Nonlinear Functional Analysis and Applications
Vol. 20, No. 1 (2015), pp. 1-25

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

ON A NONLINEAR LOVE'S EQUATION WITH MIXED NONHOMOGENEOUS CONDITIONS

Nguyen Tuan Duy[1], Le Thi Phuong Ngoc[2] and Nguyen Anh Triet[3]

[1]Department of Fundamental Sciences, University of Finance and Marketing
306 Nguyen Trong Tuyen Str., Dist. Tan Binh, Ho Chi Minh City, Vietnam

[1]Department of Mathematics and Computer Science, University of Natural Science
Vietnam National University Ho Chi Minh City
227 Nguyen Van Cu Str., Dist. 5, Ho Chi Minh City, Vietnam
e-mail: tuanduy2312@gmail.com

[2]Nha Trang Educational College
01 Nguyen Chanh Str., Nha Trang City, Vietnam
e-mail: ngoc1966@gmail.com

[3]Department of Mathematics, University of Architecture of HoChiMinh City
196 Pasteur Str., Dist.3, HoChiMinh City, Vietnam
e-mail: anhtriet1@gmail.com

Abstract. The paper is devoted to the study of a Love's equation with mixed nonhomogeneous conditions. Existence of a weak solution is proved by using Faedo-Galerkin method. Uniqueness, regularity and asymptotic behavior of solutions are also discussed.

1. Introduction

In this paper, we consider the following Love's equation with initial conditions and mixed nonhomogeneous conditions

$$\begin{gathered} u_{tt} - u_{xx} - \varepsilon u_{xxtt} + \lambda \left| u_t \right|^{q-2} u_t + K \left| u \right|^{p-2} u = F(x,t), \\ x \in \Omega = (0,1), \quad 0 < t < T, \end{gathered} \tag{1.1}$$

$$\varepsilon u_{xtt}(0,t) + u_x(0,t) = h_0 u(0,t) + g_0(t), \tag{1.2}$$

$$-\varepsilon u_{xtt}(1,t) - u_x(1,t) = h_1 u(1,t) + g_1(t), \tag{1.3}$$

[0]Received April 08, 2014. Revised February 17, 2015.
[0]2010 Mathematics Subject Classification: 35L05, 35L15, 35L70, 37B25.
[0]Keywords: Faedo-Galerkin method, global existence, nonlinear Love's equation, mixed nonhomogeneous conditions.

$$u(x,0)=\tilde{u}_0(x),\ u_t(x,0)=\tilde{u}_1(x), \tag{1.4}$$

where $\varepsilon>0$, $p>1$, $q>1$, $\lambda>0$, K, h_0, $h_1\geq 0$ are constants and $\tilde{u}_0$, $\tilde{u}_1$, F, g_0, g_1 are given functions satisfying conditions specified later.

Equation (1.1) describes vertical oscillations of a nonlinear viscous elastic bar, in which the nonlinear term $\tilde{F}(x,t)=F(x,t)-\lambda\left|u_t\right|^{q-2}u_t-K\left|u\right|^{p-2}u$ contains the external forces acting on the bar. These external forces depend on the displacement u and velocity of movement u_t. The conditions (1.2), (1.3) describe the elastic binding at the two ends of the bar.

When $F=0$, $\lambda=K=0$, $\Omega=(0,L)$, Eq.(1.1) is related to the Love's equation

$$u_{tt}-\tfrac{E}{\rho}u_{xx}-2\mu^2k^2u_{xxtt}=0, \tag{1.5}$$

presented by V. Radochová in 1978 (see [9]). This equation describes vertical oscillations of a rod, which was established from Euler's variational equation of an energy function

$$\int_0^T dt\int_0^L\left[\tfrac{1}{2}F\rho\left(u_t^2+\mu^2k^2u_{tx}^2\right)-\tfrac{1}{2}F\left(Eu_x^2+\rho\mu^2k^2u_xu_{xtt}\right)\right]dx, \tag{1.6}$$

the parameters in (1.6) have the following meanings: u is the displacement, L is the length of the rod, F is the area of cross-section, k is the cross-section radius, E is the Young modulus of the material and ρ is the mass density. By using the Fourier method, Radochová [9] obtained a classical solution of problem (1.5) associated with initial conditions (1.4) and boundary conditions

$$u(0,t)=u(L,t)=0, \tag{1.7a}$$

or

$$\begin{cases} u(0,t)=0, \\ \varepsilon u_{xtt}(L,t)+c^2u_x(L,t)=0, \end{cases} \tag{1.7b}$$

where $c^2=\frac{E}{\rho}$, $\varepsilon=2\mu^2k^2$. On the other hand, the asymptotic behaviour of solutions of problems (1.4), (1.5), (1.7) as $\varepsilon\to 0_+$ are also established by the method of small parameter.

Equations of Love waves or equations for waves of Love types have been studied by many authors, we refer to [3], [4], [8] and references therein.

In view of Mathematics, problem (1.1) with high derivative appearing in equation, which is compatible with the boundary conditions (1.2), (1.3), will usually make solutions of problem being more smooth (it means that the solution belongs to a function space narrower) than other problems without higher derivative terms.

Thus, when we consider a perturbed problem with small parameter $\varepsilon>0$, the limit of a solution as $\varepsilon\to 0_+$ in some sense, if it exists, will belong to a function space wider than the space containing the solution of the perturbed problem.

The paper consists of four sections. Using the Faedo-Galerkin method, compactness method and monotone method generated by the nonlinear component $|u_t|^{q-2}u_t$, Section 2 is devoted to the study of the existence a weak solution for problems (1.1)-(1.4) with $(\tilde{u}_0, \tilde{u}_1) \in H^1 \times H^1$, $p > 1, q > 1$. Here, a energy lemma(Lemma 2.4) is also established in order to pass the limit of a approximate problem and prove the uniqueness in case $p \geq 2$. In section 3, we consider the regularity of solution for problems (1.1)-(1.4) with $(\tilde{u}_0, \tilde{u}_1) \in H^2 \times H^2$, $p \geq 2, q \geq 2$ and some other conditions. In case $p = q = 2$, we show that the regularity of solutions depending on the regularity of data. Finally, the asymptotic behavior of solutions as $\varepsilon \to 0_+$ is discussed in Section 4. The results obtained here may be considered as the generalizations of those in [9].

2. Existence and uniqueness of a weak solution

First, we put $\Omega = (0,1)$; $Q_T = \Omega \times (0,T)$, $T > 0$ and we denote the usual function spaces used in this paper by the notations $C^m\left(\overline{\Omega}\right)$, $W^{m,p} = W^{m,p}(\Omega)$, $L^p = W^{0,p}(\Omega)$, $H^m = W^{m,2}(\Omega)$, $1 \leq p \leq \infty$, $m = 0, 1, \ldots$. Let $\langle \cdot, \cdot \rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2 and we denote by $\|\cdot\|_X$ the norm in the Banach space X. We call X' the dual space of X. We denote by $L^p(0,T;X)$, $1 \leq p \leq \infty$ for the Banach space of the real functions $u : (0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p\, dt\right)^{1/p} < \infty \quad \text{for } 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup} \|u(t)\|_X \quad \text{for } p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t)$, $u''(t) = u_{tt}(t)$, $u_x(t)$, $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

On H^1 we shall use the following norms

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}, \quad \|v\|_i = \left(v^2(i) + \|v_x\|^2\right)^{1/2}, \quad i = 0, 1.$$

Then the following lemma is known.

Lemma 2.1. *The imbedding $H^1 \hookrightarrow C^0([0,1])$ is compact and*

$$\begin{cases} \|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\, \|v\|_{H^1}, \quad \forall v \in H^1, \\ \|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\, \|v\|_i, \quad \forall v \in H^1, i = 0, 1. \end{cases} \tag{2.1}$$

We remark that, there norms $\|v\|_0$, $\|v\|_1$, $\|v\|_{H^1}$ are equivalent on H^1 and

$$\frac{1}{\sqrt{3}}\|v\|_i \leq \|v\|_{H^1} \leq \sqrt{3}\|v\|_i, \text{ for all } v \in H^1, \ i = 0, 1. \tag{2.2}$$

It is also easy to prove the result as below.

Lemma 2.2. *Let h_0, $h_1 \geq 0$, $h_0 + h_1 > 0$. Then there exists a constant $\alpha_0 > 0$ depending only on h_0, h_1 such that*

$$\|v_x\|^2 + h_0 v^2(0) + h_1 v^2(1) \geq \alpha_0 \|v\|_{H^1}^2, \textit{ for all } v \in H^1. \tag{2.3}$$

Remark 2.3. The weak formulation of the initial-boundary value problem (1.1)-(1.4) can be given in the following manner: Find $u \in L^\infty(0,T;H^1)$, with $u_t \in L^\infty(0,T;H^1)$, such that u satisfies the following variational equation

$$\begin{cases} \frac{d}{dt}\left[\langle u_t(t), w\rangle + \varepsilon\langle u_{xt}(t), w_x\rangle\right] + \langle u_x(t), w_x\rangle \\ + \sum_{i=0}^{1}\left(h_i u(i,t) + g_i(t)\right) w(i) \\ +\lambda\langle |u_t|^{q-2} u_t, w\rangle + K\langle |u|^{p-2} u, w\rangle = \langle F(t), w\rangle, \end{cases} \tag{2.4}$$

for all $w \in H^1$, a.e., $t \in (0,T)$, together with the initial conditions

$$u(0) = \tilde{u}_0, \ u_t(0) = \tilde{u}_1. \tag{2.5}$$

We need the following assumptions:

(H_1) $p > 1$, $q > 1$, $\lambda > 0$, $K > 0$, $\varepsilon > 0$;

(H_2) $h_0, h_1 \geq 0, h_0 + h_1 > 0$;

(H_3) $(\tilde{u}_0, \tilde{u}_1) \in H^1 \times H^1$;

(H_4) $F \in L^1(0,T;L^2)$;

(H_5) $g_i \in W^{1,1}(0,T)$, $i = 0, 1$.

Then, we have the following theorem.

Theorem 2.4. *Let $T > 0$. Suppose that (H_1)-(H_5) hold. Then, there exists a weak solution u of problems* (1.1)-(1.4) *such that*

$$u \in L^\infty\left(0,T;H^1\right), \ u_t \in L^\infty\left(0,T;H^1\right). \tag{2.6}$$

Furthermore, if $p \geq 2$, then the solution is unique.

Proof. The proof consits of four steps.

