. In order to study control problems involving the system (2.5) we must now define the drift and the diffusion operators $\{f, \sigma\}$ including the semigroup generator. For any $\gamma \in [0,1]$ define the family of Hilbert spaces $E_\gamma \equiv \{x \in E : A^\gamma x \in E\}$ equipped with the graph norm topology given by

$$|x|_\gamma \equiv |A^\gamma x|_E.$$

Since A is a closed operator its fractional powers are also closed and hence with respect to the given norm topology these are Banach spaces. Clearly, for $0 \leq \gamma \leq \alpha \leq 1$, we have the continuous embeddings

$$[D(A)] \equiv E_1 \hookrightarrow E_\alpha \hookrightarrow E_\gamma \hookrightarrow E_0 \equiv E \hookrightarrow E_{-\gamma} \hookrightarrow E_{-\alpha} \hookrightarrow E_{-1}.$$

The spaces with negative indices are defined as the completion of E with respect to the norm topology $|x|_r \equiv |A^{-r}x|_E$ for all $r \geq 0$ (see assumption (A1) below). In addition to these spaces we have the family of fractional (quotient) Sobolev spaces given by

$$X_\alpha \equiv W_2^{2\alpha m}(\Sigma)/Ker\mathcal{B} \text{ for } \alpha \in (0,1].$$

Recall that fractional Sobolev spaces as well as Sobolev spaces with negative exponents are generally constructed by use of the theory of Fourier transform [15, p.30, Vol.1].

Basic Assumptions:
(A1) $-A$ is the infinitesimal generator of an analytic semigroup $S(t), t \geq 0$, on the Hilbert space E satisfying

$$\sup\{\| S(t) \|_{\mathcal{L}(E)}, t \in I\} \leq M < \infty.$$

Without loss of generality we may assume that $0 \in \rho(A)$, the resolvent set of A. If not, one can choose a large enough positive number c such that $0 \in \rho(cI+A)$ and absorb it in the drift.
(A2) There exists $\gamma \in [0, 1/2)$ such that $f : I \times E_\gamma \longrightarrow E$ and it is measurable in the first argument and continuous with respect to the second. Further, there exists a constant $K \neq 0$ such that

$$|f(t,e)|_E^2 \leq K^2\{1 + |e|_{E_\gamma}^2\},\ |f(t,e_1) - f(t,e_2)|_E^2 \leq K^2\{|e_1 - e_2|_{E_\gamma}^2\}$$

for all $e, e_1, e_2 \in E_\gamma$ and $t \in I$.
(A3) The incremental covariance of the Brownian motion W_d denoted by $Q_d \in \mathcal{L}_1^+(H)$ (is positive nuclear). For $\gamma \in [0, 1/2)$, the diffusion $\sigma : I \times E_\gamma \longrightarrow \mathcal{L}(H, E)$ is measurable in the first argument and continuous with respect to the second, and there exists a constant $K_{Q_d} \neq 0$ such that for all $(t, e) \in I \times E_\gamma$ and $e_1, e_2 \in E_\gamma$,

$$|\sigma(t,e)|_{Q_d}^2 \leq K_{Q_d}^2\{1 + |e|_{E_\gamma}^2\}, \quad |\sigma(t,e_1) - \sigma(t,e_2)|_{Q_d}^2 \leq K_{Q_d}^2\{|e_1 - e_2|_{E_\gamma}^2\}$$

where $|\sigma|_{Q_d}^2 = tr(\sigma Q_d \sigma^*)$.
(A4) The indices $\{\alpha, \gamma\}$ satisfy $0 \leq \gamma < 1/2$ and $\gamma + 1/2 < \alpha < 1$.

Remark 3.1. The assumptions **(A2)** and **(A3)** are much more relaxed compared to the standard growth and Lipschitz assumptions for $\{f, \sigma\}$ over the same space E. Our assumptions admit $\{f, \sigma\}$ containing differential expressions. For example, let $F : I \times E \longrightarrow E$ and $\gamma \in (0, 1)$ and define $f(t, x) = F(t, A^\gamma x)$. Then $f : I \times E_\gamma \longrightarrow E$ and if F admits linear growth then

$$|f(t,x)|_E^2 = |F(t, A^\gamma x)|_E^2 \leq K^2(1 + |A^\gamma x|_E^2) = K^2(1 + |x|_{E_\gamma}^2).$$

For a more specific example, let $\gamma = 1/2$ and the map F be given by an expression of the form $F(t, \xi, D^{\beta}\varphi(\xi), |\beta| \leq m)$ where $F : I \times \Sigma \times R^N \longrightarrow R$ with $N = card\{|\beta| \leq m\}$ for $\beta = (\beta_1, \beta_2, \cdots, \beta_n), \beta_i \geq 0$.

For proof of the existence, uniqueness and regularity properties of solutions of the integral equation (2.5) we must introduce the appropriate spaces where they may reside. Let $B^a_{\infty}(I, E_{\gamma})$ denote the vector space of E_{γ} valued $\mathcal{F}_t$-adapted random processes having square integrable norms (with respect to the measure P) which are bounded on I. Furnished with the norm topology,

$$\parallel x \parallel_{B^a_{\infty}(I,E_{\gamma})} \equiv (\sup\{\mathbf{E}|x(t)|^2_{E_{\gamma}}, t \in I\})^{1/2},$$

$B^a_{\infty}(I, E_{\gamma})$ is a closed subspace of the Banach space $L^a_{\infty}(I, L_2(\Omega, E_{\gamma}))$ and hence it is a Banach space. For admissible controls, let $\mathcal{G}_t, t \geq 0$, denote a nondecreasing family of sub-sigma algebras of the current of sigma algebras $\mathcal{F}_t, t \geq 0$. Let U be a weakly compact convex subset of the trace space $Y_{\beta}(\partial\Sigma_c)$ for any β satisfying $1 \geq \beta > \alpha > \gamma + 1/2$. For admissible controls we choose the set $\mathcal{U}_{ad} \equiv L^a_2(I, U) \subset L^a_2(I, Y_{\beta}(\partial\Sigma_c))$ which consists of $\mathcal{G}_t$-adapted U-valued random processes with square integrable norms. With this preparation we prove the following existence result.

Theorem 3.2. *Consider the integral equation* (2.5) *modeling the controlled initial boundary problem* (2.1)*. Suppose the assumptions* (A1)-(A4) *hold with the coefficients of the differential operator satisfying* $\{a_{\theta}, |\theta| = 2m\} \in C(\Sigma) \cap L_{\infty}(\Sigma)$ *and* $\{a_{\theta}, |\theta| \leq 2m - 1\} \in L_{\infty}(\Sigma)$*. Further, suppose that the state space for the Brownian motion* W_d *is* H *with incremental covariance operator* $Q_d \in \mathcal{L}^+_1(H)$*, and that for* W_b *is the space* Y_{α} *for any* $\alpha \in (1/2, 1]$ *with incremental covariance operator* $Q_b \in \mathcal{L}^+_1(Y^*_{\alpha}, Y_{\alpha})$*. Then, for every* $\mathcal{F}_0$ *measurable* E_{γ} *valued random variable* $x_0 \in L_2(\Omega, E_{\gamma})$*, and control* $u \in \mathcal{U}_{ad}$*, the integral equation has a unique solution* $x \in B^a_{\infty}(I, E_{\gamma})$*. Further the solution has a continuous modification.*

