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FIXED POINT RESULTS IN S-METRIC SPACES

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Abstract. In this paper, the notion of S-metric spaces will be introduced. We present a coupled coincidence point theorems for multi-valued maps on complete S-metric spaces using mixed g-monotone mappings. The single-valued case and an illustrative example are given. Using a similar method as in [4] a common fixed point theorem for three single-valued mappings is obtained in S-metric spaces.

1. Introduction

The Banach contraction principle is the most celebrated fixed point theorem and has been generalized in various directions. Fixed point problems for contractive mappings in metric spaces with a partially order have been studied by many authors (see [1]-[8]). Bhaskar and Lakshmikantham [3] introduced the concept of coupled fixed point and studied the problems of a uniqueness of a coupled fixed point in partially ordered metric spaces. They applied their theorems to problems of the existence of solution for a periodic boundary

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value problem. V. Lakshmikantham and Ćirić ([6]) established some coincidence and common coupled fixed point theorems under nonlinear contractions in partially ordered metric spaces.

In the present paper, we introduce the notion of S-metric spaces and give some properties of them. A coupled coincidence point theorems for multi-valued mappings on complete S-metric spaces will be proved. In addition, we give an illustrative example for the single-valued case. Also, it will be proved a common fixed point theorem for three single-valued mappings in complete S-metric spaces.

We begin with the following definition.

Definition 1.1. Let X be a nonempty set. An *S-metric* on X is a function $S : X^3 \to [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

(1) $S(x, y, z) \geq 0$,
(2) $S(x, y, z) = 0$ if and only if $x = y = z$,
(3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an *S-metric space*.

Immediate examples of such S-metric spaces are:

(1) Let $X = \mathbb{R}^n$ and $||\cdot||$ a norm on X, then $S(x, y, z) = ||y + z - 2x|| + ||y - z||$ is an S-metric on X.
(2) Let $X = \mathbb{R}^n$ and $||\cdot||$ a norm on X, then $S(x, y, z) = ||x - z|| + ||y - z||$ is an S-metric on X.
(3) Let X be a nonempty set, d is ordinary metric on X, then $S(x, y, z) = d(x, z) + d(y, z)$ is an S-metric on X.

Lemma 1.2. *In an S-metric space, we have $S(x, x, y) = S(y, y, x)$.*

Proof. By third condition of S-metric, we have

$$S(x, x, y) \leq S(x, x, x) + S(x, x, x) + S(y, y, x) = S(y, y, x) \tag{1.1}$$

and similarly

$$S(y, y, x) \leq S(y, y, y) + S(y, y, y) + S(x, x, y) = S(x, x, y). \tag{1.2}$$

Hence by (1.1) and (1.2), we get $S(x, x, y) = S(y, y, x)$. □

Definition 1.3. Let (X, S) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with center x and radius r as follows respectively:

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\},$$
$$B_S[x, r] = \{y \in X : S(y, y, x) \leq r\}.$$

Example 1.4. Let $X = \mathbb{R}$. Denote $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Thus

$$\begin{aligned} B_S(1, 2) &= \{y \in \mathbb{R} : S(y, y, 1) < 2\} = \{y \in \mathbb{R} : |y - 1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} = (0, 2). \end{aligned}$$

Definition 1.5. Let (X, S) be an S-metric space and $A \subset X$.

(1) If for every $x \in A$ there exists $r > 0$ such that $B_S(x, r) \subset A$, then the subset A is called open subset of X.

(2) Subset A of X is said to be *S-bounded* if there exists $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$.

(3) A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\forall\ n \geq n_0 \Longrightarrow S(x_n, x_n, x) < \varepsilon$$

and we denote by $\lim_{n\to\infty} x_n = x$.

(4) Sequence $\{x_n\}$ in X is called a *Cauchy sequence* if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \varepsilon$ for each $n, m \geq n_0$.

(5) The S-metric space (X, S) is said to be *complete* if every Cauchy sequence is convergent.

(6) Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X (induced by the S-metric S).