Step 1. *The Faedo-Galerkin approximation.*

Let $\{w_j\}$ be a denumerable base of H^1. We find the approximate solution of Prob.(1.1)-(1.4) in the form

$$u_m(t) = \sum_{j=1}^{m} c_{mj}(t) w_j, \tag{2.7}$$

where the coefficient functions c_{mj} satisfy the following system of ordinary differential equations

$$\begin{cases} \langle u''_m(t), w_j\rangle + \langle u_{mx}(t) + \varepsilon u''_{mx}(t), w_{jx}\rangle + \lambda \left\langle |u'_m(t)|^{q-2} u'_m(t), w_j \right\rangle \\ \quad +K \left\langle |u_m(t)|^{p-2} u_m(t), w_j \right\rangle + \sum_{i=0}^{1} \left(h_i u_m(i,t) + g_i(t)\right) w_j(i) \\ = \langle F(t), w_j\rangle, \quad 1 \le j \le m, \\ u_m(0) = \tilde{u}_{0m},\ u'_m(0) = \tilde{u}_{1m}, \end{cases} \tag{2.8}$$

where

$$\begin{cases} \tilde{u}_{0m} = \sum_{j=1}^{m} \alpha_{mj} w_j \to \tilde{u}_0 \quad \text{strongly in } H^1, \\ \tilde{u}_{1m} = \sum_{j=1}^{m} \beta_{mj} w_j \to \tilde{u}_1 \quad \text{strongly in } H^1. \end{cases} \tag{2.9}$$

From the assumptions of Theorem 2.3, system (2.8) has a solution u_m on an interval $[0, T_m] \subset [0, T]$. The following estimates allow one to take $T_m = T$ for all m (see [2]).

Step 2. *A priori estimates.*

Multiplying the j^{th} equation of (2.8) by $c'_{mj}(t)$ and summing up with respect to j, afterwards, integrating by parts with respect to the time variable from 0 to t, after some rearrangements, we get

$$\begin{aligned} S_m(t) &= S_m(0) + 2\sum_{i=0}^{1} g_i(0)\tilde{u}_{0m}(i) + 2\int_0^t \langle F(s), u'_m(s)\rangle\, ds \\ &\quad +2\sum_{i=0}^{1} \int_0^t g'_i(s) u_m(i,s) ds - 2\sum_{i=0}^{1} g_i(t) u_m(i,t) \\ &= S_m(0) + 2\sum_{i=0}^{1} g_i(0)\tilde{u}_{0m}(i) + \sum_{j=0}^{3} I_j, \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} S_m(t) &= \|u'_m(t)\|^2 + \|u_{mx}(t)\|^2 + \varepsilon \|u'_{mx}(t)\|^2 + \sum_{i=0}^{1} h_i u_m^2(i,t) \\ &\quad + \frac{2K}{p} \|u_m(t)\|_{L^p}^p + 2\lambda \int_0^t \|u'_m(s)\|_{L^q}^q\, ds. \end{aligned} \tag{2.11}$$

By (2.9), (2.11) and the imbedding $H^1 \hookrightarrow C^0\left(\overline{\Omega}\right)$, there exists a positive constant $\bar{C}_0$ depending only on $\tilde{u}_0$, $\tilde{u}_1$, h_0, h_1, K, p, $g_0(0)$, $g_1(0)$ and ε, such that

$$\begin{aligned} & S_m(0) + 2\sum_{i=0}^{1} g_i(0)\tilde{u}_{0m}(i) + \frac{2K}{p} \|\tilde{u}_{0m}\|_{L^p}^p \\ &= \|\tilde{u}_{1m}\|^2 + \|\tilde{u}_{0mx}\|^2 + \varepsilon \|\tilde{u}_{1mx}\|^2 + \sum_{i=0}^{1} h_i \tilde{u}_{0m}^2(i) \\ &\quad +2\sum_{i=0}^{1} g_i(0)\tilde{u}_{0m}(i) + \frac{2K}{p} \|\tilde{u}_{0m}\|_{L^p}^p \\ &\le \frac{1}{2}\bar{C}_0, \quad \text{for all } m. \end{aligned} \tag{2.12}$$

Using (2.3) and the following inequalities

$$2ab \leq \beta a^2 + \tfrac{1}{\beta} b^2, \text{ for all } a,\ b \in \mathbb{R},\ \beta > 0, \tag{2.13}$$

$$\begin{aligned} S_m(t) \geq \left\| u_m'(t) \right\|^2 + \alpha_0 \left\| u_m(t) \right\|_{H^1}^2 + \varepsilon \left\| u_{mx}'(t) \right\|^2 \\ + \tfrac{2K}{p} \left\| u_m(t) \right\|_{L^p}^p + 2\lambda \textstyle\int_0^t \left\| u_m'(s) \right\|_{L^q}^q ds, \end{aligned} \tag{2.14}$$

and note that

$$|u_m(i,t)| \leq \|u_m(t)\|_{C^0(\overline{\Omega})} \leq \sqrt{2} \left\| u_m(t) \right\|_{H^1} \leq \sqrt{\tfrac{2}{\alpha_0}} \sqrt{S_m(t)},\ i = 0,\ 1, \tag{2.15}$$

we can estimate all terms in the righthand side of (2.10) as follows.

$$\begin{aligned} I_1 &= 2 \textstyle\int_0^t \left\langle F(s), u_m'(s) \right\rangle ds \\ &\leq \textstyle\int_0^t \|F(s)\| \, ds + \int_0^t \|F(s)\| \left\| u_m'(s) \right\|^2 ds \\ &\leq C_T + \textstyle\int_0^t \|F(s)\| \, S_m(s) \, ds, \end{aligned} \tag{2.16}$$

where C_T is a bound depending on T. For short, in what follows, C_T always is a constant with the same meaning.

$$\begin{aligned} I_2 &= 2 \textstyle\sum_{i=0}^1 \int_0^t g_i'(s) u_m(i,s) ds \leq 2\sqrt{\tfrac{2}{\alpha_0}} \int_0^t \sum_{i=0}^1 |g_i'(s)| \sqrt{S_m(s)} ds \\ &\leq \tfrac{2}{\alpha_0} \textstyle\int_0^t \sum_{i=0}^1 |g_i'(s)| \, ds + \int_0^t \sum_{i=0}^1 |g_i'(s)| \, S_m(s) ds \\ &\leq C_T + \textstyle\int_0^t d_T^{(1)}(s) S_m(s) \, ds, \end{aligned} \tag{2.17}$$

where $d_T^{(1)}(s) = \sum_{i=0}^1 |g_i'(s)|$, with $d_T^{(1)} \in L^1(0,T)$.

$$\begin{aligned} I_3 = -2 \textstyle\sum_{i=0}^1 g_i(t) u_m(i,t) &\leq 2\sqrt{\tfrac{2}{\alpha_0}} \textstyle\sum_{i=0}^1 \|g_i\|_{L^\infty(0,T)} \sqrt{S_m(t)} \\ &\leq \tfrac{1}{\beta} C_T + \beta S_m(t), \end{aligned} \tag{2.18}$$

for all $\beta > 0$, $C_T \geq \frac{2}{\alpha_0} \left(\sum_{i=0}^1 \|g_i\|_{L^\infty(0,T)} \right)^2$. Combining (2.10), (2.12), (2.16)-(2.18) and choose $\beta = \frac{1}{2}$, we obtain

$$S_m(t) \leq d_T^{(0)} + \textstyle\int_0^t d_T^{(2)}(s) S_m(s) \, ds,\ 0 \leq t \leq T_m, \tag{2.19}$$

where $d_T^{(0)} = \bar{C}_0 + 8C_T$, $d_T^{(2)}(s) = 2\left[2 + d_T^{(1)}(s) + \|F(s)\|\right]$, $d_T^{(2)} \in L^1(0,T)$. By Gronwall's lemma, we deduce from (2.19) that

$$S_m(t) \leq d_T^{(0)} \exp\left[\textstyle\int_0^T d_T^{(2)}(s) ds\right] \leq C_T, \text{ for all } t \in [0,T]. \tag{2.20}$$

Thus, we can take constant $T_m = T$ for all m.

Step 3. *Limiting process.*

From (2.11), (2.20), we deduce the existence of a subsequence of $\{u_m\}$ still also so denoted, such that

$$\begin{cases} u_m \to u & \text{in} \quad L^\infty(0,T;H^1) \quad \text{weakly*}, \\ u'_m \to u' & \text{in} \quad L^\infty(0,T;H^1) \quad \text{weakly*}, \\ u_m \to u & \text{in} \quad L^\infty(0,T;L^p) \quad \text{weakly*}, \\ u'_m \to u' & \text{in} \quad L^q(Q_T) \quad \text{weakly}, \\ |u_m|^{p-2}\, u_m \to \chi_0 & \text{in} \quad L^\infty(0,T;L^{p'}) \quad \text{weakly*}, \\ |u'_m|^{q-2}\, u'_m \to \chi_1 & \text{in} \quad L^{q'}(Q_T) \quad \text{weakly}. \end{cases} \tag{2.21}$$

By the compactness lemma of Lions ([7], p. 57), (2.21) leads to the existence of a subsequence of $\{u_m\}$ denoted by the same symbol such that

$$u_m \to u \quad \text{strongly in} \quad L^2(Q_T) \quad \text{and a.e. in } Q_T. \tag{2.22}$$

By means of the continuity of function $x \longmapsto |x|^{p-2}x$, we have

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{a.e. in } Q_T. \tag{2.23}$$

Using Lions's Lemma ([7], Lemma 1.3, p.12), it follows from (2.20) and (2.23) that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{in } L^{p'}(Q_T) \quad \text{weakly}. \tag{2.24}$$

By $(2.21)_5$ and (2.24), we deduce that

$$\chi_0 = |u|^{p-2}u. \tag{2.25}$$

Passing to the limit in (2.8) by (2.9), (2.21), (2.24) and (2.25), we have u satisfying the problem

$$\begin{cases} \frac{d}{dt}\left[\langle u'(t), v\rangle + \varepsilon \langle u'_x(t), v_x\rangle\right] + \langle u_x(t), v_x\rangle + \lambda \langle \chi_1(t), v\rangle \\ \quad +K\left\langle |u(t)|^{p-2}\, u(t), v\right\rangle + \sum_{i=0}^{1}\left(h_i u(i,t) + g_i(t)\right) v(i) \\ = \langle F(t), v\rangle\,, \text{ for all } v \in H^1, \\ u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1. \end{cases} \tag{2.26}$$

It remains to prove $\chi_1 = |u'|^{q-2}\, u'$. We need the following lemma.

Lemma 2.5. *Let u be the weak solution of the following problem*

$$\begin{cases} u'' - u_{xx} - \varepsilon u''_{xx} = \Phi,\ 0 < x < 1,\ 0 < t < T, \\ (-1)^i\left[\varepsilon u''_x(i,t) + u_x(i,t)\right] = G_i(t), \\ u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1, \\ u \in L^\infty(0,T;H^1), \quad u' \in L^\infty(0,T;H^1), \\ \tilde{u}_0,\ \tilde{u}_1 \in H^1,\ G_0,\ G_1 \in L^2\left(0,T\right),\ \Phi \in L^1(0,T;L^2). \end{cases} \tag{2.27}$$

Then we have

$$\begin{aligned} &\tfrac{1}{2}\left\|u'(t)\right\|^2+\tfrac{1}{2}\left\|u_x(t)\right\|^2+\tfrac{\varepsilon}{2}\left\|u'_x(t)\right\|^2 \\ &\quad+\sum_{i=0}^{1}\int_0^t G_i(s)u'(i,s)ds-\int_0^t\left\langle \Phi(s),u'(s)\right\rangle ds \\ &\geq \tfrac{1}{2}\left\|\tilde{u}_1\right\|^2+\tfrac{1}{2}\left\|\tilde{u}_{0x}\right\|^2+\tfrac{\varepsilon}{2}\left\|\tilde{u}_{1x}\right\|^2,\ \ a.e.,\ t\in[0,T]. \end{aligned} \tag{2.28}$$

Furthermore, if $\tilde{u}_0=\tilde{u}_1=0$ *then there is equality in* (2.28).