Proof. Consider the operator F defined by, for $t \geq 0$,

$$\begin{aligned}(Fx)(t) \equiv\; & S(t)x_0 + \int_0^t S(t-\tau)f(\tau, x(\tau))d\tau \\ & + \int_0^t S(t-\tau)\sigma(\tau, x(\tau))dW_d(\tau) \\ & + \int_0^t AS(t-\tau)\mathcal{R}_n dW_b(\tau) + \int_0^t AS(t-\tau)\mathcal{R}_c u(\tau)d\tau, \quad (3.1)\end{aligned}$$

for any fixed $u \in \mathcal{U}_{ad}$ and any E_{γ} valued initial state x_0 having finite second moment, that is, $\mathbf{E}|x_0|^2_{E_{\gamma}} < \infty$. Thus the question of existence of a solution of the integral equation (2.5) is equivalent to the question of existence of a fixed point of the operator F, that is an $x \in B^a_\infty(I, E_\gamma)$ so that $x = Fx$. Since both W_d and W_b are $\mathcal{F}_t$-adapted and $x(t), t \in I$, is $\mathcal{F}_t$ -adapted and $u(t)$ is $\mathcal{G}_t(\subset \mathcal{F}_t)$-adapted, we conclude that $(Fx)(t)$ is $\mathcal{F}_t$-adapted. We prove that $F : B^a_\infty(I, E_\gamma) \longrightarrow B^a_\infty(I, E_\gamma)$. Let $x \in B^a_\infty(I, E_\gamma)$ with $x(0) = x_0$ P-a.s. For convenience of presentation, we let $\{z_1, z_2, z_3, z_4, z_5\}$ denote the first, second, third, fourth and the fifth term on the right hand side of the expression (3.1). Since $S(t), t \geq 0$, is an analytic semigroup we know that $A^\alpha S(t)$ is a bounded operator in E for all $t > 0$. Hence for any $\alpha \geq 0$ there exists a positive constant C_α such that $\| A^\alpha S(t) \|_{\mathcal{L}(E)} \leq C_\alpha / t^\alpha$ for $t > 0$. Throughout the presentation we use C_α to represent this bound. Considering first $\{z_1, z_2, z_3\}$, it follows from straightforward computation using assumptions $(A1)$-$(A4)$ that

$$\begin{aligned} \mathbf{E}|z_1(t)|^2_{E_\gamma} &\equiv \mathbf{E}|A^\gamma z_1(t)|^2_E = \mathbf{E}|A^\gamma S(t) x_0|^2_E = \mathbf{E}|S(t) A^\gamma x_0|^2_E \\ &\leq M^2 \mathbf{E}|A^\gamma x_0|^2_E = M^2 \mathbf{E}|x_0|^2_{E_\gamma}, \quad \forall\, t \in I. \end{aligned} \tag{3.2}$$

For the second term, we have

$$\begin{aligned} \mathbf{E}|z_2(t)|^2_{E_\gamma} &= \mathbf{E}|A^\gamma z_2|^2_E = \mathbf{E}\Big|\int_0^t A^\gamma S(t-s) f(s, x(s)) ds\Big|^2_E \\ &\leq \{T^{1-2\gamma}/(1-2\gamma)\}(C_\gamma K)^2 \int_0^t \big(1 + \mathbf{E}|x(s)|^2_{E_\gamma}\big) ds \\ &\leq \{T^{2(1-\gamma)}/(1-2\gamma)\}(C_\gamma K)^2 \big(1 + \sup_{0 \leq s \leq t} \mathbf{E}|x(s)|^2_{E_\gamma}\big), \end{aligned} \tag{3.3}$$

for all $t \in I$. For the third term, we have

$$\begin{aligned} &\mathbf{E}|z_3(t)|^2_{E_\gamma} = \mathbf{E}|A^\gamma z_3(t)|^2_E = \mathbf{E}\Big|\int_0^t A^\gamma S(t-s)\sigma(s, x(s)) dW_d(s)\Big|^2_E \\ &= \mathbf{E}\int_0^t tr\big(A^\gamma S(t-s)\sigma(s, x(s)) Q_d \sigma^*(s, x(s)) S^*(t-s)(A^\gamma)^*\big) ds \\ &= \mathbf{E}\int_0^t |A^\gamma S(t-s)\sigma(s, x(s))|^2_{Q_d}\, ds \\ &\leq [(K_{Q_d} C_\gamma)^2/(1-2\gamma)] T^{1-2\gamma}(1 + \sup_{0 \leq s \leq t} \mathbf{E}|x(s)|^2_{E_\gamma}), \quad t \in I. \end{aligned} \tag{3.4}$$

Clearly, by assumption (A4) the expressions on the righthand side of both (3.3) and (3.4) are finite. Considering the fourth term we have

$$\mathbf{E}|z_4(t)|^2_{E_\gamma} \equiv \mathbf{E}|A^\gamma z_4(t)|^2_E = \mathbf{E}\Big|\int_0^t A^\gamma A S(t-s)\mathcal{R}_n dW_b(s)\Big|^2_E. \tag{3.5}$$

By our assumption all the coefficients $\{a_\theta, |\theta| \leq 2m\} \subset L_\infty(\Sigma)$ and therefore $X_\alpha \subset E_\alpha$ for any $\alpha \in [0, 1]$. Since $\mathcal{R}_n \in \mathcal{L}(Y_\alpha(\partial\Sigma_n), X_\alpha)$ and $A^\alpha : E_\alpha \longrightarrow E$, we have $A^\alpha \mathcal{R}_n \in \mathcal{L}(Y_\alpha(\partial\Sigma_n), E)$. Thus the identity (3.5) is equivalent to the following identity

$$\begin{aligned}\mathbf{E}|z_4(t)|^2_{E_\gamma} &\equiv \mathbf{E}|A^\gamma z_4(t)|^2_E \\ &= \mathbf{E}|\int_0^t A^\gamma A^{1-\alpha} S(t-s) A^\alpha \mathcal{R}_n dW_b(s)|^2_E. \end{aligned} \tag{3.6}$$

Hence, it follows from the assumption $(A4)$ that

$$\begin{aligned}\mathbf{E}|z_4(t)|^2_{E_\gamma} &\leq \int_0^t \parallel A^{1+\gamma-\alpha} S(t-s) \parallel^2_{\mathcal{L}(E)} tr(A^\alpha \mathcal{R}_n Q_b (A^\alpha \mathcal{R}_n)^*)\ ds \\ &\leq \int_0^t \parallel A^{1+\gamma-\alpha} S(t-s) \parallel^2_{\mathcal{L}(E)} \parallel (A^\alpha \mathcal{R}_n) \parallel^2_{Q_b} ds \\ &\leq [C^2_{1+\gamma-\alpha}/(2(\alpha-\gamma)-1)] T^{2(\alpha-\gamma)-1} \parallel A^\alpha \mathcal{R}_n \parallel^2_{Q_b}, \end{aligned} \tag{3.7}$$

for $t \in I$, where $C_{1+\gamma-\alpha}$ is the generic constant mentioned above, that is,

$$(|A^r S(t)|_{\mathcal{L}(E)} \leq C_r/t^r, \text{ for } 0 \leq r < \infty).$$

Similarly, one can verify that

$$\begin{aligned}&\mathbf{E}|z_5(t)|^2_{E_\gamma} = \mathbf{E}|A^\gamma z_5(t)|^2_E = \mathbf{E}|\int_0^t A^\gamma A^{1-\alpha} S(t-s) A^\alpha \mathcal{R}_c u(s) ds|^2_E \\ &\leq \mathbf{E}\Big(\int_0^t \parallel A^{1+\gamma-\alpha} S(t-s) \parallel_{\mathcal{L}(E)} |A^\alpha \mathcal{R}_c u(s)|_E ds\Big)^2 \\ &\leq \Big(\int_0^t \parallel A^{1+\gamma-\alpha} S(t-s) \parallel^2_{\mathcal{L}(E)} ds\Big)\Big(\mathbf{E}\int_0^t |A^\alpha \mathcal{R}_c u(s)|^2_E ds\Big) \\ &\leq \big(\frac{C^2_{1+\gamma-\alpha}}{2(\alpha-\gamma)-1}\big) T^{2(\alpha-\gamma)-1}\Big(\mathbf{E}\int_0^t |A^\alpha \mathcal{R}_c u(s)|^2_E ds\Big), \quad t \in I. \end{aligned} \tag{3.8}$$