Lemma 1.6. *Let (X, S) be an S-metric space. If $r > 0$ and $x \in X$, then the ball $B_S(x, r)$ is open subset of X.*

Proof. Let $y \in B_S(x, r)$, hence $S(y, y, x) < r$. If set $\delta = S(x, x, y)$ and $r' = \frac{r-\delta}{2}$ then we prove that $B_S(y, r') \subseteq B_S(x, r)$. Let $z \in B_S(y, r')$, then $S(z, z, y) < r'$. By third condition of S-metric we have

$$S(z, z, x) \leq S(z, z, y) + S(z, z, y) + S(x, x, y) < 2r' + \delta = r$$

Hence $B_S(y, r') \subseteq B_S(x, r)$. That is the ball $B_S(x, r)$ is a open subset of X. □

Lemma 1.7. *Let (X, S) be an S-metric space. If sequence $\{x_n\}$ in X converges to x, then x is unique.*

Proof. Let $\{x_n\}$ converges to x and y, then for each $\varepsilon > 0$ there exist $n_1, n_2 \in \mathbb{N}$ such that

$$\forall\ n \geq n_1 \Longrightarrow S(x_n, x_n, x) < \frac{\varepsilon}{4}$$

and

$$\forall\ n \geq n_2 \Longrightarrow S(x_n, x_n, y) < \frac{\varepsilon}{2}.$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ by third condition S-metric we have:

$$S(x, x, y) \leq 2S(x, x, x_n) + S(y, y, x_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $S(x, x, y) = 0$ so $x = y$. □

Lemma 1.8. *Let (X, S) be an S-metric space. If sequence $\{x_n\}$ in X is converges to x, then $\{x_n\}$ is a Cauchy sequence.*

Proof. Since $\lim_{n\to\infty} x_n = x$ then for each $\varepsilon > 0$ there exists $n_1, n_2 \in \mathbb{N}$ such that

$$n \geq n_1 \Rightarrow S(x_n, x_n, x) < \frac{\varepsilon}{4}$$

and

$$m \geq n_2 \Rightarrow S(x_m, x_m, x) < \frac{\varepsilon}{2}.$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n, m \geq n_0$ by third condition of S-metric we have:

$$S(x_n, x_n, x_m) \leq 2S(x_n, x_n, x) + S(x_m, x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence. □

Lemma 1.9. *Let (X, S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then*

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

Proof. Since $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then for each $\varepsilon > 0$ there exist $n_1, n_2 \in\ \mathbb{N}$ such that

$$\forall\ n \geq n_1 \Rightarrow S(x_n, x_n, x) < \frac{\varepsilon}{4}$$

and

$$\forall\ n \geq n_2 \Rightarrow S(y_n, y_n, y) < \frac{\varepsilon}{4}.$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n \geq n_0$ by third condition of S-metric we have:

$$\begin{aligned} S(x_n, x_n, y_n) &\leq 2S(x_n, x_n, x) + S(y_n, y_n, x) \\ &\leq 2S(x_n, x_n, x) + 2S(y_n, y_n, y) + S(x, x, y) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + S(x, x, y) = \varepsilon + S(x, x, y). \end{aligned}$$

Hence we have:

$$S(x_n, x_n, y_n) - S(x, x, y) < \varepsilon. \tag{1.3}$$

On the other hand, we have

$$\begin{aligned} S(x,x,y) &\leq 2S(x,x,x_n) + S(y,y,x_n) \\ &\leq 2S(x,x,x_n) + 2S(y,y,y_n) + S(x_n,x_n,y_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + S(x_n,x_n,y_n) = \varepsilon + S(x_n,x_n,y_n), \end{aligned}$$

that is

$$S(x,x,y) - S(x_n,x_n,y_n) < \varepsilon. \tag{1.4}$$

Therefore by relations (1.3) and (1.4) we have $|S(x_n,x_n,y_n) - S(x,x,y)| < \varepsilon$, that is

$$\lim_{n\to\infty} S(x_n,x_n,y_n) = S(x,x,y).$$

□

2. Main Results

Definition 2.1. ([3]) An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x,y) = x, \ F(y,x) = y.$$

Definition 2.2. ([6]) An element $(x,y) \in X \times X$ is called a coupled coincidence point of a mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x,y) = gx, \ F(y,x) = gy.$$

Definition 2.3. ([6]) Let X be a non-empty set and $F : X \times X \to X$ and $g : X \to X$ are mappings. We say F and g are commutative if

$$gF(x,y) = F(gx,gy)$$

for all $x, y \in X$.

Definition 2.4. ([8]) Suppose $(X, \leq)$ is a partially ordered set and $A, h : X \to X$ are mappings of X into itself. We say A is h-non-decreasing if for $x, y \in X$,

$$h(x) \leq h(y) \ \text{ implies } \ A(x) \leq A(y).$$

Definition 2.5. ([6]) Suppose $(X, \leq)$ is a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ are mappings. We say F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, if for $x_1, x_2, y_1, y_2 \in X$,

$$gx_1 \leq gx_2 \ \text{ implies } \ F(x_1,y) \leq F(x_2,y), \ \forall y \in X,$$

and

$$gy_1 \leq gy_2 \text{ implies } F(x,y_2) \leq F(x,y_1), \ \forall x \in X.$$

Example 2.6. Let $X = \mathbb{R}^+$. Define a map F on $X \times X$ as follows:

$$F(x,y) = \frac{x}{y}.$$

If define

$$gx = x^2,$$

then it is easy to see that F and g are commutative and F is g-non-decreasing and non-increasing.Also, if define

$$A = \{(x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ : xy = 1\},$$

then for every $(x_0, y_0) \in A$ is a coupled coincidence point of F and g.