Proof. The idea of the proof is the same as in ([5], Lemma 2.1, p. 79). Fix t_1, t_2, $0<t_1<t_2<T$ and let $v(x,t)$ be the function defined as follows

$$v(x,t)=\theta_m(t)[(\theta_m(t)u'(x,t))*\rho_k(t)*\rho_k(t)], \tag{2.29}$$

where
(i) θ_m is a continuous, piecewise linear function on $[0,T]$ defined as follows:

$$\theta_m(t)=\begin{cases} 0, & \text{if,}\quad t\in[0,T]\setminus[t_1+1/m,\ t_2-1/m], \\ 1, & \text{if,}\quad t\in[t_1+2/m,t_2-2/m], \\ m(t-t_1-1/m), & \text{if,}\quad t\in[t_1+1/m,t_1+2/m], \\ -m(t-t_2+1/m), & \text{if,}\quad t\in[t_2-2/m,t_2-1/m]. \end{cases} \tag{2.30}$$

(ii) $\{\rho_k\}$ is a regularizing sequence in $C_c^\infty(\mathbb{R})$, i.e.,

$$\rho_k\in C_c^\infty(\mathbb{R}),\quad \rho_k(t)=\rho_k(-t),\quad \int_{-\infty}^{+\infty}\rho_k(t)dt=1,\ \text{supp}\ \rho_k\subset[-1/k,1/k]. \tag{2.31}$$

(iii) $*$ is the convolution product in the time variable, ie.,

$$(u*\rho_k)(x,t)=\int_{-\infty}^{+\infty}u(x,t-s)\rho_k(s)ds. \tag{2.32}$$

We take the scalar product of the function $v(x,t)$ in (2.29) with equation (2.27), then integrate with respect to the time variable from 0 to T, and we have

$$X_{mk}+Y_{mk}=Z_{mk}, \tag{2.33}$$

where

$$\begin{cases} X_{mk}=\int_0^T\langle u''(t),v(t)\rangle dt, \\ Y_{mk}=-\int_0^T\langle\frac{\partial}{\partial x}\left(u_x(t)+\varepsilon u_{xtt}(t)\right),v(t)\rangle dt, \\ Z_{mk}=\int_0^T\langle\Phi(t),v(t)\rangle dt. \end{cases} \tag{2.34}$$

Using the properties of the functions $\theta_m(t)$ and $\rho_k(t)$, after some lengthy calculation, we can show that

$$\begin{cases} \lim\limits_{k\to+\infty} X_{mk} = -\int_0^T \theta_m\theta_m' \left\|u'(t)\right\|^2 dt, \\ \lim\limits_{k\to+\infty} Y_{mk} = -\int_0^T \theta_m\theta_m' \left\|u_x(t)\right\|^2 dt - \varepsilon\int_0^T \theta_m\theta_m' \left\|u_x'(t)\right\|^2 dt \\ \qquad\qquad + \sum_{i=0}^1 \int_0^T \theta_m^2 G_i(t)u'(i,t)dt, \\ \lim\limits_{k\to+\infty} Z_{mk} = \int_0^T \theta_m^2 \langle \Phi(t), u'(t)\rangle dt. \end{cases} \tag{2.35}$$

Letting $m\to\infty$, from (2.33)–(2.35) we obtain

$$\begin{aligned} &\tfrac{1}{2}\left\|u'(t_2)\right\|^2 + \tfrac{1}{2}\left\|u_x(t_2)\right\|^2 + \tfrac{\varepsilon}{2}\left\|u_x'(t_2)\right\|^2 \\ &\quad + \sum_{i=0}^1 \int_{t_1}^{t_2} G_i(t)u'(i,t)dt - \int_{t_1}^{t_2}\langle \Phi(t), u'(t)\rangle dt \\ &= \tfrac{1}{2}\left\|u'(t_1)\right\|^2 + \tfrac{1}{2}\left\|u_x(t_1)\right\|^2 + \tfrac{\varepsilon}{2}\left\|u_x'(t_1)\right\|^2, \end{aligned} \tag{2.36}$$

a.e., $t_1\ t_2 \in (0,T),\ t_1 < t_2$.

From (2.36), using the weak lower semicontinuity of the functional $v \longmapsto \|v\|^2$, we obtain (2.28) by taking $t_2 = t$ and passing to the limit as $t_1 \to 0_+$. In the case of $\tilde{u}_0 = \tilde{u}_1 = 0$, we prolong u, Φ, G_0, G_1 by 0 as $t<0$ and we deduce that equality (2.36) is also true for almost $t_1 < t_2 < T$. Then, taking $t_1 < 0$ in (2.36), its right-hand side is 0, letting $t_1 \to 0_-$, we have equality (2.28). The proof of Lemma 2.5 is completed. □

Remark 2.6. Lemma 2.5 is a relative generalization of a lemma presented in Lions's book ([7], Lemma 6.1, p. 224).

We now return to prove that $\chi_1 = |u'|^{q-2}u'$. From (2.10) and (2.11), we obtain

$$\begin{aligned} &2\lambda\int_0^t \left\langle |u_m'(s)|^{q-2}u_m'(s), u_m'(s)\right\rangle ds \\ &= 2\lambda\int_0^t \|u_m'(s)\|_{L^q}^q ds \\ &= \|\tilde{u}_{1m}\|^2 + \varepsilon\|\tilde{u}_{1mx}\|^2 + \|\tilde{u}_{0mx}\|^2 + \sum_{i=0}^1 h_i\tilde{u}_{0m}^2(i) \\ &\quad + \tfrac{2K}{p}\|\tilde{u}_{0m}\|_{L^p}^p - \|u_m'(t)\|^2 - \varepsilon\|u_{mx}'(t)\|^2 - \|u_{mx}(t)\|^2 \\ &\quad - \sum_{i=0}^1 h_i u_m^2(i,t) - \tfrac{2K}{p}\|u_m(t)\|_{L^p}^p + 2\int_0^t \langle F(s), u_m'(s)\rangle ds \\ &\quad -2\sum_{i=0}^1 \int_0^t g_i(s)u_m'(i,s)ds. \end{aligned} \tag{2.37}$$

Using Lemma 2.5, with $\Phi = F - K|u|^{p-2}u - \lambda\chi_1$, $G_i(t) = h_i u(i,t) + g_i(t)$, it follows from (2.9), (2.21), (2.28), (2.37) that

$$
\begin{aligned}
&2\lambda \limsup_{m\to\infty} \int_0^t \left\langle |u_m'(s)|^{q-2}\, u_m'(s), u_m'(s) \right\rangle ds \\
&\le \|\tilde{u}_1\|^2 + \varepsilon \|\tilde{u}_{1x}\|^2 + \|\tilde{u}_{0x}\|^2 + \textstyle\sum_{i=0}^1 h_i \tilde{u}_0^2(i) + \frac{2K}{p} \|\tilde{u}_0\|_{L^p}^p \\
&\quad - \liminf_{m\to\infty} \|u_m'(t)\|^2 - \varepsilon \liminf_{m\to\infty} \|u_{mx}'(t)\|^2 \\
&\quad - \liminf_{m\to\infty} \left(\|u_{mx}(t)\|^2 + \textstyle\sum_{i=0}^1 h_i u_m^2(i,t) \right) - \frac{2K}{p} \liminf_{m\to\infty} \|u_m(t)\|_{L^p}^p \\
&\quad + 2\int_0^t \langle F(s), u'(s)\rangle ds - 2\textstyle\sum_{i=0}^1 \int_0^t g_i(s) u'(i,s) ds \\
&\le \|\tilde{u}_1\|^2 + \varepsilon \|\tilde{u}_{1x}\|^2 + \|\tilde{u}_{0x}\|^2 + \textstyle\sum_{i=0}^1 h_i \tilde{u}_0^2(i) + \frac{2K}{p} \|\tilde{u}_0\|_{L^p}^p \\
&\quad - \|u'(t)\|^2 - \varepsilon \|u_x'(t)\|^2 - \|u_x(t)\|^2 - \textstyle\sum_{i=0}^1 h_i u^2(i,t)
\end{aligned}
\tag{2.38}
$$

and

$$
\begin{aligned}
&-\frac{2K}{p} \|u(t)\|_{L^p}^p + 2\int_0^t \langle F(s), u'(s)\rangle ds - 2\textstyle\sum_{i=0}^1 \int_0^t g_i(s) u'(i,s) ds \\
&\le \|\tilde{u}_1\|^2 + \varepsilon \|\tilde{u}_{1x}\|^2 + \|\tilde{u}_{0x}\|^2 - \|u'(t)\|^2 - \|u_x(t)\|^2 - \varepsilon \|u_x'(t)\|^2 \\
&\quad + 2\int_0^t \left\langle F(s) - K|u(s)|^{p-2} u(s) - \lambda \chi_1(s), u'(s) \right\rangle ds \\
&\quad - 2\textstyle\sum_{i=0}^1 \int_0^t \left(h_i u(i,s) + g_i(s) \right) u'(i,s) ds \\
&\quad + 2\lambda \int_0^t \langle \chi_1(s), u'(s)\rangle ds \le 2\lambda \int_0^t \langle \chi_1(s), u'(s)\rangle ds.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\Psi_m(t) &= \int_0^t \left\langle |u_m'(s)|^{q-2}\, u_m'(s) - |v(s)|^{q-2}\, v(s), u_m'(s) - v(s) \right\rangle ds \\
&\ge 0,
\end{aligned}
\tag{2.39}
$$

for all $v \in L^q(Q_T)$. Combining $(2.21)_{2,4,6}$, (2.38) and (2.39), we get

$$
\begin{aligned}
0 &\le \limsup_{m\to\infty} \Psi_m(t) \\
&\le \int_0^t \left\langle \chi_1(s) - |v(s)|^{q-2}\, v(s), u'(s) - v(s) \right\rangle ds, \quad \forall v \in L^q(Q_T).
\end{aligned}
\tag{2.40}
$$

In (2.40), choose $v(s) = u'(s) - \delta w$, with $\delta > 0$ and $w \in L^q(Q_T)$. Apply the argument of Minty and Browder (see Lions [7], p. 172), we obtain $\chi_1 = |u'|^{q-2}\, u'$. The proof of existence is completed.

