Nonlinear Functional Analysis and Applications
Vol. 20, No. 1 (2015), pp. 27-42

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

SOME RESULTS ON COUPLED BEST PROXIMITY POINTS IN ORDERED METRIC SPACES

V. Pragadeeswarar[1], Chayut Kongban[2], M. Marudai[3] and Poom Kumam[4]

[1]Department of Mathematics, Bharathidasan University
Tiruchirappalli 620 024, Tamil Nadu, India
e-mail: pragadeeswarar@gmail.com

[2]Department of Mathematics, Faculty of Science
King Mongkuts University of Technology Thonburi (KMUTT)
Bang Mod, Thrung Khru, Bangkok 10140, Thailand
e-mail: chayut_kb@hotmail.com

[3]Department of Mathematics, Bharathidasan University
Tiruchirappalli 620 024, Tamil Nadu, India
e-mail: marudaim@hotmail.com

[4]Department of Mathematics, Faculty of Science
King Mongkuts University of Technology Thonburi (KMUTT)
Bang Mod, Thrung Khru, Bangkok 10140, Thailand
e-mail: poom.kum@kmutt.ac.th

Abstract. The main purpose of this work is to prove the existence and uniqueness of coupled best proximity point for mappings satisfying proximally coupled weak contraction in a complete ordered metric space. Further, our result provides an extension of a result due to J. Harjani, B. López and K. Sadarangani.

1. Introduction and preliminaries

Let A be nonempty subset of a metric space (X, d) and $T : A \to X$ has a fixed point in A if the fixed point equation $Tx = x$ has at least one solution. That is, $x \in A$ is a fixed point of T if $d(x, Tx) = 0$. If the fixed point equation $Tx = x$ does not possess a solution, then $d(x, Tx) > 0$ for all $x \in A$. In

[0]Received April 25, 2014. Revised November 17, 2014.
[0]2010 Mathematics Subject Classification: 47H09, 47H10.
[0]Keywords: Partially ordered set, optimal approximate solution, proximally increasing mapping, coupled fixed point, coupled best proximity point.

such a situation, it is our aim to find an element $x \in A$ such that $d(x, Tx)$ is minimum in some sense. The best approximation theory and best proximity pair theorems are studied in this direction. Here we state the following well-known best approximation theorem due to Ky Fan [5].

Theorem 1.1. ([5]) *Let A be a nonempty compact convex subset of a normed linear space X and $T : A \to X$ be a continuous function. Then there exists $x \in A$ such that $\|x - Tx\| = d(Tx, A) := \inf\{\|Tx - a\| : a \in A\}$.*

Such an element $x \in A$ in Theorem 1.1 is called a best approximant of T in A. Note that if $x \in A$ is a best approximant, then $\|x - Tx\|$ need not be the optimum. Best proximity point theorems have been explored to find sufficient conditions so that the minimization problem $\min_{x \in A} \|x - Tx\|$ has at least one solution. To have a concrete lower bound, let us consider two nonempty subsets A, B of a metric space X and a mapping $T : A \to B$. The natural question is whether one can find an element $x_0 \in A$ such that $d(x_0, Tx_0) = \min\{d(x, Tx) : x \in A\}$. Since $d(x, Tx) \geq d(A, B)$,the optimal solution to the problem of minimizing the real valued function $x \to d(x, Tx)$ over the domain A of the mapping T will be the one for which the value $d(A, B)$ is attained. A point $x_0 \in A$ is called a best proximity point of T if $d(x_0, Tx_0) = d(A, B)$. Note that if $d(A, B) = 0$, then the best proximity point is nothing but a fixed point of T.

The existence and convergence of best proximity points is an interesting topic of optimization theory which recently attracted the attention of many authors [3, 4, 9, 10, 15, 17, 18, 21]. Also one can find the existence of best proximity point in the setting of partially order metric space in [1, 2, 13, 14, 16, 20].

On the other hand, Bhaskar and Lakshmikantham were initiated the concept called mixed monotone mapping and proved some coupled fixed point theorems for mappings satisfying the mixed monotone property which is used to investigate a large class of problems and discussed the existence and uniqueness of a solution for a periodic boundary value problem. For more details on this concept one may go through the references [6, 7, 11, 12, 19].

More precisely about the definition of coupled fixed point, let X be a nonempty set and $F : X \times X \to X$ be a given mapping. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping F if $F(x, y) = x$ and $F(y, x) = y$.

They also introduced the notion of mixed monotone mapping. If $(X, \leq)$ is a partially ordered set, the mapping F is said to have the mixed monotone property if

$$x_1, x_2 \in X, \;\; x_1 \leq x_2 \quad \Rightarrow \quad F(x_1, y) \leq F(x_2, y), \;\; \forall y \in X$$

and

$$y_1, y_2 \in X, \;\; y_1 \leq y_2 \quad \Rightarrow \quad F(x, y_1) \geq F(x, y_2), \;\; \forall x \in X.$$

Khan et al. [8] introduced the use of control function in metric fixed point theory, which they called an altering distance function is follows:

Definition 1.2. A function $\psi : [0, \infty) \to [0, \infty)$ is said to be an altering distance function or control functions if it satisfies the following conditions.

(i) ψ is continuous and non-decreasing.
(ii) $\psi(t) = 0$ if and only if $t = 0$.