Note that for any $u \in \mathcal{U}_{ad}$ the expression on the righthand side of the above inequality is finite. Using the bounds (3.2),(3.3),(3.4),(3.7) and (3.8), it is easy to see that $Fx \in B^a_\infty(I, E_\gamma)$ for every $x \in B^a_\infty(I, E_\gamma)$. Next we show that F has a unique fixed point in $B^a_\infty(I, E_\gamma)$. Define the interval $I_T \equiv [0, T]$ for any $T > 0$ finite. We show that, for T sufficiently small, F is a contraction in the Banach space $B^a_\infty(I_T, E_\gamma)$. Taking any pair of elements $x, y \in B^a_\infty(I_T, E_\gamma)$ satisfying $x(0) = y(0) = x_0$, and using the same basic assumptions one can easily verify that there exists a constant $\eta(T)$, dependent on T, such that

$$\mathbf{E}|(Fx)(t) - (Fy)(t)|^2_{E_\gamma} \ \leq\ \eta(T) \sup_{0\leq s\leq t\leq T} \{|x(s) - y(s)|^2_{E_\gamma}\} \tag{3.9}$$

where

$$\eta(T) = \{(C_\gamma K)^2 T/(1-2\gamma) + (C_\gamma K_{Q_d})^2/(1-2\gamma)\} T^{1-2\gamma}. \tag{3.10}$$

Clearly, it follows from (3.9) that

$$\| Fx - Fy \|_{B^a_\infty(I_T, E_\gamma)} \ \leq \ \sqrt{\eta(T)} \ \| x - y \|_{B^a_\infty(I_T, E_\gamma)} .$$

By assumption $(A4)$, $0 \leq \gamma < (1/2)$. Thus η is a continuous increasing function of T starting from $\eta(0) = 0$. Hence for $T = T_1 > 0$, sufficiently small, $\eta(T_1) < 1$ and therefore F is a contraction on the Banach space $B^a_\infty(I_{T_1}, E_\gamma)$ and hence by Banach fixed point theorem F has a unique fixed point in $B^a_\infty(I_{T_1}, E_\gamma)$. Since the interval I is compact, it can be covered by a finite number of intervals of suitable length on each of which F is a contraction. Thus we conclude that F has a unique fixed point in $B^a_\infty(I, E_\gamma)$. Hence for each given initial state and control, the integral equation (2.5) has a unique solution. That the solution has a continuous modification follows from the well known factorization technique due to Da Prato and Zabczyk [10]. This completes the proof. □

For any fixed $\mathcal{F}_0$ measurable random variable $x_0 \in L_2(\Omega, E_\gamma)$, let $x(u) \in B^a_\infty(I, E_\gamma)$ denote the solution of the integral equation (2.5) corresponding to the control $u \in \mathcal{U}_{ad}$. Then we have the following result as a corollary of Theorem 3.2.

Corollary 3.3. *Suppose the assumptions of Theorem* 3.2 *hold with the admissible controls* $\mathcal{U}_{ad}$. *Then the solution set* $\Xi \equiv \{x(u), u \in \mathcal{U}_{ad}\}$ *is a bounded subset of* $B^a_\infty(I, E_\gamma)$.

Proof. For any $u \in \mathcal{U}_{ad}$, let $x(u) \in B^a_\infty(I, E_\gamma)$ denote the unique solution of equation (2.5). Then, following similar procedure as in the proof of Theorem 3.2, one can establish the following inequality

$$\mathbf{E}|x(u)(t)|^2_{E_\gamma} \leq c_1 + c_2 \int_0^t \mathbf{E}|x(u)(s)|^2_{E_\gamma} \, ds, \tag{3.11}$$

where

$$\begin{aligned} c_1 = 2^4 \Big\{ & M^2 \mathbf{E}|x_0|^2_{E_\gamma} + ((C_\gamma K)^2/(1-2\gamma)) T^{2(1-\gamma)} \\ & + [(K_{Q_d} C_\gamma)^2/(1-2\gamma)] T^{1-2\gamma} \\ & + [C^2_{1+\gamma-\alpha}/(2(\alpha-\gamma)-1)] T^{2(\alpha-\gamma)-1} \parallel A^\alpha \mathcal{R}_n \parallel^2_{Q_b} \\ & + \Big(\frac{C^2_{1+\gamma-\alpha}}{2(\alpha-\gamma)-1} \Big) T^{2(\alpha-\gamma)-1} \sup_{u \in \mathcal{U}_{ad}} \big(\parallel A^\alpha \mathcal{R}_c u \parallel^2_{L^a_2(I\times\Omega, E)} \big) \Big\} \end{aligned}$$

and

$$c_2 \equiv 2^4 \{ [(C_\gamma K)^2/(1-2\gamma)] T^{2(1-\gamma)} + [(K_{Q_d} C_\gamma)^2/(1-2\gamma)] T^{1-2\gamma} \}.$$

Since $\mathcal{U}_{ad} = L_2^a(I,U)$ with U a bounded subset of $Y_\alpha(\partial\Sigma_c)$, and $A^\alpha\mathcal{R}_c$ is a bounded operator from Y_β to $Y_{\beta-\alpha}(\partial\Sigma_c) \subset E$, we have

$$\sup_{u\in\mathcal{U}_{ad}} \| A^\alpha\mathcal{R}_c u \|^2_{L_2^a(I\times\Omega,E)} < \infty.$$

It is clear from our assumptions $(A1)$-$(A3)$ that the Lipschitz and growth properties of f and σ are uniform with respect to the control. Hence the constants $\{c_1, c_2\}$ are independent of control, and therefore it follows from Gronwall inequality that

$$\sup\{\| x(u) \|_{B^a_\infty(I,E_\gamma)} \colon u \in \mathcal{U}_{ad}\} \leq c_1 \exp c_2 T < \infty.$$

Hence the set $\{x(u), u \in \mathcal{U}_{ad}\}$ is a bounded subset of $B^a_\infty(I,E_\gamma)$. This completes the proof. □

4. Existence of optimal control

For study of optimal controls we need the continuity of the map $u \longrightarrow x$, that is, the control to solution map. This is crucial in the proof of existence of optimal controls. Since continuity is critically dependent on the topology, we must mention the topologies used for the control space and the solution space. For the solution space we have the norm topology on $B^a_\infty(I,E_\gamma)$ as seen in section 3. So we must consider an admissible topology for the control space. Let $\mathcal{G}_t, t \geq 0$, denote a nondecreasing family of sub-sigma algebras of the current of sigma algebras $\mathcal{F}_t, t \geq 0$. Since U is a closed bounded convex subset of the trace space $Y_\beta(\Sigma_c)$ which is a Hilbert space, it is weakly compact and convex for any β satisfying $1 \geq \beta > \alpha > \gamma + 1/2$. For admissible controls we consider the set $\mathcal{U}_{ad} \equiv L_2^a(I,U) \subset L_2^a(I, Y_\beta(\partial\Sigma_c))$ which consist of $\mathcal{G}_t$-adapted U-valued random processes with square integrable norms. Let λ denote the Lebesgue measure on I and $\lambda \times P$ the product measure on the sigma algebra of subsets of the set $I\times\Omega$. Let μ denote the restriction of the product measure $\lambda \times P$ on the sigma algebra of $\mathcal{G}_t$-predictable subsets of the set $I \times \Omega$ denoted by $\mathcal{P}$. We furnish $\mathcal{U}_{ad}$ with the topology of weak convergence in U for μ almost all $(t,\omega) \in I \times \Omega$. In other words a generalized sequence $u^n \longrightarrow u^o$ in this topology if and only if for any $v^* \in Y^*_\beta(\partial\Sigma)$, the dual of $Y_\beta(\partial\Sigma_c)$,

$$< v^*, u^n(t) > \longrightarrow < v^*, u^o(t) >$$

μ almost everywhere. Since U is a weakly compact subset of a separable Hilbert space $Y_\beta(\partial\Sigma_c)$ it follows from a well known theorem [Dunford 12, Theorem V.6.8, p.434] that the weak topology is metrizable with a metric, say, d. Using this metric topology we may define a metric topology for $\mathcal{U}_{ad}$ as follows. For any pair $u, v \in \mathcal{U}_{ad}$ define

$$D(u,v) \equiv \mu\{(t,\omega) \in I \times \Omega : d(u(t,\omega), v(t,\omega)) \neq 0\}.$$

Clearly, this is simply the measure of the sets in $\mathcal{P}$ on which u differs from v. Thus we set $u = v$ if and only if $D(u, v) = 0$. The reader can easily verify that D satisfies all the axioms of a metric space. We denote this metric topology by τ_D.