Step 4. *Uniqueness of the solution.*

Assume now that $p \ge 2$ holds. Let u_1, u_2 be two weak solutions of problems (1.1)-(1.4), such that

$$
u_i \in L^\infty(0,T;H^1), u_i' \in L^\infty(0,T;H^1), \quad i = 1,2. \tag{2.41}
$$

Then $u = u_1 - u_2$ is the weak solution of the following problem

$$\begin{cases} u'' - u_{xx} - \varepsilon u''_{xx} = -\lambda \left[|u'_1|^{q-2} u'_1 - |u'_2|^{q-2} u'_2 \right] \\ \qquad\qquad\qquad\qquad -K \left[|u_1|^{p-2} u_1 - |u_2|^{p-2} u_2 \right], \\ 0 < x < 1,\, 0 < t < T, \\ (-1)^i \left[\varepsilon u''_x(i,t) + u_x(i,t) \right] = h_i u(i,t), \\ u(0) = u'(0) = 0, \\ u \in L^\infty(0,T;H^1), \quad u' \in L^\infty(0,T;H^1). \end{cases} \tag{2.42}$$

Using again Lemma 2.5 with $\tilde{u}_0 = \tilde{u}_1 = 0$, $\Phi = -\lambda(|u'_1|^{q-2} u'_1 - |u'_2|^{q-2} u'_2)$ $-K(|u_1|^{p-2} u_1 - |u_2|^{p-2} u_2)$, $G_i(t) = h_i u(i,t)$, $i = 0, 1$, we have

$$Z(t) = -2K \int_0^t \left\langle |u_1(s)|^{p-2} u_1(s) - |u_2(s)|^{p-2} u_2(s), u'(s) \right\rangle ds, \tag{2.43}$$

where

$$\begin{aligned} Z(t) &= \|u'(t)\|^2 + \|u_x(t)\|^2 + \varepsilon \|u'_x(t)\|^2 + \textstyle\sum_{i=0}^1 h_i u^2(i,t) \\ &\quad +2\lambda \int_0^t \left\langle |u'_1(s)|^{q-2} u'_1(s) - |u'_2(s)|^{q-2} u'_2(s), u'(s) \right\rangle ds. \end{aligned} \tag{2.44}$$

Applying the following inequality, for all $p \geq 2$,

$$\begin{aligned} &\left| \, |x|^{p-2}x - |y|^{p-2}y \right| \\ &\leq (p-1)M^{p-2} |x-y|, \quad \forall x, y \in [-M, M], \quad \forall M > 0, \end{aligned} \tag{2.45}$$

with $M = \sqrt{2} \max_{i=1,2} \|u_i\|_{L^\infty(0,T;H^1)}$, and note that

$$Z(t) \geq \|u'(t)\|^2 + \alpha_0 \|u(t)\|_{H^1}^2 \geq 2\sqrt{\alpha_0} \|u'(t)\| \|u(t)\|_{H^1}, \tag{2.46}$$

we deduce from (2.43), (2.46) that

$$\begin{aligned} Z(t) &= -2K \int_0^t \left\langle |u_1(s)|^{p-2} u_1(s) - |u_2(s)|^{p-2} u_2(s), u'(s) \right\rangle ds \\ &\leq 2K(p-1)M^{p-2} \int_0^t \|u(s)\| \|u'(s)\| ds \\ &\leq K(p-1)M^{p-2} \frac{1}{\sqrt{\alpha_0}} \int_0^t Z(s) ds. \end{aligned} \tag{2.47}$$

By Gronwall's lemma, (2.47) gives $Z \equiv 0$, i.e., $u_1 \equiv u_2$. Theorem 2.4 follows. □

3. The regularity of solutions

In this section, we study the regularity of solutions of Prob.(1.1)-(1.4) corresponding to $(\tilde{u}_0, \tilde{u}_1) \in H^2 \times H^2$.

Henceforth, we strengthen the hypotheses as follows:

(H_1') $p \geq 2,\ q \geq 2,\ \lambda > 0,\ K > 0,\ \varepsilon > 0;$

(H_3') $(\tilde{u}_0, \tilde{u}_1) \in H^2 \times H^2;$

(H_4') $F,\ F' \in L^1(0, T; L^2);$

(H_5') $g_i \in W^{2,1}(0, T),\ i = 0, 1.$

Then, we have the following theorem.

Theorem 3.1. *Let $T > 0$. Suppose that $(H_1'), (H_2), (H_3') - (H_5')$ hold. Then Prob.(1.1)-(1.4) has a unique weak solution*

$$u \in L^\infty\left(0, T; H^2\right), \text{ such that } u_t, u_{tt} \in L^\infty\left(0, T; H^2\right). \tag{3.1}$$

Remark 3.2. (i) The regularity obtained by (3.1) shows that Prob.(1.1)-(1.4) has a unique strong solution

$$u \in C^1\left(0, T; H^2\right),\ \ u_{tt} \in L^\infty\left(0, T; H^2\right). \tag{3.2}$$

(ii) In [1], Browder has studied the differential equation $u_{tt} + Au + M(u) = 0$, $t > 0$, with the Cauchy initial conditions $u(0_+) = u_0$, $u_t(0_+) = u_1$, where A is a positive densely defined self-adjoint linear operator in a Hilbert space H with $A^{1/2}$ being its positive square root, $M(u)$ is a (possibly) nonlinear function from $D\left(A^{1/2}\right)$ to H and some other conditions. In general, the results in the Theorem 3.1 and in ([1], [6]) overlap and do not include each other as particular cases.

Proof. The proof consists of four steps.

Step 1. *The Faedo-Galerkin approximation.*

Let $\{w_j\}$ be a denumerable base of H^2. We find the approximate solution $u_m(t)$ of Prob.(1.1)-(1.4) in the form (2.7), where the coefficient functions c_{mj} satisfy the system of ordinary differential equations $(2.8)_1$, where

$$\begin{cases} \tilde{u}_{0m} = \sum_{j=1}^m \alpha_{mj} w_j \to \tilde{u}_0 \quad \text{strongly in } H^2, \\ \tilde{u}_{1m} = \sum_{j=1}^m \beta_{mj} w_j \to \tilde{u}_1 \quad \text{strongly in } H^2. \end{cases} \tag{3.3}$$

Step 2. *A priori estimates I.*

Proceeding as in the proof of Theorem 2.4, we get, after using assumptions (H_1'), (H_2) and $(H_3') - (H_5')$,

$$\begin{aligned} S_m(t) &= \|u_m'(t)\|^2 + \|u_{mx}(t)\|^2 + \varepsilon \|u_{mx}'(t)\|^2 + \textstyle\sum_{i=0}^1 h_i u_m^2(i, t) \\ &\quad + \tfrac{2K}{p} \|u_m(t)\|_{L^p}^p + 2\lambda \textstyle\int_0^t \|u_m'(s)\|_{L^q}^q\, ds \leq C_T, \end{aligned} \tag{3.4}$$

for all $t \in [0,T]$ and for all m, and C_T always indicates a bound depending on T.

A priori estimates II.

Now differentiating $(2.8)_1$ with respect to t, we have

$$\begin{aligned} &\langle u_m'''(t), w_j\rangle + \langle u_{mx}'(t) + \varepsilon u_{mx}'''(t), w_{jx}\rangle + \lambda(q-1)\left\langle |u_m'(t)|^{q-2}\, u_m''(t), w_j\right\rangle \\ &\quad + K(p-1)\left\langle |u_m(t)|^{p-2}\, u_m'(t), w_j\right\rangle \\ &\quad + \textstyle\sum_{i=0}^{1} \left(h_i u_m'(i,t) + g_i'(t)\right) w_j(i) = \langle F'(t), w_j\rangle\,, \text{ for all } 1 \le j \le m. \end{aligned} \tag{3.5}$$

Multiplying the j th equation of (3.5) by $c_{mj}''(t)$, summing up with respect to j and then integrating with respect to the time variable from 0 to t, after some rearrangements, the result is

$$\begin{aligned} X_m(t) &= X_m(0) + 2\textstyle\sum_{i=0}^{1} g_i'(0)\tilde{u}_{1m}(i) + 2\int_0^t \langle F'(s), u_m''(s)\rangle\, ds \\ &\quad -2K(p-1)\textstyle\int_0^t \left\langle |u_m(s)|^{p-2}\, u_m'(s), u_m''(s)\right\rangle ds \\ &\quad -2\textstyle\sum_{i=0}^{1} g_i'(t) u_m'(i,t) + 2\int_0^t \sum_{i=0}^{1} g_i''(s) u_m'(i,s) ds \\ &\equiv X_m(0) + 2\textstyle\sum_{i=0}^{1} g_i'(0)\tilde{u}_{1m}(i) + \sum_{j=1}^{4} J_j, \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} X_m(t) &= \|u_m''(t)\|^2 + \|u_{mx}'(t)\|^2 + \varepsilon\, \|u_{mx}''(t)\|^2 + \textstyle\sum_{i=0}^{1} h_i\, |u_m'(i,t)|^2 \\ &\quad + 2\lambda(q-1)\textstyle\int_0^t ds \int_0^1 |u_m'(x,s)|^{q-2}\, |u_m''(s,s)|^2\, ds. \end{aligned} \tag{3.7}$$

First, we are going to estimate $\xi_m = \|u_m''(0)\|^2 + \varepsilon\, \|u_{mx}''(0)\|^2$. Letting $t \to 0_+$ in Eq. $(2.8)_1$, multiplying the result by $c_{mj}''(0)$, we get

$$\begin{aligned} &\|u_m''(0)\|^2 + \varepsilon\, \|u_{mx}''(0)\|^2 + \langle \tilde{u}_{0mx}, u_{mx}''(0)\rangle \\ &\quad + \textstyle\sum_{i=0}^{1} \left(h_i \tilde{u}_{0m}(i) + g_i(0)\ \right) u_m''(i,0) + K\left\langle |\tilde{u}_{0m}|^{p-2}\, \tilde{u}_{0m}, u_m''(0)\right\rangle \\ &\quad + \lambda\left\langle |\tilde{u}_{1m}|^{q-2}\, \tilde{u}_{1m}, u_m''(0)\right\rangle = \langle F(0), u_m''(0)\rangle\,. \end{aligned} \tag{3.8}$$

Note that

$$\begin{aligned} |u_m''(i,0)| &\le \|u_m''(0)\|_{C^0([0,1])} \le \sqrt{2}\, \|u_m''(0)\|_{H^1} \\ &= \sqrt{2}\sqrt{\|u_m''(0)\|^2 + \|u_{mx}''(0)\|^2} \\ &\le \sqrt{2}\max\{1, \tfrac{1}{\sqrt{\varepsilon}}\}\sqrt{\|u_m''(0)\|^2 + \varepsilon\, \|u_{mx}''(0)\|^2} \\ &= \sqrt{2}\max\{1, \tfrac{1}{\sqrt{\varepsilon}}\}\sqrt{\xi_m}. \end{aligned} \tag{3.9}$$