The main theoretical results of Harjani et.al in [7] are the following two coupled fixed point theorems.

Theorem 1.3. ([7]) *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be mapping having the mixed monotone property on X and continuous such that*

$$\begin{aligned} &\psi(d(F(x, y), F(u, v))) \\ &\leq \psi(\max(d(x, u), d(y, v))) - \phi(\max(d(x, u), d(y, v))), \end{aligned} \tag{1.1}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$, where ψ and ϕ are altering distance functions. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$.

Theorem 1.4. ([7]) *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property:*

(i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,*
(ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \geq y_n$ for all n.*

Let $F : X \times X \to X$ be mapping having the mixed monotone property on X and continuous such that

$$\psi(d(F(x, y), F(u, v))) \leq \psi(\max(d(x, u), d(y, v))) - \phi(\max(d(x, u), d(y, v))),$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$, where ψ and ϕ are altering distance functions. If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$.

Motivated by the above theorems, we introduce the concept of proximal mixed monotone property and proximally coupled weak contraction. We also explore the existence and uniqueness of coupled best proximity points in the setting of partially ordered metric spaces, thereby producing optimal approximate solutions for that function with respect to both coordinates. Further, we attempt to give the generalization of the Theorem 1.3 and Theorem 1.4.

Let X be a non-empty set such that $(X, \leq)$ is a poset and (X, d) is a metric space. Unless otherwise specified, it is assumed throughout this section that A and B are non-empty subsets of the metric space (X, d), the following notions are used subsequently:

$$d(A, B) := \inf\{d(x, y) : x \in A \text{ and } y \in B\},$$

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

In [10], the authors discussed sufficient conditions which guarantee the non-emptiness of A_0 and B_0. Also, in [15], the authors proved that A_0 is contained in the boundary of A.

Definition 1.5. Let $(X, d, \leq)$ be a ordered metric space and A, B are nonempty subset of X. A mapping $F : A \times A \to B$ is said to be proximal mixed monotone property if $F(x, y)$ is proximally nondecreasing in x and is proximally non-increasing in y, that is, for all $x, y \in A$.

$$\left.\begin{array}{r} x_1 \leq x_2 \\ d(u_1, F(x_1, y)) = d(A, B) \\ d(u_2, F(x_2, y)) = d(A, B) \end{array}\right\} \implies u_1 \leq u_2$$

and

$$\left.\begin{array}{r} y_1 \leq y_2 \\ d(u_3, F(x, y_1)) = d(A, B) \\ d(u_4, F(x, y_2)) = d(A, B) \end{array}\right\} \implies u_4 \leq u_3.$$

where $x_1, x_2, y_1, y_2, u_1, u_2, u_3, u_4 \in A$.

One can see that, if $A = B$ in the above definition, the notion of proximal mixed monotone property reduces to that of mixed monotone property.

Lemma 1.6. *Let $(X, d, \leq)$ be an ordered metric space and A, B are nonempty subset of X. Assume A_0 is nonempty. A mapping $F : A \times A \to B$ has proximal mixed monotone property with $F(A_0 \times A_0) \subseteq B_0$ then for any x_0, x_1, x_2, y_0 and y_1 are elements in A_0*

$$\left.\begin{array}{r} x_0 \leq x_1 \text{ and } y_0 \geq y_1 \\ d(x_1, F(x_0, y_0)) = d(A, B) \\ d(x_2, F(x_1, y_1)) = d(A, B) \end{array}\right\} \implies x_1 \leq x_2. \tag{1.2}$$

Proof. By hypothesis $F(A_0 \times A_0) \subseteq B_0$, therefore $F(x_1, y_0) \in B_0$. Hence there exists $x_1^* \in A$ such that

$$d(x_1^*, F(x_1, y_0)) = d(A, B). \tag{1.3}$$

Using F is proximal mixed monotone (In particular F is proximally non decreasing in x) to (1.2) and (1.3), we get

$$\left.\begin{aligned} x_0 \leq x_1 \\ d(x_1, F(x_0, y_0)) = d(A, B) \\ d(x_1^*, F(x_1, y_0)) = d(A, B) \end{aligned}\right\} \Longrightarrow x_1 \leq x_1^*. \tag{1.4}$$

Analogously , using F is proximal mixed monotone (In particular F is proximally non increasing in y) to (1.2) and (1.3), we get

$$\left.\begin{aligned} y_1 \leq y_0 \\ d(x_2, F(x_1, y_1)) = d(A, B) \\ d(x_1^*, F(x_1, y_0)) = d(A, B) \end{aligned}\right\} \Longrightarrow x_1^* \leq x_2. \tag{1.5}$$

From (1.4) and (1.5), one can conclude the $x_1 \leq x_2$. Hence the proof. ☐

Lemma 1.7. *Let $(X, d, \leq)$ be an ordered metric space and A, B are nonempty subset of X. Assume A_0 is nonempty. A mapping $F : A \times A \to B$ has proximal mixed monotone property with $F(A_0 \times A_0) \subseteq B_0$ then for any x_0, x_1, y_0, y_1 and y_2 are elements in A_0*

$$\left.\begin{aligned} x_0 \leq x_1 \text{ and } y_0 \geq y_1 \\ d(y_1, F(y_0, x_0)) = d(A, B) \\ d(y_2, F(y_1, x_1)) = d(A, B) \end{aligned}\right\} \Longrightarrow y_1 \geq y_2. \tag{1.6}$$