Now we present a result on continuity of the control to solution map.

Theorem 4.1. *Consider the control system* (2.5) *corresponding to any control* $u \in \mathcal{U}_{ad}$ *which is equipped with the metric topology* τ_D*, and suppose the assumptions of Theorem* 3.2 *hold and that, for each* $\theta \in (0, 1]$*, the injection* $E_\theta \hookrightarrow E$ *is compact. Then the control to solution map* $u \longrightarrow x$ *is continuous with respect to the* τ_D *topology on* $\mathcal{U}_{ad}$ *and the strong (norm) topology on* $B^a_\infty(I, E_\gamma)$.

Proof. Let $u^n \in \mathcal{U}_{ad}$ be a generalized sequence and suppose $u^n \xrightarrow{\tau_D} u^o$. Let $\{x^n, x^o\} \in B^a_\infty(I, E_\gamma)$ with, $x^n(0) = x^o(0) = x_0$, denote the solutions of the integral equation (2.5) corresponding to the controls $\{u^n, u^o\}$ respectively. We show that $x^n \xrightarrow{s} x^o$ in $B^a_\infty(I, E_\gamma)$. Clearly it follows from equation (2.5) corresponding to the controls $\{u^n, u^o\}$ that

$$\begin{aligned} x^n(t) - x^o(t) &= \int_0^t S(t-s)\big(f(s, x^n(s)) - f(s, x^o)\big)\, ds \\ &\quad + \int_0^t S(t-s)\big(\sigma(s, x^n(s)) - \sigma(s, x^o)\big)\, dW_d(s) \\ &\quad + \int_0^t AS(t-s)\mathcal{R}_c(u^n(s) - u^o(s))ds, \quad t \in I. \end{aligned} \tag{4.1}$$

Following similar computations as in the proof of Theorem 3.2, we use (4.1) to derive the following inequality

$$\begin{aligned} &\mathbf{E}|x^n(t) - x^o(t)|^2_{E_\gamma} \\ &\le 4\eta(t) \int_0^t \mathbf{E}|x^n(s) - x^o(s)|^2_{E_\gamma} ds \\ &\quad + 4\frac{t^{2(\alpha-\gamma)-1}}{2(\alpha-\gamma)-1} \mathbf{E} \int_0^t |A^\alpha \mathcal{R}_c(u^n(s) - u^o(s))|^2_E ds, \end{aligned} \tag{4.2}$$

where

$$\eta(t) \equiv \big\{((C_\gamma K)^2/(1-2\gamma))t^{2(1-\gamma)} + ((C_\gamma K_{Q_d})^2/(1-2\gamma))t^{(1-2\gamma)}\big\}.$$

By assumption $(A4)$ the expression on the righthand side of the inequality (4.2) is well defined for all $t \ge 0$. It follows from the above inequality that for any nonnegative $T < \infty$ we have

$$\begin{aligned}
&\| x^n - x^o \|^2_{B^a_\infty(I_T,E_\gamma)} \\
&\leq 4\eta(T)T \| x^n - x^o \|^2_{B^a_\infty(I_T,E_\gamma)} \\
&\quad + \frac{4T^{2(\alpha-\gamma)-1}}{2(\alpha-\gamma)-1}\mathbf{E}\int_0^T |A^\alpha \mathcal{R}_c(u^n(s) - u^o(s))|^2_E ds. \qquad (4.3)
\end{aligned}$$

Recall that the function η given by the expression (3.10) is a nondecreasing continuous function of its argument satisfying $\eta(0) = 0$. Thus we can choose $T = T_1$ sufficiently small so that $4\eta(T_1)T_1 < 1$. For such a choice of T we have

$$\begin{aligned}
&(1 - 4\eta(T_1)T_1) \| x^n - x^o \|^2_{B^a_\infty(I_{T_1},E_\gamma)} \\
&\leq \frac{4T_1^{2(\alpha-\gamma)-1}}{(2(\alpha-\gamma)-1)}\mathbf{E}\int_0^{T_1} |A^\alpha \mathcal{R}_c(u^n(s) - u^o(s))|^2_E ds. \qquad (4.4)
\end{aligned}$$

Now if we show that

$$\lim_{n\to\infty} \mathbf{E}\int_0^{T_1} |A^\alpha \mathcal{R}_c(u^n(s) - u^o(s))|^2_E ds = 0, \qquad (4.5)$$

then we will have proved the continuity as stated in the Theorem for the time interval $I_{T_1} = [0, T_1]$. We prove this after we have shown that this process can be extended to cover any given closed bounded interval $I_T \equiv [0, T]$. Considering once again the integral equation (2.5) starting from time T_1 corresponding to controls $\{u^n, u^o\}$ and subtracting one from the other we have

$$\begin{aligned}
x^n(t) - x^o(t) &= S(t-T_1)(x^n(T_1) - x^o(T_1)) \\
&\quad + \int_{T_1}^t S(t-s)\big(f(s, x^n(s)) - f(s, x^o)\big)\, ds \\
&\quad + \int_{T_1}^t S(t-s)\big(\sigma(s, x^n(s)) - \sigma(s, x^o)\big)\, dW_d(s) \\
&\quad + \int_{T_1}^t AS(t-s)\mathcal{R}_c(u^n(s) - u^o(s))ds, \qquad (4.6)
\end{aligned}$$

for all $t \geq T_1$. Clearly, it follows from the assumption $(A1)$ applied to the first term on the righthand side of (4.6) that for all $t \geq T_1$ and $t \in I$,

$$\begin{aligned}
\mathbf{E}|S(t-T_1)(x^n(T_1) - x^o(T_1)|^2_{E_\gamma} &\equiv \mathbf{E}|A^\gamma S(t-T_1)(x^n(T_1) - x^o(T_1)|^2_E \\
&\leq M^2\, \mathbf{E}|x^n(T_1) - x^o(T_1)|^2_{E_\gamma}.
\end{aligned}$$

By virtue of the continuity result for the interval $[0, T_1]$ and the fact that the processes $\{x^n, x^o\}$ have continuous modifications, it follows from the above inequality that as $n \to \infty$,

$$\mathbf{E}|A^\gamma S(t-T_1)(x^n(T_1) - x^o(T_1)|^2_E \quad \to \quad 0$$

for any $t \geq T_1$. Thus for the next interval, we can ignore the first term on the righthand side of (4.6). Then following the same procedure as for the interval $[0, T_1]$, we arrive at the following expression

$$\begin{aligned} &\sup_{T_1 \leq s \leq t} \mathbf{E}|x^n(s) - x^o(s)|^2_{E_\gamma} \\ &\leq 4\eta(t - T_1)(t - T_1) \sup_{T_1 \leq s \leq t} \mathbf{E}|x^n(s) - x^o(s)|^2_{E_\gamma} \\ &\quad + 4\mathbf{E}|\int_{T_1}^{t} A^{1+\gamma-\alpha} S(t-s) A^\alpha \mathcal{R}_c(u^n(s) - u^0(s) ds|^2_E. \end{aligned} \tag{4.7}$$