This implies that

$$\begin{aligned}
\xi_m &= \|u_m''(0)\|^2 + \varepsilon \|u_{mx}''(0)\|^2 \\
&\leq \|\tilde{u}_{0mx}\| \|u_{mx}''(0)\| + \sum_{i=0}^{1} |h_i \tilde{u}_{0m}(i) + g_i(0)| |u_m''(i,0)| \\
&\quad + \left[\lambda \left\| |\tilde{u}_{1m}|^{q-1} \right\| + K \left\| |\tilde{u}_{0m}|^{p-1} \right\| + \|F(0)\| \ \right] \|u_m''(0)\| \\
&\leq \tfrac{1}{2\beta_1} \|\tilde{u}_{0mx}\|^2 + \tfrac{\beta_1}{2} \|u_{mx}''(0)\|^2 \\
&\quad + \tfrac{1}{2\beta_1} \left(\sum_{i=0}^{1} |h_i \tilde{u}_{0m}(i) + g_i(0)| \right)^2 + \beta_1 \|u_m''(0)\|_{H^1}^2 \\
&\quad + \tfrac{1}{2\beta_1} \left[\lambda \left\| |\tilde{u}_{1m}|^{q-1} \right\| + K \left\| |\tilde{u}_{0m}|^{p-1} \right\| + \|F(0)\| \ \right]^2 + \tfrac{\beta_1}{2} \|u_m''(0)\|^2 \\
&\leq \tfrac{1}{2\beta_1} \|\tilde{u}_{0mx}\|^2 + \tfrac{\beta_1}{2\varepsilon} \xi_m \\
&\quad + \tfrac{1}{2\beta_1} \left(\sum_{i=0}^{1} |h_i \tilde{u}_{0m}(i) + g_i(0)| \right)^2 + \beta_1 \max\{1, \tfrac{1}{\varepsilon}\} \xi_m \\
&\quad + \tfrac{1}{2\beta_1} \left[\lambda \left\| |\tilde{u}_{1m}|^{q-1} \right\| + K \left\| |\tilde{u}_{0m}|^{p-1} \right\| + \|F(0)\| \ \right]^2 + \tfrac{\beta_1}{2} \xi_m \\
&\leq \tfrac{1}{2\beta_1} \|\tilde{u}_{0mx}\|^2 + \tfrac{1}{2\beta_1} \left(\sum_{i=0}^{1} |h_i \tilde{u}_{0m}(i) + g_i(0)| \right)^2 \\
&\quad + \tfrac{1}{2\beta_1} \left[\lambda \left\| |\tilde{u}_{1m}|^{q-1} \right\| + K \left\| |\tilde{u}_{0m}|^{p-1} \right\| + \|F(0)\| \ \right]^2 \\
&\quad + \tfrac{\beta_1}{2} \left[1 + \tfrac{1}{\varepsilon} + 2 \max\{1, \tfrac{1}{\varepsilon}\}\right] \xi_m, \text{ for all } \beta_1 > 0.
\end{aligned} \tag{3.10}$$

Choose $\beta_1 > 0$, such that $\frac{\beta_1}{2} \left[1 + \frac{1}{\varepsilon} + 2 \max\{1, \frac{1}{\varepsilon}\}\right] \leq \frac{1}{2}$, we have

$$\begin{aligned}
\xi_m &= \|u_m''(0)\|^2 + \varepsilon \|u_{mx}''(0)\|^2 \\
&\leq \tfrac{1}{\beta_1} \|\tilde{u}_{0mx}\|^2 + \tfrac{1}{\beta_1} \left(\sum_{i=0}^{1} |h_i \tilde{u}_{0m}(i) + g_i(0)| \right)^2 \\
&\qquad + \tfrac{1}{\beta_1} \left[\lambda \left\| |\tilde{u}_{1m}|^{q-1} \right\| + K \left\| |\tilde{u}_{0m}|^{p-1} \right\| + \|F(0)\| \ \right]^2 \\
&\leq \overline{X}_0, \quad \text{for all } m,
\end{aligned} \tag{3.11}$$

where $\overline{X}_0$ is a constant depending only on p, q, K, λ, F, $\tilde{u}_0$, $\tilde{u}_1$, h_0, h_1, $g_0(0)$, $g_1(0)$ and ε. By (3.3), (3.7) and (3.11), we obtain

$$\begin{aligned}
& X_m(0) + 2 \sum_{i=0}^{1} g_i'(0) \tilde{u}_{1m}(i) \\
& = \xi_m + \|\tilde{u}_{1mx}\|^2 + \sum_{i=0}^{1} h_i |\tilde{u}_{1m}(i)|^2 + 2 \sum_{i=0}^{1} g_i'(0) \tilde{u}_{1m}(i) \\
& \leq \tfrac{1}{2} X_0, \quad \text{for all } m,
\end{aligned} \tag{3.12}$$

where X_0 is a constant depending only on p, q, K, λ, F, $\tilde{u}_0$, $\tilde{u}_1$, h_0, h_1, $g_0(0)$, $g_1(0)$ and ε. By (2.1), (2.3), (2.15), (3.7) and also note

$$\begin{aligned}
X_m(t) \geq \|u_m''(t)\|^2 + \alpha_0 \|u_m'(t)\|_{H^1}^2 + \varepsilon \|u_{mx}''(t)\|^2 \\
+ 2\lambda(q-1) \int_0^t ds \int_0^1 |u_m'(x,s)|^{q-2} |u_m''(s,s)|^2 \, ds,
\end{aligned} \tag{3.13}$$

$$\begin{aligned} |u_m'(i,t)| &\le \|u_m'(t)\|_{C^0(\overline{\Omega})} \le \sqrt{2}\,\|u_m'(t)\|_{H^1} \\ &\le \sqrt{\tfrac{2}{\alpha_0}}\sqrt{X_m(t)}, \quad i = 0,\, 1, \end{aligned} \tag{3.14}$$

we estimate all terms on the right-hand side of (3.6) as follows

$$\begin{aligned} J_1 &= 2\int_0^t \langle F'(s), u_m''(s)\rangle\, ds \le \|F'\|_{L^1(0,T;L^2)} + \int_0^t \|F'(s)\|\, X_m(s)\, ds \\ &\le C_T + \int_0^t \|F'(s)\|\, X_m(s)\, ds; \end{aligned} \tag{3.15}$$

where C_T always indicates a bound depending on T;

$$\begin{aligned} J_2 &= 2K(p-1)\int_0^t \left\langle |u_m(s)|^{p-2}\, u_m'(s), u_m''(s)\right\rangle ds \\ &\le 2K(p-1)\left(\sqrt{\tfrac{2}{\alpha_0}}\right)^{p-2}\int_0^t \left(\sqrt{S_m(s)}\right)^{p-2}\sqrt{S_m(s)}\sqrt{X_m(s)}ds \\ &\le 2(p-1)\left(\sqrt{\tfrac{2}{\alpha_0}}\right)^{p-2}\sqrt{C_T^{p-1}}\int_0^t \sqrt{X_m(s)}ds \\ &\le C_T + \int_0^t X_m(s)ds; \end{aligned} \tag{3.16}$$

$$\begin{aligned} J_3 &= -2\sum_{i=0}^1 g_i'(t)u_m'(i,t) \le 2\sum_{i=0}^1 |g_i'(t)|\,|u_m'(i,t)| \\ &\le 2\sqrt{\tfrac{2}{\alpha_0}}\sum_{i=0}^1 |g_i'(t)|\sqrt{X_m(t)} \\ &\le \tfrac{1}{\beta}\tfrac{2}{\alpha_0}\left(\sum_{i=0}^1 \|g_i'\|_{L^\infty(0,T)}\right)^2 + \beta X_m(t) \le \tfrac{1}{\beta}C_T + \beta X_m(t); \end{aligned} \tag{3.17}$$

$$\begin{aligned} J_4 &= 2\sum_{i=0}^1 \int_0^t g_i''(s)u_m'(i,s)ds \le 2\sqrt{\tfrac{2}{\alpha_0}}\sum_{i=0}^1 \int_0^t |g_i''(s)|\sqrt{X_m(s)}ds \\ &\le \sqrt{\tfrac{2}{\alpha_0}}\sum_{i=0}^1 \int_0^t |g_i''(s)|\,[1 + X_m(s)]\, ds \\ &\le C_T\left[1 + \int_0^t \bar{g}(s)X_m(s)\, ds\right], \end{aligned} \tag{3.18}$$

where $\bar{g}(s) = \sum_{i=0}^1 |g_i''(s)|$, $\bar{g} \in L^1(0,T)$. Combining (3.6), (3.12), (3.15)–(3.18) and choose $\beta = \frac{1}{2}$, we get after some rearrangements

$$X_m(t) \le C_T + C_T\int_0^t (1 + \bar{g}(s) + \|F'(s)\|)\, X_m(s)\, ds,\ 0 \le t \le T. \tag{3.19}$$

By Gronwall's Lemma, (3.19) gives

$$X_m(t) \le C_T \exp\left[C_T\int_0^T (1 + \bar{g}(s) + \|F'(s)\|\ )\, ds\right] \le C_T, \forall t \in [0,T]. \tag{3.20}$$

Step 3. *Limiting process.*

From (3.4), (3.7), (3.20), we deduce the existence of a subsequence of $\{u_m\}$ still also so denoted, such that

$$\begin{cases} u_m \to u & \text{in} \quad L^\infty(0,T;H^1) \quad \text{weakly*}, \\ u'_m \to u' & \text{in} \quad L^\infty(0,T;H^1) \quad \text{weakly*}, \\ u''_m \to u'' & \text{in} \quad L^\infty(0,T;H^1) \quad \text{weakly*}. \end{cases} \tag{3.21}$$

By the compactness lemma of Lions ([7], p. 57), (3.21) leads to the existence of a subsequence of $\{u_m\}$ denoted by the same symbol such that

$$\begin{cases} u_m \to u & \text{strongly in} \quad L^2(Q_T) \quad \text{and a.e. in } Q_T, \\ u'_m \to u' & \text{strongly in} \quad L^2(Q_T) \quad \text{and a.e. in } Q_T. \end{cases} \tag{3.22}$$

Using again the inequality (2.45), with $M = \sqrt{2}C_T$, we deduce from (3.22) that

$$|u_m|^{p-2}u_m \to |u|^{p-2}u \quad \text{strongly in} \quad L^2(Q_T). \tag{3.23}$$

Similarly

$$|u'_m|^{q-2}u'_m \to |u'|^{q-2}u' \quad \text{strongly in} \quad L^2(Q_T). \tag{3.24}$$

Passing to the limit in (2.8), by (2.3), (2.21) – (2.24), we have u satisfying the problem

$$\begin{cases} \langle u''(t), v\rangle + \langle u_x(t) + \varepsilon u''_x(t), v_x\rangle + \lambda \left\langle |u'(t)|^{q-2}u'(t), v\right\rangle \\ \quad + \sum_{i=0}^{1} \left(h_i u(i,t) + g_i(t)\right) v(i) + K\left\langle |u(t)|^{p-2}\, u(t), v\right\rangle \\ = \langle F(t), v\rangle\,, \quad \forall v \in H^1, \\ u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1. \end{cases} \tag{3.25}$$

On the other hand, we have from (3.21), $(3.25)_1$ that

$$\begin{aligned} \frac{\partial^2}{\partial x^2}\left(u + \varepsilon u_{tt}\right) &= u_{tt} + \lambda|u_t|^{q-2}u_t + K\,|u|^{p-2}\,u - F(t) \\ &\in L^\infty(0,T;L^2). \end{aligned} \tag{3.26}$$

So

$$u + \varepsilon u_{tt} \equiv \Phi \in L^\infty(0,T;H^2). \tag{3.27}$$

Furthermore, by $u_{tt} + \frac{1}{\varepsilon}u \equiv \frac{1}{\varepsilon}\Phi$, it follows that

$$\begin{aligned} u(t) = \cos&\left(\sqrt{\tfrac{1}{\varepsilon}}t\right)\tilde{u}_0 + \sqrt{\varepsilon}\sin\left(\sqrt{\tfrac{1}{\varepsilon}}t\right)\tilde{u}_1 \\ &+ \sqrt{\varepsilon}\int_0^t \sin\left(\sqrt{\tfrac{1}{\varepsilon}}(t-s)\right)\tfrac{1}{\varepsilon}\Phi(s)ds \in L^\infty(0,T;H^2). \end{aligned} \tag{3.28}$$

Then

$$\begin{aligned} &u_{tt} = \tfrac{1}{\varepsilon}\left(\Phi - u\right) \in L^\infty(0,T;H^2), \\ &u_t = \tilde{u}_1 + \int_0^t u_{tt}(s)ds \in L^\infty(0,T;H^2). \end{aligned} \tag{3.29}$$

Thus u, u_t, $u_{tt} \in L^\infty(0,T;H^2)$ and the existence of a weak solution is proved completely.