Proof. The prove is same as the Lemma 1.6. ☐

Definition 1.8. Let $(X, d, \leq)$ be an ordered metric space and A, B are nonempty subset of X. A mapping $F : A \times A \to B$ is said to be proximally coupled weak contraction if it satisfies the following condition:

$$\begin{aligned} &\left.\begin{aligned} x_1 \leq x_2 \text{ and } y_1 \geq y_2 \\ d(u_1, F(x_1, y_1)) = d(A, B) \\ d(u_2, F(x_2, y_2)) = d(A, B) \end{aligned}\right\} \\ &\Longrightarrow \psi(d(u_1, u_2)) \leq \psi(\max(d(x_1, x_2), d(y_1, y_2))) \\ &\qquad\qquad - \phi(\max(d(x_1, x_2), d(y_1, y_2))) \end{aligned} \tag{1.7}$$

for all $x_1, x_2, y_1, y_2, u_1, u_2 \in A$, where ψ is altering distance function, ϕ is nondecreasing function also $\phi(t) = 0$ iff $t = 0$.

One can see that, if $A = B$ in the above definition, the notion of proximally coupled weak contraction reduces to that coupled weak contraction (or equation (1.1)).

2. Main Results

Let $(X, d, \leq)$ be a partially ordered complete metric space. Further, we endow the product space $X \times X$ with the following partial order:

$$\text{for } (x,y),(u,v) \in X \times X, \ \ (u,v) \leq (x,y) \quad \Leftrightarrow \quad x \geq u, \ \ y \leq v.$$

Theorem 2.1. *Let $(X, \leq, d)$ be a partially ordered complete metric space. Let A and B be non-empty closed subsets of the metric space (X, d) such that $A_0 \neq \emptyset$. Let $F : A \times A \to B$ satisfy the following conditions.*

(i) *F is continuous having the proximal mixed monotone property and proximally coupled weak contraction on A such that $F(A_0 \times A_0) \subseteq B_0$.*

(ii) *There exist elements (x_0, y_0) and (x_1, y_1) in $A_0 \times A_0$ such that*

$$d(x_1, F(x_0, y_0)) = d(A, B) \text{ with } x_0 \leq x_1 \text{ and}$$
$$d(y_1, F(y_0, x_0)) = d(A, B) \text{ with } y_0 \geq y_1.$$

Then there exist $(x, y) \in A \times A$ such that $d(x, F(x, y)) = d(A, B)$ and $d(y, F(y, x)) = d(A, B)$.

Proof. By hypothesis there exist elements (x_0, y_0) and (x_1, y_1) in $A_0 \times A_0$ such that

$$d(x_1, F(x_0, y_0)) = d(A, B) \text{ with } x_0 \leq x_1 \text{ and}$$
$$d(y_1, F(y_0, x_0)) = d(A, B) \text{ with } y_0 \geq y_1.$$

Because of the fact that $F(A_0 \times A_0) \subseteq B_0$, there exists an element (x_2, y_2) in $A_0 \times A_0$ such that

$$d(x_2, F(x_1, y_1) = d(A, B) \text{ and}$$
$$d(y_2, F(y_1, x_1) = d(A, B).$$

Hence from Lemma 1.6 and Lemma 1.7, we obtain $x_1 \leq x_2$ and $y_1 \geq y_2$.
Continuing this process, we can construct the sequences (x_n) and (y_n) in A_0 such that

$$d(x_{n+1}, F(x_n, y_n)) = d(A, B), \ \forall n \in \mathbb{N} \tag{2.1}$$
$$\text{with } x_0 \leq x_1 \leq x_2 \leq \cdots x_n \leq x_{n+1} \cdots \text{ and}$$
$$d(y_{n+1}, F(y_n, x_n)) = d(A, B), \ \forall n \in \mathbb{N} \tag{2.2}$$
$$\text{with } y_0 \geq y_1 \geq y_2 \geq \cdots y_n \geq y_{n+1} \cdots .$$

Since $d(x_n, F(x_{n-1}, y_{n-1})) = d(A, B)$, $d(x_{n+1}, F(x_n, y_n)) = d(A, B)$ and also we have $x_{n-1} \leq x_n$, $y_{n-1} \geq y_n$, $\forall n \in \mathbb{N}$. Now using F is proximally coupled weak contraction on A we get,

$$\begin{aligned} \psi(d(x_n, x_{n+1})) \leq\ & \psi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))) \\ & - \phi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))). \end{aligned} \tag{2.3}$$