Clearly, we can choose $t = T_2 > T_1$ such that $4\eta(T_2 - T_1)(T_2 - T_1) < 1$. Then it follows from (4.7) that

$$\begin{aligned} &\big(1 - 4\eta(T_2 - T_1)(T_2 - T_1)\big) \parallel x^n - x^o \parallel_{B^\alpha_\infty([T_1, T_2], E_\gamma)} \\ &\leq \left\{ \frac{(T_2 - T_1)^{2(\alpha-\gamma)-1}}{2(\alpha - \gamma) - 1} \right\} \mathbf{E} \int_{T_1}^{T_2} |A^\alpha \mathcal{R}_c(u^n(s) - u^o(s))|^2_E ds. \end{aligned} \tag{4.8}$$

Thus, again it boils down to the question of convergence of the integral

$$\mathbf{E} \int_{T_1}^{T_2} |A^\alpha \mathcal{R}_c(u^n(s) - u^o(s))|^2_E ds \quad \longrightarrow \quad 0.$$

So it suffices to prove this for the whole interval $I \equiv I_T = [0, T]$. For convenience of reference let us denote this expression by Z_n,

$$Z_n \equiv \mathbf{E} \int_I |A^\alpha \mathcal{R}_c(u^n(s) - u^o(s))|^2_E ds. \tag{4.9}$$

Recall that the map $\mathcal{R}_c$ is the restriction of the map $\mathcal{R}$ to $Y_\beta(\partial\Sigma_c)$ and that

$$\mathcal{R}_c : Y_\beta(\partial\Sigma_c) \longrightarrow X_\beta$$

for any $\beta \in (0, 1]$. For $\alpha, \beta \in [0, 1]$, with $\beta > \alpha$, the operator $A^\alpha : X_\beta \longrightarrow X_{\beta-\alpha}$, and therefore the composition map $A^\alpha \mathcal{R}_c : Y_\beta(\partial\Sigma_c) \longrightarrow X_{\beta-\alpha}$. In the proof of Theorem 3.2 we noted that $X_{\beta-\alpha} \subset E_{\beta-\alpha}$. Since by assumption the embedding $E_{\beta-\alpha} \hookrightarrow E$ is compact, and the topology τ_D is equivalent to weak convergence μ-a.e, we conclude that

$$A^\alpha \mathcal{R}_c(u^n(s) - u^o(s)) \xrightarrow{s} 0 \text{ in } E \ \mu - a.e.$$

Further, since the set U is bounded, and $\parallel A^\alpha \mathcal{R}_c \parallel_{\mathcal{L}(Y_\beta(\partial\Sigma_c), E)} < \infty$, there exists a finite positive number b such that

$$\sup\{|A^\alpha \mathcal{R}_c v|_E, v \in U\} \leq b.$$

Thus by Lebesgue (bounded) convergence theorem we have $\lim_{n\to\infty} Z_n = 0$. This proves the continuity as stated. □

Remark 4.2. The topology used is weak convergence μ almost everywhere. Given the topology on $B^a_\infty(I, E_\gamma)$, it does not seem possible to relax this topology further.

Control Problem: Now we consider the control problem. For the cost or payoff functional, we choose the Bolza problem,

$$J(u) = \mathbf{E}\bigg\{ \int_I \ell(t, x(t), u(t))dt + \Phi(x(T))\bigg\} \longrightarrow \inf, \tag{4.10}$$

where $x \in B^a_\infty(I, E_\gamma)$ is the solution of the integral equation (2.5) (mild solution of the controlled version of system (2.1)) corresponding to the control $u \in \mathcal{U}_{ad}$. The objective is to find a control $u^o \in \mathcal{U}_{ad}$ that minimizes the functional J. The first problem we consider is the question of existence of such controls.

Theorem 4.3. *Consider the system* (2.1)*, equivalently, the integral equation* (2.5) *with the cost functional given by* (4.10)*. Suppose the assumptions of Theorem* 4.1 *hold and the function $\ell : I \times E_\gamma \times U \longrightarrow \overline{R}$ is Borel measurable in all the variables and lower semicontinuous in $(x, u) \in E_\gamma \times U$ with respect to the norm topology on E_γ and weak topology on U for almost all $t \in I$. The function Φ is lower semi continuous on E_γ, and further there exists a finite positive number C such that*

$$|\ell(t, x, \xi)| \le C\{1 + |x|^2_{E_\gamma}\}, \quad |\Phi(x)| \le C(1 + |x|^2_{E_\gamma}), \ \ \forall\ (t, x, \xi) \in I \times E \times U.$$

Then there exists an optimal control for the problem (4.10).

Proof. Since the set of admissible controls $\mathcal{U}_{ad}$ is compact in the τ_D topology, it suffices to prove that $u \longrightarrow J(u)$ is lower semicontinuous in this topology. Let u^α, $\alpha \in D$, be a net that converges in the τ_D topology to $u^o \in \mathcal{U}_{ad}$. Let $\{x^\alpha, x^o\}$ denote the solutions corresponding to the controls $\{u^\alpha, u^o\}$ respectively. Then by Theorem 4.1, $x^\alpha \xrightarrow{s} x^o$ in $B^a_\infty(I, E_\gamma)$ as $u^\alpha \xrightarrow{\tau_D} u^o$. Hence, along a subnet if necessary, $x^\alpha(t) \xrightarrow{s} x^o(t)$ in E_γ almost surely for all $t \in I$. Thus, for almost all $t \in I$, it follows from our assumption on lower semicontinuity of ℓ that

$$\ell(t, x^o(t), u^o(t)) \le \underline{\lim}\, \ell(t, x^\alpha(t), u^\alpha(t)),\ \ \mu\ a.e. \tag{4.11}$$

By our assumption we have $|\ell(t, x^\alpha(t), u^\alpha(t))| \le C\{1 + |x^\alpha(t)|^2_{E_\gamma}\}$ μ *a.e*, and by Corollary 3.3, the solution set is bounded and therefore there exists an $\mathcal{F}_t$-adapted nonnegative integrable process $L(t), t \in I$, so that

$$\sup_{\alpha \in D} \bigg\{ |\ell(t, x^\alpha(t), u^\alpha(t))|, |\ell(t, x^o(t), u^o(t))| \bigg\} \le L(t).$$

Hence, by generalized Fatou's Lemma we conclude that

$$\mathbf{E}\int_I \ell(t, x^o(t), u^o(t))\ dt \quad \le \quad \underline{\lim}\, \mathbf{E}\int_I \ell(t, x^\alpha(t), u^\alpha(t))\ dt. \tag{4.12}$$

Since Φ is also lower semicontinuous on E_γ and by Theorem 4.1, $x^\alpha(T) \xrightarrow{s} x^o(T)$ in E_γ-P-a.s it follows from the growth property of Φ that Fatou's lemma holds and we have

$$\mathbf{E}\Phi(x^o(T)) \quad \le \quad \underline{\lim}\, \mathbf{E}\Phi(x^\alpha(T).$$

Thus we have proved that each component of the functional J given by (4.10) is lower semicontinuous with respect to the τ_D topology and hence J itself is lower semicontinuous in this topology. Since $\mathcal{U}_{ad}$ is compact in this topology, J attains its minimum on $\mathcal{U}_{ad}$. Hence an optimal control exists. This completes the proof. □