Step 4. *Uniqueness of the solution.*

Let u_1, u_2 be two weak solutions of problem (1.1)–(1.4) such that

$$u_i \in C^1\left(0, T; H^2\right), \; u_i'' \in L^\infty\left(0, T; H^2\right), \; i = 1, 2. \tag{3.30}$$

Then $w = u_1 - u_2$ verifies

$$\begin{cases} \langle w''(t), v\rangle + \langle w_x(t) + \varepsilon w_x''(t), v_x\rangle + \lambda \left\langle |u_1'|^{q-2}u_1' - |u_2'|^{q-2}u_2', v\right\rangle \\ \quad + \sum_{i=0}^{1} h_i w(i,t) v(i) \\ = -K\left\langle |u_1|^{p-2}u_1 - |u_2|^{p-2}u_2, v\right\rangle, \text{ for all } v \in H^1, \\ w(0) = w'(0) = 0. \end{cases} \tag{3.31}$$

We take $v = w = u_1 - u_2$ in (3.30) and integrating with respect to t, we obtain

$$\sigma(t) = -2K \int_0^t \left\langle |u_1(s)|^{p-2}u_1(s) - |u_2(s)|^{p-2}u_2(s), w'(s)\right\rangle ds, \tag{3.32}$$

where

$$\begin{aligned} \sigma(t) &= \|w'(t)\|^2 + \varepsilon \|w_x'(t)\|^2 + \|w_x(t)\|^2 + \textstyle\sum_{i=0}^{1} h_i w^2(i,t) \\ &\quad + 2\lambda \int_0^t \left\langle |u_1'(s)|^{q-2}\, u_1'(s) - |u_2'(s)|^{q-2}\, u_2'(s), w'(s)\right\rangle ds. \end{aligned} \tag{3.33}$$

Using again the inequality (2.45), with $M = M_1 = \sqrt{2} \max_{i=1,2} \|u_i\|_{L^\infty(0,T;H^1)}$, we deduce that

$$\begin{aligned} &\left|\, |u_1(s)|^{p-2}u_1(s) - |u_2(s)|^{p-2}u_2(s)\right| \\ &\leq (p-1)M_1^{p-2} \left|w(s)\right|, \quad \forall (x,s) \in Q_T, \end{aligned} \tag{3.34}$$

and the following inequalities

$$\sigma(t) \geq \|w'(t)\|^2 + \varepsilon \|w_x'(t)\|^2 + \alpha_0 \|w(t)\|_{H^1}^2, \tag{3.35}$$

$$\|w(t)\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\, \|w(t)\|_{H^1} \leq \sqrt{\tfrac{2}{\alpha_0}} \sqrt{\sigma(t)}, \tag{3.36}$$

we obtain

$$\begin{aligned} &-2K \int_0^t \left\langle |u_1(s)|^{p-2}u_1(s) - |u_2(s)|^{p-2}u_2(s), w'(s)\right\rangle ds \\ &\leq 2K(p-1)M_1^{p-2} \int_0^t \|w(s)\|\, \|w'(s)\|\, ds \\ &\leq 2K(p-1)M_1^{p-2} \sqrt{\tfrac{2}{\alpha_0}} \int_0^t \sigma(s) ds \equiv K_T \int_0^t \sigma(s) ds. \end{aligned} \tag{3.37}$$

Hence

$$\sigma(t) \leq K_T \int_0^t \sigma(s) ds. \tag{3.38}$$

By Gronwall's Lemma, it follows from (3.38) that $\sigma \equiv 0$, i.e., $u_1 \equiv u_2$. Theorem 3.1 is proved completely. □

Now, we continue to study the regularity of solutions of problems (1.1)-(1.4), corresponding to $p = q = 2$.

$$\begin{cases} Lu \equiv u'' - u_{xx} - \varepsilon u''_{xx} + \lambda u' + Ku = F(x,t), \\ 0 < x < 1,\, 0 < t < T, \\ L_i u \equiv (-1)^i \left[\varepsilon u''_x(i,t) + u_x(i,t)\right] - h_i u(i,t) = g_i(t),\; i = 0,1, \\ u(0) = \tilde{u}_0,\; u'(0) = \tilde{u}_1. \end{cases} \tag{3.39}$$

For this purpose, we assume that the constants ε, K, λ, h_0, h_1 satisfy the conditions (H_1'), (H_2). Furthermore, we will impose the following stronger assumptions. With $r \in N$, we assume that

$(H_3^{[r]})$ $(\tilde{u}_0, \tilde{u}_1) \in H^{r+2} \times H^{r+2}$.

$(H_4^{[r]})$ The function $F(x,t)$ satisfies

$$\begin{cases} \frac{\partial^j F}{\partial t^j} \in L^\infty(0,T;H^r), \quad 0 \le j \le r, \\ \frac{\partial^{r+1} F}{\partial t^{r+1}} \in L^1(0,T;H^r). \end{cases}$$

$(H_5^{[r]})$ $g_0,\; g_1 \in W^{r+1,1}\left(0,T\right),\; r \ge 1.$

First, we define the sequences $\{\tilde{u}_0^{[k]}\}$, $\{\tilde{u}_1^{[k]}\}$, $k = 0, 1, ..., r{+}2$ by the following recurrent formulas

$$\begin{cases} \tilde{u}_0^{[0]} = \tilde{u}_0,\; \tilde{u}_1^{[0]} = \tilde{u}_1, \\ \tilde{u}_0^{[k]} = \tilde{u}_1^{[k-1]},\; k \in \{1, 2, ..., r+1\},\; r \ge 1, \end{cases} \tag{3.40}$$

where $\tilde{u}_0^{[k]}$ is defined by the following problem

$$\begin{cases} -\varepsilon \Delta \tilde{u}_0^{[k]} + \tilde{u}_0^{[k]} = \frac{\partial^{k-2} F}{\partial t^{k-2}}(\cdot, 0) + \Delta \tilde{u}_0^{[k-2]} - K\tilde{u}_0^{[k-2]} - \lambda \tilde{u}_1^{[k-2]} \\ \qquad\qquad\qquad \equiv \Phi^{[k]},\; 0 < x < 1, \\ (-1)^i \varepsilon \tilde{u}_{0x}^{[k]}(i) = -(-1)^i \tilde{u}_{0x}^{[k-2]}(i) + h_i \tilde{u}_0^{[k-2]}(i) + \frac{d^{k-2} g_i}{dt^{k-2}}(0) \\ \qquad\qquad\qquad \equiv \Phi_i^{[k]},\; i = 0, 1. \end{cases} \tag{3.41}$$

Then, we have the following lemma.

Lemma 3.3. *Suppose that* $(H_3^{[r]}) - (H_5^{[r]})$ *hold. Then problem* (3.41) *has a unique weak solution* $\tilde{u}_0^{[k]} \in H^1$. *Furthermore, we have* $\tilde{u}_0^{[k]} \in H^{r+2}$, $k = 2, 3, ..., r+1$.

Proof. A weak solution of problem (3.41) is obtained from the following variational problem. Find $U \in H^1$ such that

$$a(U, w) = \langle \tilde{L}, w \rangle, \text{ for all } w \in H^1, \tag{3.42}$$

where

$$\begin{cases} a(U,w) = \langle \varepsilon U_x, w_x \rangle + \langle U, w \rangle, \\ \langle \tilde{L}, w \rangle = \langle \Phi^{[k]}, w \rangle - \sum_{i=0}^{1} \Phi_i^{[k]} w(i). \end{cases} \tag{3.43}$$

Using the Lax-Milgram's theorem, problem (3.42) has a unique weak solution $\tilde{u}_0^{[k]} \in H^1$.

We shall prove that

$$\tilde{u}_0^{[k]} \in H^{r+2},\ k \in \{1,2,...,r+1\},\ r \geq 1. \tag{3.44}$$

(i) $k = 1 : \tilde{u}_0^{[1]} = \tilde{u}_1^{[0]} = \tilde{u}_1 \in H^{r+2}$. (by $(H_3^{[r]})$).
(ii) Suppose by induction that $\tilde{u}_0^{[1]}, ..., \tilde{u}_0^{[k-1]} \in H^{r+2}$ hold. We shall prove that $\tilde{u}_0^{[k]} \in H^{r+2}$ holds.