As $\phi \geq 0$,

$$\psi(d(x_n, x_{n+1})) \leq \psi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n)))$$

and, using the fact that ϕ is nondecreasing, we have

$$d(x_n, x_{n+1}) \leq \max(d(x_{n-1}, x_n), d(y_{n-1}, y_n)). \tag{2.4}$$

Similarly, since $x_{n-1} \leq x_n$, $y_{n-1} \geq y_n$, we get

$$\begin{aligned}\psi(d(y_n, y_{n+1})) &\leq \psi(\max(d(y_{n-1}, y_n), d(x_{n-1}, x_n))) \\ &\quad - \phi(\max(d(y_{n-1}, y_n), d(x_{n-1}, x_n))), \\ &\leq \psi(\max(d(y_{n-1}, y_n), d(x_{n-1}, x_n)))\end{aligned} \tag{2.5}$$

and consequently,

$$d(y_n, y_{n+1}) \leq \max(d(y_{n-1}, y_n), d(x_{n-1}, x_n)). \tag{2.6}$$

By (2.4) and (2.6),we get

$$\max(d(x_n, x_{n+1}), d(y_n, y_{n+1})) \leq \max(d(x_{n-1}, x_n), d(y_{n-1}, y_n)),$$

and, thus, the sequence $\{\max(d(x_n, x_{n+1}), d(y_n, y_{n+1}))\}$ is nonnegative decreasing. This implies that there exists $r \geq 0$ such that

$$\lim_{n\to\infty} \max(d(x_n, x_{n+1}), d(y_n, y_{n+1})) = r. \tag{2.7}$$

One can see that if $\psi : [0, \infty] \to [0, \infty]$ is nondecreasing, $\psi(\max(a, b)) = \max(\psi(a), \psi(b))$ for $a, b \in [0, \infty]$. Taking into account this and (2.3) and (2.5), we get

$$\begin{aligned}\max(\psi(d(x_n, x_{n+1})), \psi(d(y_n, y_{n+1}))) &= \psi(\max(d(x_n, x_{n+1}), d(y_n, y_{n+1}))) \\ &\leq \psi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))) \\ &\quad - \phi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))) \\ &\leq \psi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))).\end{aligned}$$

Letting $n \to \infty$ and taking into account (2.7), we get

$$\psi(r) \leq \psi(r) - \lim_{n\to\infty} \phi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))) \leq \psi(r)$$

and this implies

$$\lim_{n\to\infty} \phi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))) = 0. \tag{2.8}$$

But, as $0 < r \leq \max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))$ and ϕ is nondecreasing function,

$$0 < \phi(r) \leq \phi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))),$$

and this gives us $\lim_{n\to\infty} \phi(\max(d(x_{n-1}, x_n), d(y_{n-1}, y_n))) \geq \phi(r) > 0$ which contradicts to (2.8). Hence,

$$\lim_{n\to\infty} \max(d(x_n, x_{n+1}), d(y_n, y_{n+1})) = 0. \tag{2.9}$$

Now to prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences. Assume that at least one of the sequences $\{x_n\}$ or $\{y_n\}$ is not a Cauchy sequence. This implies that $\lim_{n,m\to\infty} d(x_n, x_m) \nrightarrow 0$ or $\lim_{n,m\to\infty} d(y_n, y_m) \nrightarrow 0$, and, consequently,

$$\lim_{n,m\to\infty} \max(d(x_n, x_m), d(y_n, y_m)) \nrightarrow 0.$$

Then there exists $\epsilon > 0$ for which we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ such that $n(k)$ is smallest index for which $n(k) > m(k) > k$,

$$\max(d(x_{m(k)}, x_{n(k)}), d(y_{m(k)}, y_{n(k)})) \geq \epsilon. \tag{2.10}$$

This means that

$$\max(d(x_{m(k)}, x_{n(k)-1}), d(y_{m(k)}, y_{n(k)-1})) < \epsilon. \tag{2.11}$$

Since $x_{n(k)-1} \geq x_{m(k)-1}$ and $y_{n(k)-1} \leq y_{m(k)-1}$, using the proximally coupled weak contraction, we obtain

$$\begin{aligned} &\psi(d(x_{n(k)}, x_{m(k)})) \\ &\leq \psi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))) \\ &\quad - \phi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} &\psi(d(y_{n(k)}, y_{m(k)})) \\ &\leq \psi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))) \\ &\quad - \phi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))). \end{aligned} \tag{2.13}$$

By (2.12) and (2.13), we get

$$\begin{aligned} &\max(\psi(d(x_{n(k)}, x_{m(k)}), \psi(d(y_{n(k)}, y_{m(k)})) \\ &\leq \psi(max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))) \\ &\quad - \phi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))). \end{aligned} \tag{2.14}$$

On the other hand, the triangular inequality and (2.11) give us

$$\begin{aligned} d(x_{n(k)}, x_{m(k)}) &\leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)}) \\ &< d(x_{n(k)}, x_{n(k)-1}) + \epsilon \end{aligned} \tag{2.15}$$

and

$$\begin{aligned} d(y_{n(k)}, y_{m(k)}) &\leq d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)}) \\ &< d(y_{n(k)}, y_{n(k)-1}) + \epsilon. \end{aligned} \tag{2.16}$$

From (2.10), (2.15) and (2.16), we get

$$\begin{aligned} \epsilon &\leq \max(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) \\ &\leq \max(d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})) + \epsilon. \end{aligned}$$

Letting $k \to \infty$ in the last inequality and using (2.9), we have

$$\lim_{k\to\infty} \max(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) = \epsilon. \tag{2.17}$$