5. Control by Borel measures on the boundary

Point controls or Dirac measures as controls are special cases of Borel measures [16]. In Theorem 4.1 we considered controls which are $\mathcal{G}_t$-adapted measurable random processes with values in $U \subset Y_\beta(\Sigma_c)$. Here in this section we want to consider controls which take values in the space of Borel measures $\mathcal{M}(\partial\Sigma_c)$. The possibility of such extension depends on the embedability of $\mathcal{M}(\partial\Sigma_c) \subset Y_\beta(\partial\Sigma_c)$ for some β satisfying $1 \ge \beta > \alpha$. If no such β exists, the extension is not possible because we must satisfy the requirement that $\mathcal{R} \in iso(Y_\beta(\partial\Sigma), X_\beta)$. This is due to Sobolev embedding theorem. Indeed, for each $j \in \{1, 2, \cdots, m\}$ suppose $(m_j + 1/2) - 2\beta m > (n-1)/2$. Then by a Sobolev embedding theorem, $W_2^{(m_j+1/2)-2\beta m}(\partial\Sigma) \hookrightarrow C_j(\partial\Sigma)$ where $C_j(\partial\Sigma)$ denotes the space of continuous bounded functions on $\partial\Sigma$. Let $M_j(\partial\Sigma)$ denote the space of Borel measures on $\partial\Sigma$ representing the dual of $C_j(\partial\Sigma)$. As a corollary of this result we have the following diagram,

$$\begin{array}{lcr} W_2^{(m_j+1/2)-2\beta m}(\partial\Sigma) \hookrightarrow C_j(\partial\Sigma) \\ \downarrow * \qquad\qquad\qquad\qquad\quad \downarrow * \\ W_2^{2\beta m-(m_j+1/2)}(\partial\Sigma) \hookleftarrow M_j(\partial\Sigma), \end{array} \tag{5.1}$$

where $\hookrightarrow$ denotes continuous and dense embedding and the $\downarrow *$ denotes the map that assigns the topological dual of the space behind the arrow. Define the space of vector measures $\mathcal{M}_m(\partial\Sigma_c) \equiv \Pi_{j=1}^m M_j(\partial\Sigma_c)$. Clearly, the predual of this space is given by $C_m(\partial\Sigma_c) = \Pi_{j=1}^m C_j(\partial\Sigma_c)$. It follows from Riesz representation theorem that a continuous linear functional ℓ on $C_m(\partial\Sigma_c)$ is given by

$$\ell(\varphi) = \ell_\mu(\varphi) = \int_{\partial\Sigma_c} \sum_{j=1}^{m} \varphi_j(\xi)\mu_j(d\xi) \equiv \int_{\partial\Sigma_c} < \varphi(\xi), \mu(d\xi) >$$

for some $\mu \in \mathcal{M}_m$. Note that this space contains Dirac measures (or point measures) as special case. For example, let $\{\zeta_r, r = 1, 2 \cdots k\}$ be a set of distinct points in $\partial\Sigma_c$ and define the measure $\mu_j(d\xi) \equiv \sum_{r=1}^{k} \varpi_r^j \, \delta_{\zeta_r}(d\xi)$ where δ_{ζ_r} is the Dirac measure concentrated at ζ_r, and $\varpi_r^j \in R$. Then

$$\ell_\mu(\varphi) = \sum_{r=1}^{k} \sum_{j=1}^{m} \varpi_r^j \varphi_j(\zeta_r), \quad \text{for } \varphi \in C_m(\partial\Sigma_c).$$

Returning to control problem, we choose any (weak star) closed bounded (bounded in variation norm) subset Γ of $\mathcal{M}_m(\partial\Sigma)$. The boundary controls are then the weak star measurable $\mathcal{G}_t$-adapted random processes defined on I and taking values in the space of vector measures $\Gamma \subset \mathcal{M}_m(\partial\Sigma)$. By Alaoglu's theorem, Γ is weak star compact and so also is the set $L_\infty^a(I, \Gamma) \equiv \mathcal{U}_{ad}$ chosen as the set of admissible controls. Since $\partial\Sigma_c$ is a closed bounded subset of $\partial\Sigma$, and $C_m(\partial\Sigma_c)$ is a separable Banach space, again we can introduce a metric topology on $L_\infty^a(I, \Gamma)$ precisely as in section 4. Here we use Theorem V.1.1 [12, p.426] and define the metric on Γ by

$$d(u, v) \equiv \sum_{n=1}^{\infty} (1/2^n) \frac{|u(\varphi_n) - v(\varphi_n)|}{1 + |u(\varphi_n) - v(\varphi_n)|} \quad \text{for } u, v \in \Gamma, \tag{5.2}$$

where $u(\varphi) \equiv \int_{\partial\Sigma_c} \varphi(\eta) u(d\eta)$ and $\{\varphi_n\}$ is any dense sequence from the unit ball of $C_m(\partial\Sigma_c)$. Using this metric we define the metric D_* on $\mathcal{U}_{ad}$ precisely as in section 4 giving

$$D_*(u, v) \equiv \mu\{(t, \omega) \in I \times \Omega : d(u(t, \omega), v(t, \omega)) \neq 0\}, \tag{5.3}$$

where the measure μ is the restriction of the product measure $dt \times dP$ on the sigma algebra of predictable sets $\mathcal{P}$ of the set $I \times \Omega$ with respect to the current of sigma algebras $\mathcal{G}_t, t \geq 0$. We denote this metric topology by τ_{D_*} In this case, by convergence of the net $u^\alpha \xrightarrow{\tau_{D_*}} u^o$ we mean weak star convergence μ a.e. Instead of repeating the proofs, we simply state that Theorem 4.1, asserting continuity of the control to solution map $u \longrightarrow x$, remains valid with τ_D replaced by τ_{D_*}. Similarly, Theorem 4.3, asserting the existence of an optimal control, remains valid with the metric topology τ_{D_*} replacing τ_D.

Remark 5.1. In order to compute the optimal controls one needs necessary conditions of optimality, see [1,2,6,7]. We leave this problem for future work.

5.1. **Two Examples:**

(E1) Euler Plate Equation: This example is intended to illustrate the possibility of using Borel measures (including point measures) as controls on the boundary. Consider the stochastically perturbed Euler plate equation in a bounded domain $\Sigma \subset R^2$ as follows:

$$\begin{aligned} & y_{tt} + \Delta^2 y - \rho \Delta y_t \\ & = F(t,\xi,y,\Delta y,y_t) + G(t,\xi,y)N \quad \text{for} \ \ (t,\xi) \in I \times \Sigma \end{aligned} \tag{5.4}$$

with nonhomogeneous boundary conditions given by

$$\begin{aligned} \mathcal{B}_1(y,y_t)|_{\partial\Sigma} &\equiv \sum_{|\theta|\le 3} \beta_{1,\theta}(\xi) D^\theta y + \sum_{|\theta|\le 1} \gamma_{1,\theta}(\xi) D^\theta y_t \\ &= u_1, (t,\xi) \in I \times \partial\Sigma, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \mathcal{B}_2(y,y_t)|_{\partial\Sigma} &\equiv \sum_{|\theta|\le 3} \beta_{2,\theta}(\xi) D^\theta y + \sum_{|\theta|\le 1} \gamma_{2,\theta}(\xi) D^\theta y_t \\ &= u_2, (t,\xi) \in I \times \partial\Sigma \end{aligned} \tag{5.6}$$

and initial conditions given by

$$y(0,\xi) = y_1(\xi), \quad y_t(0,\xi) = y_2(\xi), \quad \xi \in \Sigma. \tag{5.7}$$

Defining $X \equiv (y, y_t)'$ we can rewrite the above equation as a first order equation as follows

$$\partial X/\partial t + \mathcal{A}X = f(X) + g(X)N \tag{5.8}$$

where $\mathcal{A} \equiv \begin{pmatrix} 0 & -1 \\ \Delta^2 & -\rho\Delta \end{pmatrix}$, $f(X) = \begin{pmatrix} 0 \\ F(X_1,X_2) \end{pmatrix}$ and $g(X) = \begin{pmatrix} 0 \\ G(X_1,X_2) \end{pmatrix}$. Using the above notation the boundary conditions are given by

$$\begin{aligned} \mathcal{B}_j(X)|_{\partial\Sigma} &= \sum_{|\theta|\le m_j} \beta_{j,\theta}(\xi) D^\theta X_1 + \sum_{|\theta|\le 1} \gamma_{j,\theta}(\xi) D^\theta X_2 \\ &= u_j, \quad (t,\xi) \in I \times \partial\Sigma \end{aligned} \tag{5.9}$$

for $j = 1, 2$. We choose the energy space for the state space. This is given by the Hilbert space