In fact, by $(H_4^{[r]})$, we have $\frac{\partial^{k-2} F}{\partial t^{k-2}}(\cdot,0) \in H^r$, $2 \leq k \leq r+2$. Hence, by induction we obtain

$$\Phi^{[k]} = \frac{\partial^{k-2} F}{\partial t^{k-2}}(\cdot,0) + \Delta\tilde{u}_0^{[k-2]} - K\tilde{u}_0^{[k-2]} - \lambda\tilde{u}_0^{[k-1]} \in H^r. \tag{3.45}$$

On the other hand, by $\tilde{u}_0^{[k]} \in H^1$ and (3.45),

$$\varepsilon\Delta\tilde{u}_0^{[k]} = \tilde{u}_0^{[k]} - \Phi^{[k]} \in H^1. \tag{3.46}$$

Then $\tilde{u}_0^{[k]} \in H^3$. Similarly, we have also $\tilde{u}_0^{[k]} \in H^{2s+1}$, with $s \in \mathbb{N}$, $2s-1 \leq r < 2s+1$. Then

$$\varepsilon\Delta\tilde{u}_0^{[k]} = \tilde{u}_0^{[k]} - \Phi^{[k]} \in H^r. \tag{3.47}$$

Thus

$$\tilde{u}_0^{[k]} \in H^{r+2}. \tag{3.48}$$

Lemma 3.3 is proved. □

Now, formally differentiating problem (3.39) with respect to time up to order r and letting $u^{[r]} = \frac{\partial^r u}{\partial t^r}$ we are led to consider the solution $u^{[r]}$ of problem $(Q^{[r]})$:

$$(Q^{[r]}) \begin{cases} Lu^{[r]} = \frac{\partial^r F}{\partial t^r}(x,t), \quad (x,t) \in Q_T, \\ L_i u^{[r]} = \frac{d^r g_i}{dt^r}(t), \quad i = 0,1, \\ u^{[r]}(0) = \tilde{u}_0^{[r]}, \quad u_t^{[r]}(0) = \tilde{u}_1^{[r]}, \end{cases} \tag{3.49}$$

where

$$\begin{cases} Lw = w'' - \Delta w - \varepsilon\Delta w'' + Kw + \lambda w', \\ L_i w = (-1)^i \left[\varepsilon w_x''(i) + w_x(i)\right] - h_i w(i), \quad i = 0,1. \end{cases} \tag{3.50}$$

From assumptions $(H_1^{[r]}) - (H_3^{[r]})$ we deduce that $\tilde{u}_0^{[r]}$, $\tilde{u}_1^{[r]}$, $\frac{\partial^r F}{\partial t^r}$, $\frac{d^r g_0}{dt^r}$ and $\frac{d^r g_1}{dt^r}$ satisfy the conditions of Theorem 3.1. Hence, the problem $(Q^{[r]})$ has a unique weak solution $u^{[r]}$ such that

$$u^{[r]} \in C^1\left(0,T;H^2\right), \quad u_{tt}^{[r]} \in L^\infty\left(0,T;H^2\right). \tag{3.51}$$

Moreover, from the uniqueness of a weak solution we have $u^{[r]} = \frac{\partial^r u}{\partial t^r}$. Hence we deduce from (3.51) that the solution u of problem (3.39) satisfy

$$u \in C^{r+1}\left(0,T;H^2\right), \ \frac{\partial^{r+2} u}{\partial t^{r+2}} \in L^\infty\left(0,T;H^2\right). \tag{3.52}$$

Next we shall prove by induction on r that

$$u \in C^{r+1}\left(0,T;H^{r+2}\right), \ \frac{\partial^{r+2} u}{\partial t^{r+2}} \in L^\infty(0,T;H^{r+2}), \ r \geq 1. \tag{3.53}$$

In the case of $r=1$, the proof of (3.53) is easy, we omit the details. We now prove with $r \geq 2$. Suppose by induction that (3.53) holds for $r-1$. i.e.,

$$u \in C^{r}\left(0,T;H^{r+1}\right), \ \frac{\partial^{r+1} u}{\partial t^{r+1}} \in L^\infty(0,T;H^{r+1}). \tag{3.54}$$

We shall prove that (3.53) holds. To achieve this, we have to prove that

$$\begin{cases} \frac{\partial^{r} u}{\partial t^{r}} \in L^\infty(0,T;H^{r+2}), \\ \frac{\partial^{r+1} u}{\partial t^{r+1}} \in L^\infty(0,T;H^{r+2}), \\ \frac{\partial^{r+2} u}{\partial t^{r+2}} \in L^\infty(0,T;H^{r+2}), \ r \geq 1. \end{cases} \tag{3.55}$$

By $(Q^{[r]})_1$,

$$\left(u^{[r]} - \varepsilon\Delta u^{[r]}\right)'' - \Delta u^{[r]} + K u^{[r]} + \lambda u_t^{[r]} = \frac{\partial^r F}{\partial t^r}. \tag{3.56}$$

Put

$$\begin{cases} W = u^{[r]} - \varepsilon\Delta u^{[r]}, \\ \tilde{w}_0 = \tilde{u}_0^{[r]} - \varepsilon\Delta\tilde{u}_0^{[r]}, \\ \tilde{w}_1 = \tilde{u}_1^{[r]} - \varepsilon\Delta\tilde{u}_1^{[r]} = \tilde{u}_0^{[r+1]} - \varepsilon\Delta\tilde{u}_0^{[r+1]}. \end{cases} \tag{3.57}$$

Then

$$\begin{cases} W'' + \frac{1}{\varepsilon}W = \frac{1}{\varepsilon}u^{[r]} - K u^{[r]} - \lambda u_t^{[r]} + \frac{\partial^r F}{\partial t^r} \equiv \Psi^{[r]} \in L^\infty(0,T;H^r), \\ W(0) = \tilde{w}_0 \in H^r, \\ W'(0) = \tilde{w}_1 \in H^r. \end{cases} \tag{3.58}$$

Consequently

$$\begin{aligned} W(t) = {} & \cos\left(\sqrt{\tfrac{1}{\varepsilon}}t\right)\tilde{w}_0 + \sqrt{\varepsilon}\sin\left(\sqrt{\tfrac{1}{\varepsilon}}t\right)\tilde{w}_1 \\ & + \sqrt{\varepsilon}\int_0^t \sin\left(\sqrt{\tfrac{1}{\varepsilon}}(t-s)\right)\Psi^{[r]}(s)ds \in L^\infty(0,T;H^r). \end{aligned} \tag{3.59}$$

By (3.54) and (3.59), it follows that

$$\Delta u^{[r]} = \tfrac{1}{\varepsilon} u^{[r]} - \tfrac{1}{\varepsilon} W \in L^\infty(0,T;H^r). \tag{3.60}$$

Thus

$$u^{[r]} \in L^\infty(0,T;H^{r+2}). \tag{3.61}$$

On the other hand, by $(3.58)_1$, we obtain

$$W'' = -\tfrac{1}{\varepsilon} W + \Psi^{[r]} \in L^\infty(0,T;H^r). \tag{3.62}$$

It follows from (3.52), (3.62) and $r \geq 2$ that

$$\Delta u_{tt}^{[r]} = \tfrac{1}{\varepsilon} u_{tt}^{[r]} - \tfrac{1}{\varepsilon} W'' \in L^\infty(0,T;H^2). \tag{3.63}$$

Thus

$$u_{tt}^{[r]} \in L^\infty(0,T;H^4). \tag{3.64}$$

Similarly, we also have $u_{tt}^{[r]} \in L^\infty(0,T;H^{2s})$, with $s \in \mathbb{N}$, $2s-2 \leq r < 2s$. Then

$$\Delta u_{tt}^{[r]} = \tfrac{1}{\varepsilon} u_{tt}^{[r]} - \tfrac{1}{\varepsilon} W'' \in L^\infty(0,T;H^r). \tag{3.65}$$

Thus

$$u_{tt}^{[r]} \in L^\infty(0,T;H^{r+2}). \tag{3.66}$$

On the other hand

$$u_t^{[r]} = \tilde{u}_1^{[r]} + \textstyle\int_0^t u_{tt}^{[r]}(s)ds \in L^\infty(0,T;H^{r+2}). \tag{3.67}$$

Combining (3.61), (3.66) and (3.67), by induction arguments on r, we conclude that (3.53) holds. The following theorem follows.

Theorem 3.4. *Let $(H_3^{[r]}) - (H_5^{[r]})$ hold. Then the unique solution $u(x,t)$ of problem* (3.39) *satisfies* (3.53).

4. Asymptotic behavior of solutions as $\varepsilon \to 0_+$

In this part, we assume that $p > 2$, $q > 1$, $\lambda > 0$, $K > 0$ and h_0, h_1, $\tilde{u}_0$, $\tilde{u}_1$, F, g_0, g_1 satisfy the assumptions $(H_2) - (H_5)$. Let $\varepsilon > 0$. By Theorem 2.4, problem(1.1)–(1.4) has a unique weak solution $u = u_\varepsilon$ depending on ε.

We consider the following perturbed problem, where ε is a small parameter:

$$(P_\varepsilon) \begin{cases} u_{tt} - u_{xx} - \varepsilon u_{xxtt} + \lambda \left|u_t\right|^{q-2} u_t + K \left|u\right|^{p-2} u = F(x,t), \\ \qquad\qquad 0 < x < 1, 0 < t < T, \\ (-1)^i \left[\varepsilon u_x''(i,t) + u_x(i,t)\right] = h_i u(i,t) + g_i(t), \\ u(0) = \tilde{u}_0,\ u'(0) = \tilde{u}_1. \end{cases} \tag{4.1}$$

We shall study asymptotic behavior of the solution of (P_ε) as $\varepsilon \to 0_+$.

Theorem 4.1. *Let $T > 0$, $p > 2$, $q > 1$, $\lambda > 0$, $K > 0$. Let $(H_2) - (H_5)$ hold. Then*

(i) *Problem* (P_0) *corresponding to* $\varepsilon = 0$ *has a unique weak solution* $\bar{u}_0$ *satisfying*

$$\bar{u}_0 \in L^\infty\left(0, T; H^1\right), \ \bar{u}_0' \in L^\infty\left(0, T; L^2\right). \tag{4.2}$$

(ii) *The solution* u_ε *converges to* $\bar{u}_0$*, as* $\varepsilon \to 0_+$*, in the following sense*

$$\begin{aligned} u_\varepsilon &\to \bar{u}_0 \quad \text{in} \quad L^\infty(0,T;H^1) \quad \text{weakly*}, \\ u_\varepsilon' &\to \bar{u}_0' \quad \text{in} \quad L^\infty(0,T;L^2) \quad \text{weakly*}. \end{aligned}$$

(iii) *If* $\bar{u}_0'' \in L^2(0,T;H^2)$*, then solution* u_ε *converges strongly in* $W(Q_T)$ *to* $\bar{u}_0$*, as* $\varepsilon \to 0_+$*, where*

$$W(Q_T) = \{v \in L^\infty(0,T;H^1) : v' \in L^\infty(0,T;L^2)\}. \tag{4.3}$$

Furthermore, we have the estimation

$$\|u_\varepsilon' - \bar{u}_0'\|_{L^\infty(0,T;L^2)} + \|u_\varepsilon - \bar{u}_0\|_{L^\infty(0,T;H^1)} \le C_T\sqrt{\varepsilon}, \tag{4.4}$$

where C_T *is a posistive constant depending only on* T*.*

Proof. First, we note that if the small parameter $\varepsilon > 0$ satisfy $0 < \varepsilon < 1$ then a priori estimates of the sequence $\{u_m\}$ in the proof of Theorem 2.4 for Prob. (P_ε) satisfy

$$\begin{aligned} &\|u_m'(t)\|^2 + \|u_m(t)\|_{H^1}^2 + \varepsilon\,\|u_{mx}'(t)\|^2 \\ &+ \|u_m(t)\|_{L^p}^p + \lambda \textstyle\int_0^t \|u_m'(s)\|_{L^q}^q\,ds \le C_T, \end{aligned} \tag{4.5}$$

for all $t \in [0,T]$ and for all m, and C_T is a constant depending only on T, p, K, h_0, h_1, $\tilde{u}_0$, $\tilde{u}_1$, F, g_0, g_1 (independent of ε). Hence, the limit $u = u_\varepsilon$ of the sequence $\{u_m\}$ as $m \to +\infty$, in suitable function spaces is a unique weak solution of Prob. (P_ε) satisfying

$$\begin{aligned} &\|u_\varepsilon'(t)\|^2 + \|u_\varepsilon(t)\|_{H^1}^2 + \varepsilon\,\|u_{\varepsilon x}'(t)\|^2 \\ &+ \|u_\varepsilon(t)\|_{L^p}^p + \lambda \textstyle\int_0^t \|u_\varepsilon'(s)\|_{L^q}^q\,ds \le C_T, \end{aligned} \tag{4.6}$$

for all $t \in [0,T]$ and for all $\varepsilon \in (0,1)$.