Again, the triangular inequality and (2.11) give us

$$\begin{aligned} d(x_{n(k)-1}, x_{m(k)-1}) &\leq d(x_{n(k)-1}, x_{m(k)}) + d(x_{m(k)}, x_{m(k)-1}) \\ &< \epsilon + d(x_{m(k)}, x_{m(k)-1}) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} d(y_{n(k)-1}, y_{m(k)-1}) &\leq d(y_{n(k)-1}, y_{m(k)}) + d(y_{m(k)}, y_{m(k)-1}) \\ &< \epsilon + d(y_{m(k)}, y_{m(k)-1}). \end{aligned} \tag{2.19}$$

By (2.18) and (2.19), we get

$$\begin{aligned} &\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) \\ &< \max(d(x_{m(k)}, x_{m(k)-1}), d(y_{m(k)}, y_{m(k)-1})) + \epsilon. \end{aligned} \tag{2.20}$$

Using the triangular inequality we have

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)})$$

and

$$d(y_{n(k)}, y_{m(k)}) \leq d(y_{n(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)})$$

and by the two last inequalities and (2.10) we get

$$\begin{aligned} \epsilon &\leq \max(d(x_{n(k)}, x_{m(k)}), d(y_{n(k)}, y_{m(k)})) \\ &\leq \max(d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})) \\ &\quad + \max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) \\ &\quad + \max(d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})). \end{aligned}$$

By (2.20) and (2.21), we get

$$\begin{aligned} &\epsilon - \max(d(x_{n(k)}, x_{n(k)-1}), d(y_{n(k)}, y_{n(k)-1})) \\ &\quad - \max(d(x_{m(k)-1}, x_{m(k)}), d(y_{m(k)-1}, y_{m(k)})) \\ &\leq \max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) \\ &< \max(d(x_{m(k)}, x_{m(k)-1}), d(y_{m(k)}, y_{m(k)-1})) + \epsilon. \end{aligned}$$

Letting $k \to \infty$ in the last inequality and using (2.9), we have

$$\lim_{k\to\infty} \max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) = \epsilon. \tag{2.21}$$

Finally, letting $k \to \infty$ in (2.14) and using (2.17), (2.21) and the continuity of ψ, we get

$$\psi(\epsilon) \leq \psi(\epsilon) - \lim_{k\to\infty} \phi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))) \leq \psi(\epsilon)$$

and this implies

$$\lim_{k\to\infty} \phi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))) = 0. \tag{2.22}$$

But, from $\lim_{k\to\infty} \max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})) = \epsilon$, we can find $k_0 \in \mathbb{N}$ such that for any $k \geq k_0$

$$\frac{\epsilon}{2} \leq \max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))$$

and consequently,

$$0 < \phi(\frac{\epsilon}{2}) \leq \phi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1}))) \text{ for } k \geq k_0.$$

Therefore,

$$0 < \phi(\frac{\epsilon}{2}) \leq \phi(\max(d(x_{n(k)-1}, x_{m(k)-1}), d(y_{n(k)-1}, y_{m(k)-1})))$$

and this contradicts (2.22). Therefore, the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy. Since A is closed subset of a complete metric space X, these sequences have limits. Thus, there exists $x, y \in A$ such that $x_n \to x$ and $y_n \to y$. Therefore $(x_n, y_n) \to (x, y)$ in $A \times A$. Since F is continuous, we have $F(x_n, y_n) \to F(x, y)$ and $F(y_n, x_n) \to F(y, x)$. Hence the continuity of the metric function d implies that $d(x_{n+1}, F(x_n, y_n)) \to d(x, F(x, y))$ and $d(y_{n+1}, F(y_n, x_n)) \to d(y, F(y, x))$.

But from equations (2.1) and (2.2) we get, the sequences $d(x_{n+1}, F(x_n, y_n))$ and $d(y_{n+1}, F(y_n, x_n))$ are constant sequences with the value $d(A, B)$. Therefore, $d(x, F(x, y)) = d(A, B)$ and $d(y, F(y, x)) = d(A, B)$. This completes the proof of the theorem. □

Corollary 2.2. *Let $(X, \leq, d)$ be a partially ordered complete metric space. Let A be non-empty closed subsets of the metric space (X, d) . Let $F : A \times A \to A$ satisfy the following conditions.*

(i) *F is continuous having the proximal mixed monotone property and proximally coupled weak contraction on A.*
(ii) *There exist (x_0, y_0) and (x_1, y_1) in $A \times A$ such that $x_1 = F(x_0, y_0)$ with $x_0 \leq x_1$ and $y_1 = F(y_0, x_0)$ with $y_0 \geq y_1$.*

Then there exist $(x, y) \in A \times A$ such that $d(x, F(x, y)) = 0$ and $d(y, F(y, x)) = 0$.

In what follows we prove that Theorem 2.1 is still valid for F not necessarily continuous, assuming the following hypothesis in A. A has the property that

$$\begin{aligned} &\{x_n\} \text{ is a non-decreasing sequence in } A \text{ such that } x_n \to x, \\ &\text{then } x_n \leq x. \end{aligned} \tag{2.23}$$

$$\{y_n\} \text{ is a non-increasing sequence in } A \text{ such that } y_n \to y, \text{ then } y \leq y_n. \tag{2.24}$$

Theorem 2.3. *Assume the condition* (2.23), (2.24) *and* A_0 *is closed in* X *instead of continuity of* F *in the Theorem* 2.1.