$$E \equiv H^2(\Sigma) \cap H_0^1(\Sigma) \times L_2(\Sigma)$$

equipped with the scalar product $(X,Y)_E \equiv (\Delta X_1, \Delta Y_1) + (X_2, Y_2)_{L_2(\Sigma)}$. Here the first term represents the elastic potential energy and the second the kinetic energy. Because of the particular structure of the nonlinear operators containing spatial derivatives of order no more than 2 and the time derivative of order no more than 1, we can take $\gamma = 0$ and take $E_\gamma = E_0 = E$ as the state space. Now we proceed to formulate this as an ordinary differential equation on the Hilbert space E. Denote $x(t) \equiv X(t, \cdot) \in E$ and define the operator A by

$$D(A) \equiv \{X \in E : \mathcal{A}X \in L_2(\Sigma) \ \& \ \mathcal{B}(X) = 0\}$$

and set $A\varphi = \mathcal{A}\varphi$ for $\varphi \in D(A)$. Similarly, define B by $B \equiv \mathcal{B}|_{Ker\mathcal{A}}$ with $Ker(\mathcal{A}) = \{X \in E : \mathcal{A}X = 0\}$. Considering N as the distributional derivative of $L_2(\Sigma)$-valued Wiener process on a complete probability space we can rewrite equation (5.8) as a stochastic differential equation on E

$$dx + Axdt = f(x)dt + g(x)dW, \tag{5.10}$$

subject to the boundary condition $B(x) = u$. Using the well known decomposition given by the expression 3.4 [4, p.60], the solution x admits a decomposition given by $x = x_1 + x_2$ with $x_1 \in Ker\mathcal{B}$ and $x_2 \in Ker(\mathcal{A})$. Using this fact, we can rewrite equation (5.10) and the Boundary condition $Bx = Bx_2 = u$ in one single equation as follows:

$$\begin{aligned} dx_1 + Ax_1dt &= -dx_2 + f(x)dt + g(x)dW, \\ &= -d(\mathcal{R}u) + f(x)dt + g(x)dW, \end{aligned}$$

where $\mathcal{R} = (\mathcal{B}|_{Ker\mathcal{A}})^{-1} = B^{-1}$ is the Dirichlet map solving the elliptic problem: $\mathcal{A}X = 0, \mathcal{B}(X) = u$ giving $X = \mathcal{R}u$. It follows from a result of Chen and Triggiani [8, Proposition 3.1, p24] that, due to the presence of structural damping provided by the term $\rho\Delta$, the operator A generates a stable analytic semigroup $S(t), t \geq 0$, in E. Hence the (mild) solution of this equation is given by the solution of the integral equation

$$\begin{aligned} x(t) &= S(t)x_0 + \int_0^t S(t-s)f(x(s))ds + \int_0^t S(t-s)g(x(s))dW(s) \\ &\quad + \int_0^t AS(t-s)\mathcal{R}u_s ds, \quad t \in I. \end{aligned}$$

For point controls on the boundary (the edges), it is necessary that the diagram (5.1) holds. Let us verify this. For plate equation we have $n = 2$, $m = 2$, and (from the boundary conditions) $m_j = m_1 = m_2 = 3$, and $dim(\partial\Sigma) = 1$. Thus for $\beta < 3/4$, for example, $\beta = 0.7$, we have $m_j + (1/2) - 2m\beta > (n-1)/2$ is satisfied. So by Sobolev embedding theorem we have $W_2^{(m_j+1/2)-2\beta m}(\partial\Sigma) \hookrightarrow C_j(\partial\Sigma)$ and therefore the embedding diagram (5.1) holds. Thus Borel measures supported on $\partial\Sigma$, including linear combination of Dirac measures (point measures) on it, are admissible as controls. In this case the cost functional

(4.10) is given by

$$J(u) = \mathbf{E}\Big\{ \int_{I\times\partial\Sigma} \ell(t,x(t),\xi)u_t(d\xi)dt + \Phi(x(T))\Big\}$$
$$\equiv \mathbf{E}\Big\{ \int_{I} \ell(t,x(t),u_t)dt + \Phi(x(T))\Big\} \quad (5.11)$$

with $u \in \mathcal{U}_{ad} \equiv L^a_\infty(I,\Gamma)$, where $\Gamma \subset \mathcal{M}_2(\partial\Sigma)$. The proof of existence of optimal controls in this case is identical to that of Theorem 4.3.

(E2) Kuramoto-Sivashinsky Equation: In the preceding example we did not need the full power of Theorem 3.2 since both f and g are bounded nonlinear operators in the energy space which is the state space. On the contrary, in this example as seen later, the drift is a bounded nonlinear operator from a smaller space to a larger one. Therefore we need the full power of Theorem 3.2. Here we wish to consider control problems for the well known Kuramoto-Sivashinsky like equation in one dimension given by

$$\partial_t v + a\Delta^2 v + b\Delta v = h(v,Dv) + g(\xi)n_o(t), \quad \xi \in \Sigma \equiv (0,1), \quad (5.12)$$
$$\mathcal{B}v = u \quad \text{for } \xi \in \partial\Sigma = \{0,1\} \quad (5.13)$$

with the boundary operator $\mathcal{B}$ given by

$$(\mathcal{B}_1 v)(\xi) \equiv \beta_0 v(\xi), \ (\mathcal{B}_2 v)(\xi) \equiv \beta_1 v(\xi) + \beta_2 D_\nu v(\xi) \quad (5.14)$$

for $\xi \in \partial\Sigma$ with $\beta_0, \beta_1, \beta_3 \neq 0$. Here we have used D^k to denote the spatial derivative of order k and D_ν the derivative at the point $\xi \in \partial\Sigma$ along the normal pointing outward of the boundary. The control $u(t,\xi) = (u_1(t,\xi), u_2(t,\xi))'$ for $\xi \in \partial\Sigma$. The coefficients $\{a,b\}$ are real positive and those of the boundary operator $\mathcal{B}$ are assumed to be nonzero. The function $h : R^2 \longrightarrow R$ is continuous with respect to its arguments, $g \in L_2(\Sigma)$ and n_0 is the standard white noise. Note that for $a > 0$ the operator $a\Delta^2$ is dissipative while for $b > 0$ the operator $b\Delta$ is accretive or anti-dissipative. Define the differential operator $\mathcal{A}$ by $\mathcal{A}\varphi = a\Delta^2\varphi + b\Delta\varphi$. Then define the operator A by setting

$$D(A) = \{\varphi \in E : \mathcal{A}\varphi \in E \ \& \ \varphi|_{\partial\Sigma} = D\varphi|_{\partial\Sigma} = 0\} = H^4 \cap H^2_0.$$

We show that under these assumptions the operator $-A$ generates an analytic semigroup on the Hilbert space $E \equiv L_2(\Sigma)$. Indeed, by simple integration by parts one can easily verify that

$$(A\varphi,\varphi) + (b/2\varepsilon)|\varphi|^2_E \geq (a - b\varepsilon/2)|\Delta\varphi|^2_E \quad (5.15)$$

for all ε satisfying $0 < \varepsilon < 2a/b$ and all $\varphi \in D(A)$. Choosing $\varepsilon = a/b$ we obtain

$$(A\varphi,\varphi) + (b^2/2a)|\varphi|^2_E \geq (a/2)|\Delta\varphi|^2_E. \quad (5.16)$$