Let $\{\varepsilon_m\}$ be a sequence such that $\varepsilon_m > 0$, $\varepsilon_m \to 0$ as $m \to +\infty$. We put $u_m = u_{\varepsilon_m}$, we deduce from (4.6), that, there exists a subsequence of the sequence $\{u_m\}$ still denoted by $\{u_m\}$, such that

$$\begin{cases} u_m \to \bar{u}_0 & \text{in} \quad L^\infty(0,T;H^1) \quad \text{weakly*}, \\ u'_m \to \bar{u}'_0 & \text{in} \quad L^\infty(0,T;L^2) \quad \text{weakly*}, \\ \sqrt{\varepsilon_m} u'_m \to \zeta & \text{in} \quad L^\infty(0,T;H^1) \quad \text{weakly*}, \\ u_m \to \bar{u}_0 & \text{in} \quad L^\infty(0,T;L^p) \quad \text{weakly*}, \\ u'_m \to \bar{u}'_0 & \text{in} \quad L^q(Q_T) \quad \text{weakly}, \\ |u_m|^{p-2}\, u_m \to \chi_0 & \text{in} \quad L^\infty(0,T;L^{p'}) \quad \text{weakly*}, \\ |u'_m|^{p-2}\, u'_m \to \chi_1 & \text{in} \quad L^{q'}(Q_T) \quad \text{weakly}. \end{cases} \tag{4.7}$$

By the compactness lemma of Lions ([7]: p. 57), we can deduce from $(4.7)_{1,2}$ the existence of a subsequence still denoted by $\{u_m\}$, such that

$$u_m \to \bar{u}_0 \quad \text{strongly in } L^2(Q_T) \text{ and a.e. in } \quad Q_T. \tag{4.8}$$

It follows from $(4.7)_{2,3}$, that $\zeta = 0$. Hence, we obtain from $(4.7)_3$ that

$$\sqrt{\varepsilon_m} u'_m \to 0 \quad \text{in } L^\infty(0,T;H^1) \text{ weakly*}. \tag{4.9}$$

Similarly

$$|u_m|^{p-2} u_m \to |\bar{u}_0|^{p-2} \bar{u}_0 \quad \text{strongly in} \quad L^2(Q_T), \tag{4.10}$$

and

$$\chi_1 = |\bar{u}'_0|^{q-2} \bar{u}'_0. \tag{4.11}$$

By passing to the limit, as in the proof of Theorem 2.3, we conclude that $\bar{u}_0$ is a unique weak solution of Prob. (P_0) corresponding to $\varepsilon = 0$ satisfying

$$\bar{u}_0 \in L^\infty\left(0,T;H^1\right), \; \bar{u}'_0 \in L^\infty\left(0,T;L^2\right). \tag{4.12}$$

Hence, (i) and (ii) are proved.

Next, put $u = u_\varepsilon - \bar{u}_0$, then u is the weak solution of the following problem

$$(P_\varepsilon) \begin{cases} u'' - \Delta u - \varepsilon \Delta u'' + \lambda \left(|u'_\varepsilon|^{q-2}\, u'_\varepsilon - |\bar{u}'_0|^{q-2}\, \bar{u}'_0 \right) \\ +K \left(|u_\varepsilon|^{p-2}\, u_\varepsilon - |\bar{u}_0|^{p-2}\, \bar{u}_0 \right) = \varepsilon \Delta \bar{u}''_0, \; 0 < x < 1, \; 0 < t < T, \\ (-1)^i \left[\varepsilon u''_x(i,t) + u_x(i,t)\right] = h_i u(i,t) - (-1)^i \varepsilon \bar{u}''_{0x}(i,t), \\ u(0) = u'(0) = 0. \end{cases} \tag{4.13}$$

Using again Lemma 2.3, we prove in a manner similar to the above part and the result is

$$\begin{aligned} \sigma(t) = 2\varepsilon \textstyle\int_0^t \langle \Delta \bar{u}''_0, u'(s) \rangle\, ds + 2\varepsilon \sum_{i=0}^1 \int_0^t (-1)^i \bar{u}''_{0x}(i,s) u'(i,s) ds \\ -2K \textstyle\int_0^t \left\langle |u_\varepsilon(s)|^{p-2}\, u_\varepsilon(s) - |\bar{u}_0(s)|^{p-2}\, \bar{u}_0(s), u'(s) \right\rangle ds, \end{aligned} \tag{4.14}$$

where

$$\begin{aligned}\sigma(t) = \|u'(t)\|^2 + \varepsilon\|u'_x(t)\|^2 + \|u_x(t)\|^2 + \textstyle\sum_{i=0}^1 h_i u^2(i,t) \\ +2\lambda\int_0^t \left\langle |u'_\varepsilon(s)|^{q-2} u'_\varepsilon(s) - |\bar{u}'_0(s)|^{q-2}\bar{u}'_0(s), u'(s)\right\rangle ds.\end{aligned} \tag{4.15}$$

Note that

$$\begin{cases} \int_0^t \left\langle |u'_\varepsilon(s)|^{q-2} u'_\varepsilon(s) - |\bar{u}'_0(s)|^{q-2}\bar{u}'_0(s), u'(s)\right\rangle \geq 0, \\ \sigma(t) \geq \|u'(t)\|^2 + \varepsilon\|u'_x(t)\|^2 \geq \varepsilon\|u'(t)\|^2_{H^1}, \\ \sigma(t) \geq \|u'(t)\|^2 + \alpha_0\|u(t)\|^2_{H^1} \geq 2\sqrt{\alpha_0}\,\|u(t)\|_{H^1}\|u'(t)\|. \end{cases} \tag{4.16}$$

By (2.41), (4.6), (4.16), we estimate all terms in the righthand side of (4.14) as follows

$$\begin{aligned} 2\varepsilon\int_0^t \langle\Delta\bar{u}''_0(s), u'(s)\rangle ds &\leq 2\varepsilon\int_0^t \|\Delta\bar{u}''_0(s)\|\,\|u'(s)\|\,ds \\ &\leq 2\varepsilon\int_0^t \|\bar{u}''_0(s)\|_{H^2}\|u'(s)\|\,ds \\ &\leq \varepsilon^2\int_0^t \|\bar{u}''_0(s)\|^2_{H^2} ds + \int_0^t \|u'(s)\|^2 ds \\ &\leq \varepsilon^2\|\bar{u}''_0\|^2_{L^2(0,T;H^2)} + \int_0^t \sigma(s)ds, \end{aligned} \tag{4.17}$$

$$\begin{aligned} &2\varepsilon\textstyle\sum_{i=0}^1 \int_0^t (-1)^i \bar{u}''_{0x}(i,s)u'(i,s)ds \\ &\leq 8\varepsilon\int_0^t \|\bar{u}''_{0x}(s)\|_{H^1}\|u'(s)\|_{H^1} ds \\ &\leq 8\varepsilon\int_0^t \|\bar{u}''_0(s)\|_{H^2}\|u'(s)\|_{H^1} ds \\ &\leq 16\varepsilon\int_0^t \|\bar{u}''_0(s)\|^2_{H^2} ds + \varepsilon\int_0^t \|u'(s)\|^2_{H^1} ds \\ &\leq \varepsilon\|\bar{u}''_0\|^2_{L^2(0,T;H^2)} + \int_0^t \sigma(s)ds, \end{aligned} \tag{4.18}$$

$$\begin{aligned} &-2K\int_0^t \left\langle |u_\varepsilon(s)|^{p-2} u_\varepsilon(s) - |\bar{u}_0(s)|^{p-2}\bar{u}_0(s), u'(s)\right\rangle ds \\ &\leq 2K(p-1)C_T^{p-2}\int_0^t \|u(s)\|\,\|u'(s)\|\,ds \\ &\leq K(p-1)C_T^{p-2}\frac{1}{\sqrt{\alpha_0}}\int_0^t \sigma(s)ds. \end{aligned} \tag{4.19}$$

Combining (4.14), (4.17), (4.18), (4.19), the result is

$$\sigma(t) \leq 2\varepsilon\|\bar{u}''_0\|^2_{L^2(0,T;H^2)} + \left[2 + (p-1)C_T^{p-2}\frac{K}{\sqrt{\alpha_0}}\right]\int_0^t \sigma(s)ds. \tag{4.20}$$

By Gronwall's Lemma, (4.20) leads to

$$\begin{aligned} \sigma(t) &\leq 2\varepsilon\|\bar{u}''_0\|^2_{L^2(0,T;H^2)} \exp\left(T\left[2 + (p-1)C_T^{p-2}\frac{K}{\sqrt{\alpha_0}}\right]\right) \\ &\equiv \bar{C}_T\varepsilon, \quad \forall t \in [0,T]. \end{aligned} \tag{4.21}$$

Consequently

$$\|u'_\varepsilon - \bar{u}'_0\|_{L^\infty(0,T;L^2)} + \|u_\varepsilon - \bar{u}_0\|_{L^\infty(0,T;H^1)} \leq C_T\sqrt{\varepsilon}, \tag{4.22}$$

where C_T is a constant depending only on T. Thus, (iii) is proved. Theorem 4.1 is proved completely. □

Acknowledgments: The authors wish to express their sincere thanks to the referees and the editor for their valuable comments and remarks.

References

[1] F.E. Browder, *On nonlinear wave equations*, Math. Z., **80** (1962), 249–264.
[2] E.L.A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, 1955, p. 43.
[3] A. Chattopadhyay, S. Gupta, A.K. Singh and S.A. Sahu, *Propagation of shear waves in an irregular magnetoelastic monoclinic layer sandwiched between two isotropic half-spaces*, International Journal of Engineering, Science and Technology, **1**(1) (2009), 228–244.
[4] Subhas Dutta, *On the propagation of Love type waves in an infinite cylinder with rigidity and density varying linearly with the radial distance*, Pure and Applied Geophysics, **98**(1) (1972), 35–39.
[5] J.L. Lions and W.A. Strauss, *Some nonlinear evolution equations*, Bull. Soc. Math., France, **93** (1965), 43–96.
[6] J.L. Lions, *Equations differentielles operationelles et problèmes aux limites*, Springer-Verlag, Berlin, 1961, p. 96, Section 7.
[7] J.L. Lions, *Quelques méthodes de ré solution des problèmes aux limites nonlinéaires*, Dunod; Gauthier–Villars, Paris, 1969.
[8] Mrinal K. Paul, *On propagation of love-type waves on a spherical model with rigidity and density both varying exponentially with the radial distance*, Pure and Applied Geophysics, **59**(1) (1964), 33–37.
[9] Věra Radochová, *Remark to the comparison of solution properties of Love's equation with those of wave equation*, Applications of Mathematics, **23**(3) (1978), 199–207.