Proof. Following the proof of Theorem 2.1, there exists sequences $\{x_n\}$ and $\{y_n\}$ in A satisfying the following condition

$$d(x_{n+1}, F(x_n, y_n)) = d(A, B) \text{ with } x_n \leq x_{n+1}, \quad \forall n \in \mathbb{N} \tag{2.25}$$

and

$$d(y_{n+1}, F(y_n, x_n)) = d(A, B) \text{ with } y_n \geq y_{n+1}, \quad \forall n \in \mathbb{N}. \tag{2.26}$$

Also x_n converges to x and y_n converges to y in A. From (2.23) and (2.24), we get $x_n \leq x$ and $y_n \geq y$. Note that the sequences $\{x_n\}$ and $\{y_n\}$ are in A_0 and A_0 is closed. Therefore, $(x, y) \in A_0 \times A_0$. Since $F(A_0 \times A_0) \subseteq B_0$, we get $F(x, y)$ and $F(y, x)$ are in B_0. Therefore, there exists $(x^*, y^*) \in A_0 \times A_0$ such that

$$d(x^*, F(x, y)) = d(A, B) \tag{2.27}$$

and

$$d(y^*, F(y, x)) = d(A, B). \tag{2.28}$$

Since $x_n \leq x$ and $y_n \geq y$. By using F is proximally coupled weak contraction for (2.25) and (2.27), we get

$$\begin{aligned} \psi(d(x_{n+1}, x^*)) \leq \psi(&\max(d(x_n, x), d(y_n, y))) \\ &- \phi(\max(d(x_n, x), d(y_n, y))). \end{aligned} \tag{2.29}$$

Letting $n \to \infty$ in (2.29) and using continuity of ψ, we get

$$\psi(d(x, x^*)) \leq 0 - \lim_{n \to \infty} \phi(\max(d(y, y_n), d(x, x_n))) \leq 0.$$

Using $\psi(t) = 0$ iff $t = 0$, we get $d(x, x^*) = 0$, consequently, $x = x^*$. Similarly it can be proved that $y = y*$. Using these to (2.27) and (2.28), we get $d(x, F(x, y)) = d(A, B)$ and $d(y, F(y, x)) = d(A, B)$. □

Corollary 2.4. *Assume the condition* (2.23) *and* (2.24) *instead of continuity of* F *in the Corollary* 2.2.

Now, we present an example where it can be appreciated that hypotheses in Theorem 2.1 and Theorem 2.3 do not guarantee uniqueness of the coupled best proximity point.

Example 2.5. Let $X = \{(0,1),(1,0),(-1,0),(0,-1)\} \subset \mathbb{R}^2$ and consider the usual order $(x,y) \preceq (z,t) \Leftrightarrow x \leq z$ and $y \leq t$.

Then, $(X, \preceq)$ is a partially ordered set. Besides, (X, d_2) is a complete metric space considering d_2 the euclidean metric. Let $A = \{(0,1),(1,0)\}$ and $B = \{(0,-1),(-1,0)\}$ be a closed subset of X. Then, $d(A,B) = \sqrt{2}, A = A_0$ and $B = B_0$. Let $F : A \times A \to B$ be defined as $F((x_1,x_2),(y_1,y_2)) = (-x_2,-x_1)$. Then, it can be seen that F is continuous such that $F(A_0 \times A_0) \subseteq B_0$. The only comparable pairs of points in A are $x \preceq x$ for $x \in A$, hence proximal mixed monotone property is satisfied trivially and also proximally coupled weak contraction is fulfilled for arbitrary control functions.

It can be shown that the other hypotheses of the theorem are also satisfied. However, F has three coupled best proximity points $((0,1),(0,1)), ((0,1),(1,0))$ and $((1,0),(1,0))$.

One can prove that the coupled best proximity point is in fact unique, provided that the product space $A \times A$ endowed with the partial order mentioned earlier has the following property:

$$\text{Every pair of elements has either a lower bound or an upper bound.} \tag{2.30}$$

It is known that this condition is equivalent to :
For every pair of $(x,y),(x^*,y^*) \in A \times A$, there exists a (z_1,z_2) in $A \times A$.

$$\text{that is comparable to } (x,y) \text{ and } (x^*,y^*). \tag{2.31}$$

Theorem 2.6. *In addition to the hypothesis of Theorem 2.1(resp. Theorem 2.3), suppose that for every (x,y) and (x^*,y^*) in $A_0 \times A_0$*

$$\textit{there exists } (z_1,z_2) \in A_0 \times A_0 \textit{ that is comparable to } (x,y) \textit{ and } (x^*,y^*) \tag{2.32}$$

then F has a unique coupled best proximity point of F.