For every $\varphi \in D(A)$ it follows form elementary computation (or Poincaré inequality) that there exists a positive constant c such that $|\varphi|_E \leq c|\Delta\varphi|_E$. From the above inequalities we obtain the following resolvent inequality

$$|(\lambda I + A)^{-1}|_{\mathcal{L}(E)} \leq \frac{1}{\lambda + r_0}, \quad \forall\, \lambda > -r_0 \tag{5.17}$$

where $r_0 = (a^2 - b^2c^2)/2ac^2$. Note that the destabilizing influence of the anti-dissipative term is very well reflected in the resolvent inequality. From now on we use the same symbol A to denote the closed extension of A in E as an unbounded operator. One can easily verify that the operator A is self adjoint on the Hilbert space E but not positive. It is clear from the inequality (5.16) or (5.17) that for $\beta > ((b^2c^2 - a^2)/2ac^2)$, the operator $A_\beta \equiv (\beta I + A)$ is an unbounded positive self adjoint operator in E. Then it follows from (5.17) that the resolvent of the operator A_β satisfies the inequality $|(\lambda I + A_\beta)^{-1}|_{\mathcal{L}(E)} \leq 1/\lambda$, for $\lambda > 0$. Since A_β is closed and densely defined it follows from Hille-Yosida theorem that $-A_\beta$ generates a C_0-semigroup $S_\beta(t), t \geq 0$, of contractions on E. With this modification, we can rewrite the system (5.12) as an ordinary differential equation on the Hilbert space E in the abstract form

$$(d/dt)v + A_\beta v = f(v) + \dot{W} \tag{5.18}$$

where $f(v) = \beta v + h(v, Dv)$ and $\dot{W} \equiv gn_0$ is the space time white noise. Let C denote the field of complex numbers. Then, for $\lambda \in C$ given by $\lambda = \nu + i\tau$ with $\nu > 0$, one can easily verify that

$$|((\lambda I + A_\beta)\varphi, \varphi)| \geq |\tau||\varphi|_E^2$$

and hence $|(\lambda I + A_\beta)\varphi| \geq |\tau||\varphi|_E$ for all $\varphi \in D(A) = D(A_\beta)$. From this we obtain

$$|(\lambda I + A_\beta)^{-1}|_{\mathcal{L}(E)} \leq 1/|\tau|$$

for all $Re\lambda > 0$ and $\tau \neq 0$. Thus it follows from Hille's characterization of analytic semigroups [5, Theorem 3.2.7, p.82] see also [Pazy, 17, Theorem 5.2, p.61] that $-A_\beta$ generates an analytic semigroup $S_\beta(t), t \geq 0$ in E. As a result, $-A$ it self generates an analytic semigroup $S(t) = S_\beta(t)e^{\beta t}$. Then the mild solution of equation (5.18) with homogeneous boundary condition $Bv = 0$ is given by the solution of the integral equation

$$v(t) = S_\beta(t)v_0 + \int_0^t S_\beta(t-s)f(v(s))ds + \int_0^t S_\beta(t-s)dW \tag{5.19}$$

in the Hilbert space $E_\gamma \equiv D(A^\gamma)$ for a suitable $\gamma \in [0,1]$ that admits f containing first partial of the state. Since A_β is positive self adjoint with compact resolvent, it follows from spectral theory that fractional powers of A_β are well defined and that $A_\beta^{1/4}$ is equivalent to the operator given by the spatial derivative D. Thus, for this example, $\gamma = (1/4)$ is suitable. Suppose f satisfies the following growth and Lipschitz conditions

$$(F1) : |f(v)|_E^2 \leq K^2(1 + |v|_E^2 + |Dv|_E^2),$$
$$(F2) : |f(v) - f(w)|_E^2 \leq K^2(|v - w|_E^2 + |Dv - Dw|_E^2)$$

for some $K \neq 0$. Using the continuous embedding $E_\gamma \hookrightarrow E$, with embedding constant $\tilde{c}$, it is easy to verify that the above inequalities are equivalent to the following ones

$$(F1) : |f(v)|_E^2 \leq C^2(1 + |v|_{E_\gamma}^2),$$
$$(F2) : |f(v) - f(w)|_E^2 \leq C^2(|v - w|_{E_\gamma}^2),$$

where $C^2 = K^2(1 + \tilde{c}^2)$. Considering the stochastic term

$$Z(t) = \int_0^t S_\beta(t - s)dW(s),$$

it is easy to verify that

$$\mathbf{E}|A^\gamma Z(t)|_E^2 \equiv \mathbf{E}|Z(t)|_{E_\gamma}^2 \leq |g|_E^2 \int_0^t |A^\gamma S_\beta(t - s)|_{\mathcal{L}(E)}^2 ds \leq 2C_\gamma^2 |g|_E^2 t^{1/2}.$$

Hence $Z \in B_\infty^a(I, E_\gamma)$. Thus, by Theorem 3.2 of section 3, the integral equation (5.19) has a unique solution in $B_\infty^a(I, E_\gamma)$. Hence the differential equation (5.12) with $\mathcal{B}v = 0$ has a mild solution.

(E2-A) Point Controls: For this (homogeneous) boundary problem, we may consider distributed controls, in particular, point controls or more generally signed Borel measures as controls. Let Σ_0 be a closed subset of Σ and consider the space of Borel measures $\mathcal{M}(\Sigma_0)$. Let $F \in C(\Sigma_0, E)$ be an E-valued continuous function satisfying $\sup\{|F(\xi)|_E, \xi \in \Sigma_0\} < \infty$. Consider the controlled version of equation (5.18) in the form

$$(d/dt)v + A_\beta v = f(v) + Fu_s + \dot{W}, \quad v(0) = v_0 \tag{5.20}$$

where $Fu \equiv \int_{\Sigma_0} F(\xi)u(d\xi)$ for $u \in \mathcal{M}(\Sigma_0)$. The corresponding integral equation is given by

$$\begin{aligned} v(t) &= S_\beta(t)v_0 + \int_0^t S_\beta(t - s)f(v(s))ds \\ &\quad + \int_0^t S_\beta(t - s)Fu_s ds + \int_0^t S_\beta(t - s)dW. \end{aligned} \tag{5.21}$$

For admissible controls, one may choose a bounded (in variation norm) $w*$-closed subset $M_0 \subset \mathcal{M}(\Sigma_0)$ and take $\mathcal{U}_{ad} \equiv L_\infty^a(I, M_0) \subset L_\infty^a(I, \mathcal{M}(\Sigma_0))$. Note that by $L_\infty^a(I, M_0)$ we mean the class of M_0-valued stochastic processes adapted in the weak star sense to the family of sub-sigma algebras $\mathcal{G}_t, t \geq 0$, of the current of sigma algebras $\mathcal{F}_t, t \geq 0$. For any $u \in \mathcal{U}_{ad}$, again it follows from Theorem 3.2 that equation (5.21) has a unique solution in $B^a_\infty(I, E_\gamma)$ and therefore the evolution equation (5.20) has a unique mild solution. The pay-off functional for this control problem is a modified version of the expression (4.10) given by

$$\begin{aligned} J(u) &= \mathbf{E}\Big\{\int_I \ell(t, x(t), u_t)dt + \Phi(x(T))\Big\} \\ &= \mathbf{E}\Big\{\int_{I\times\Sigma_0} \ell(t, x(t), \xi)u_t(d\xi)dt + \Phi(x(T))\Big\}. \end{aligned} \tag{5.22}$$

As in Theorem 4.1, the map $u \longrightarrow x$ is continuous with respect to a similar metric topology $D_\rho(\mu, \nu)$ determined by the metric ρ given by

$$\rho(\mu,\nu) \equiv \sum_{n=1}^{\infty}(1/2^n)\frac{|\mu(\varphi_n) - \nu(\varphi_n)|}{1 + |\mu(\varphi_n) - \nu(\varphi_n)|}, \quad \text{for } \mu, \nu \in \mathcal{M}(\Sigma_0) \tag{5.23}$$

where $\{\varphi_n\}$ is a dense subset of the unit ball in $C(\Sigma_0)$. The fact that ρ is a metric follows from Theorem V.1.1 [12, p.426] of Dunford and Schwartz. Under the same assumptions (see Theorem 4.3), with weak topology replaced by weak star topology, existence of an optimal control for the problem (5.22) follows from Theorem 4.3.

(E2-B): Since $\Sigma = (0, 1) \subset R^1$, and, for $\gamma = 1/4$, $E_\gamma = H^1$ and $H^1 \hookrightarrow C(\Sigma)$, the boundary controls are point controls and so it is covered in **(E2-A)**.

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