Proof. From Theorem 2.1(resp. Theorem 2.3), the set of coupled best proximity points of F is non-empty. Suppose that there exist (x,y) and (x^*,y^*) in A which are coupled best proximity points. That is,

$$d(x, F(x,y)) = d(A,B), \quad d(y, F(y,x)) = d(A,B)$$

and

$$d(x^*, F(x^*,y^*)) = d(A,B), \quad d(y^*, F(y^*,x^*)) = d(A,B).$$

We distinguish two cases:
Case 1. If (x,y) is comparable to (x^*,y^*) with respect to the ordering in $A \times A$. Using F is proximally coupled weak contraction to $d(x, F(x,y)) = d(A,B)$ and $d(x^*, F(x^*,y^*)) = d(A,B)$, we get

$$\psi(d(x,x^*)) \leq \psi(\max(d(x,x^*), d(y,y^*))) - \phi(\max(d(x,x^*), d(y,y^*))). \tag{2.33}$$

Similarly, one can prove that

$$\psi(d(y,y^*)) \leq \psi(\max(d(y,y^*),d(x,x^*))) - \phi(\max(d(y,y^*),d(x,x^*))). \quad (2.34)$$

From (2.33) and (2.34), we get

$$\max(\psi(d(x,x^*)),\psi(d(y,y^*)) \leq \psi(\max(d(y,y^*),d(x,x^*))) \\ - \phi(\max(d(y,y^*),d(x,x^*))).$$

Using $\psi(\max(a,b)) = \max(\psi(a),\psi(b))$ for $a,b \in [0,\infty]$, we get

$$\psi(\max(d(x,x^*),d(y,y^*))) \leq \psi(\max(d(y,y^*),d(x,x^*))) \\ - \phi(\max(d(y,y^*),d(x,x^*))).$$

This implies that $\phi(\max(d(y,y^*),d(x,x^*))) \leq 0$, using the property of ϕ, we get $\max(d(y,y^*),d(x,x^*)) = 0$. Hence, $x = x^*$ and $y = y*$.

Case 2. If (x,y) is not comparable to (x^*,y^*), then there exists $(u_1,v_1) \in A_0 \times A_0$ which is comparable to (x,y) and (x^*,y^*).

Since $F(A_0 \times A_0) \subseteq B_0$, there exists $(u_2,v_2) \in A_0 \times A_0$ such that $d(u_2,F(u_1,v_1)) = d(A,B)$ and $d(v_2,F(v_1,u_1)) = d(A,B)$. With out loss of generality assume that $(u_1,v_1) \leq (x,y)$(i.e., $x \geq u_1$ and $y \leq v_1$.) Note that $(u_1,v_1) \leq (x,y)$ implies that $(y,x) \leq (v_1,u_1)$. From Lemma 1.6 and Lemma 1.7, we get

$$\left.\begin{aligned} u_1 \leq x \text{ and } v_1 \geq y \\ d(u_2,F(u_1,v_1)) = d(A,B) \\ d(x,F(x,y)) = d(A,B) \end{aligned}\right\} \implies u_2 \leq x$$

and

$$\left.\begin{aligned} u_1 \leq x \text{ and } v_1 \geq y \\ d(v_2,F(v_1,u_1)) = d(A,B) \\ d(y,F(y,x)) = d(A,B) \end{aligned}\right\} \implies v_2 \geq y.$$

From the above to inequalities, we obtain $(u_2,v_2) \leq (x,y)$. Continuing this process, we get sequences $\{u_n\}$ and $\{v_n\}$ such that $d(u_{n+1},F(u_n,v_n)) = d(A,B)$ and $d(v_{n+1},F(v_n,u_n)) = d(A,B)$ with $(u_n,v_n) \leq (x,y)$, $\forall n \in \mathbb{N}$. Using F is proximally coupled weak contraction, we get

$$\left.\begin{aligned} u_n \leq x \text{ and } v_n \geq y \\ d(u_n,F(u_{n-1},v_{n-1})) = d(A,B) \\ d(x,F(x,y)) = d(A,B) \end{aligned}\right\} \\ \implies \psi(d(u_n,x)) \leq \psi(\max(d(u_{n-1},x),d(v_{n-1},y))) \\ - \phi(\max(d(u_{n-1},x),d(v_{n-1},y))). \quad (2.35)$$

Similarly, we can prove that

$$\left.\begin{aligned} y \le v_n \text{ and } x \ge u_n \\ d(y, F(y,x)) = d(A,B) \\ d(v_n, F(v_{n-1}, u_{n-1})) = d(A,B) \end{aligned}\right\} \tag{2.36}$$
$$\implies \psi(d(y,v_n)) \le \psi(\max(d(y,v_{n-1}), d(x,u_{n-1}))) - \phi(\max(d(y,v_{n-1}), d(x,u_{n-1}))).$$

From (2.35) and (2.36), we obtain

$$\max(\psi(d(u_n,x)), \psi(d(y,v_n))) \le \psi(\max(d(u_{n-1},x), d(v_{n-1},y))) - \phi(\max(d(u_{n-1},x), d(v_{n-1},y))).$$

But, $\psi(\max(a,b)) = \max(\psi(a), \psi(b))$ for $a, b \in [0, \infty]$, hence

$$\begin{aligned} \psi(\max(d(u_n,x), d(y,v_n))) &\le \psi(\max(d(u_{n-1},x), d(v_{n-1},y))) \\ &\quad - \phi(\max(d(u_{n-1},x), d(v_{n-1},y))) \\ &\le \psi(\max(d(u_{n-1},x), d(v_{n-1},y))). \end{aligned} \tag{2.37}$$

By using ψ is nondecreasing function, we get the sequence $\{\max(d(u_n,x), d(y,v_n))\}$ is nonnegative decreasing and bounded. This implies that there exists $r \ge 0$ such that

$$\lim_{n\to\infty} \max(d(u_n,x), d(y,v_n)) = r \ge 0. \tag{2.38}$$

Suppose $\lim_{n\to\infty} \max(d(u_n,x), d(y,v_n)) = r > 0$.

Letting $n \to \infty$ in (2.37) and using the continuity of ψ, we get

$$\psi(r) \le \psi(r) - \lim_{n\to\infty} \phi(\max(d(u_{n-1},x), d(v_{n-1},y))) \le \psi(r).$$

This implies that

$$\lim_{n\to\infty} \phi(\max(d(u_{n-1},x), d(v_{n-1},y))) = 0. \tag{2.39}$$

But $0 < r \le \max(d(u_n,x), d(y,v_n))$ and ϕ is nondecreasing function, hence

$$0 < \phi(r) \le \phi(\max(d(u_n,x), d(y,v_n)))$$

and this gives us $\lim_{n\to\infty} \phi(\max(d(u_{n-1},x), d(v_{n-1},y))) \ge \phi(r) > 0$ which contradicts (2.39). Hence,

$$\lim_{n\to\infty} \max(d(u_n,x), d(y,v_n)) = 0.$$

That is $u_n \to x$ and $v_n \to y$. Analogously, one can prove that $u_n \to x^*$ and $v_n \to y^*$. But the limit of the sequence is unique in metric space. Therefore, $x = x^*$ and $y = y^*$. Hence the proof. □

The following result, due to Harjani et. al in [7], as a corollary from the Theorem 2.6, by taking $A = B$.

Corollary 2.7. *In addition to the hypothesis of Corollary* 2.2*(resp. Corollary* 2.4*), suppose that for any two elements* (x, y) *and* (x^*, y^*) *in* $A \times A$, *there exists* $(z_1, z_2) \in A \times A$ *such that* (z_1, z_2) *is comparable to* (x, y) *and* (x^*, y^*) *then* F *has a unique coupled fixed point.*

References

[1] A. Abkar and M. Gabeleh, *Best proximity points for cyclic mappings in ordered metric spaces*, J. Optim. Theory Appl., **150**(1) (2011), 188–193.

[2] A. Abkar and M. Gabeleh, *Generalized cyclic contractions in partially ordered metric spaces*, Optim. Lett., (2011) doi: 10.1007/s11590-011-0379-y.

[3] M.A. Al-Thagafi and N. Shahzad, *Convergence and existence results for best proximity points*, Nonlinear Anal., **70** (2009), 3665–3671.

[4] A. Anthony Eldred and P. Veeramani, *Existence and convergence of best proximity points*, J. Math. Anal. Appl., **323** (2006), 1001–1006.

[5] Ky Fan, *Extensions of two fixed point theorems of F.E. Browder*, Math. Z., **122** (1969), 234–240.

[6] T. Gnana Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal., **65** (2006), 1379–1393.

[7] J. Harjani, B. López and K. Sadarangani, *Fixed point theorems for mixed monotone operators and applications to integral equations*, Nonlinear Anal., **74** (2011), 1749–1760.

[8] M.S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Austral. Math. Soc., **30**(1) (1984), 1–9.

[9] W.K. Kim and K.H. Lee, *Existence of best proximity pairs and equilibrium pairs*, J. Math. Anal. Appl., **316**(2) (2006), 433–446.

[10] W.A. Kirk, S. Reich and P. Veeramani, *Proximinal retracts and best proximity pair theorems*, Numer. Funct. Anal. Optim., **24** (2003), 851–862.

[11] V. Lakshmikanthama and L. Ćirić, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal., **70**(12) (2009), 4341–4349.

[12] N. V. Luong and N. X. Thuan, *Coupled fixed point theorems for mixed monotone mappings and an application to integral equations*, Comput. Math. Appl., **62**(11) (2011), 4238–4248.

[13] V. Pragadeeswarar and M. Marudai, *Best proximity points: approximation and optimization in partially ordered metric spaces*, Optim. Lett., doi:10.1007s11590-012-0529-x (2012).

[14] V. Pragadeeswarar and M. Marudai, *Best proximity points for generalized proximal weak contractions in partially ordered metric spaces*, Optim. Lett., doi:10.1007/s11590-013-0709-3 (2013).

[15] S. Sadiq Basha and P. Veeramani, *Best proximity pair theorems for multifunctions with open fibres*, J. Approx. Theory, **103** (2000), 119–129.

[16] S. Sadiq Basha, *Discrete optimization in partially ordered sets*, J. Global Optim., (2011) doi:10.1007/s10898-011-9774-2.

[17] V. Sankar Raj, *A best proximity point theorem for weakly contractive non-self-mappings*, Nonlinear Anal., **74** (2011), 4804–4808.

[18] M. De la Sen and R.P. Agarwal, *Some fixed point-type results for a class of extended cyclic self-mappings with a more general contractive condition*, Fixed Point Theory Appl., **59** (2011).

[19] W. Shatanawi, *Partially ordered metric spaces and coupled fixed point results*, Comput. Math. Appl., **60** (2010), 2508–2515.

[20] W. Sintunavarat and P. Kumam, *Coupled best proximity point theorem in metric spaces*, Fixed Point Theory Appl., **2012** (2012), 93.

[21] P.S. Srinivasan and P. Veeramani, *On existence of equilibrium pair for constrained generalized games*, Fixed Point Theory Appl., **1** (2004), 21–29.