In the proof of our first theorem we use the following lemma:

Lemma 2.7. *Let (X,S) be an S-metric space. If there exist sequences $\{x_n\}$ in X such that for every $n \in \mathbb{N}$*

$$S(x_n, x_n, x_{n+1}) \leq lS(x_{n-1}, x_{n-1}, x_n)$$

for every $0 < l < 1$, then sequence $\{x_n\}$ is a Cauchy sequence.

Proof. For every $n \in \mathbb{N}$ and $x_n, x_{n+1} \in X$, we have

$$\begin{aligned} S(x_n, x_n, x_{n+1}) &\leq lS(x_{n-1}, x_{n-1}, x_n) \\ &\leq l^2 S(x_{n-2}, x_{n-2}, x_{n-1}) \\ &\vdots \\ &\leq l^n S(x_0, x_0, x_1). \end{aligned}$$

Hence for every $m > n$ and $0 < l < 1$ we have, by the triangle inequality,

$$\begin{aligned} S(x_n, x_n, x_m) &\leq 2\sum_{i=n}^{m-2} S(x_i, x_i, x_{i+1}) + S(x_{m-1}, x_{m-1}, x_m) \\ &\leq 2[l^n + l^{n+1} + \cdots + l^{m-1}]S(x_0, x_0, x_1) \\ &\leq \frac{2l^n}{1-l} S(x_0, x_0, x_1) \longrightarrow 0. \end{aligned}$$

Therefore, for each $\epsilon > 0$ there exits $n_0 \in \mathbb{N}$ such that, for each $n, m \geq n_0$

$$S(x_n, x_n, x_m) < \varepsilon.$$

These show that $\{x_n\}$ is a Cauchy sequence in X. $\square$

In the next Theorem, if we write $(x,y) \preceq_g (u,v)$ for every $x,y,u,v \in X$, that is $g(x) \leq g(u)$ *and* $g(y) \geq g(v)$, where $(X, \leq)$ is a partially ordered set and $g : X \to X$ be a mapping.

Definition 2.8. Define $\Psi = \{\psi : [0,\infty) \to [0,\infty) : \psi$ is continuous and $\psi(t) \leq kt$ for some $k \in (0, \frac{1}{2})$, with $\psi(t) = 0$ if and only if $t = 0\}$.

Theorem 2.9. *Let $(X, \leq)$ be a partially ordered set and (X, S) be a complete S-metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two functions satisfying:*

(i) *$F(X \times X) \subseteq g(X)$ and g is continuous and commutes with F,*
(ii) *F has the mixed g-monotone property,*
(iii) *$S(F(x,y), F(x,y), F(x',y')) \leq \psi(S(gx,gx,gx') + S(gy,gy,gy'))$ for all $x,y,x',y' \in X$ and $\psi \in \Psi$ for which $(x,y) \preceq_g (x',y')$,*
(iv) *if $(x_n, y_n) \in X \times X$ are two sequences in X such that $(x_n, y_n) \preceq_g (x_{n+1}, y_{n+1})$ and $gx_n \to gx$ and $gy_n \to gy$, then $(x_n, y_n) \preceq_g (x,y)$ for all $n \in \mathbb{N}$,*
(v) *if there exists $(x_0, y_0) \in X \times X$ with $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$.*

Then F and g have a coupled coincidence point. That is there exist $u, v \in X$ such that $gu = F(u,v)$ and $gv = F(v,u)$.

Proof. Since $F(X \times X) \subseteq g(X)$, by (v) we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Again from $F(X \times X) \subseteq g(X)$ we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing this process we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n) \text{ and } gy_{n+1} = F(y_n, x_n) \text{ for all } n \geq 0. \tag{2.1}$$

Since $g(x_0) \leq F(x_0, y_0)$ and $g(x_1) = F(x_0, y_0)$, we have $g(x_0) \leq g(x_1)$. Then from (ii) we have,

$$F(x_0, y_0) \leq F(x_1, y_0).$$

Similarly, since $g(y_0) \geq F(y_0, x_0)$ and $g(y_1) = F(y_0, x_0)$, we have $g(y_1) \leq g(y_0)$. Then from (ii) we have,

$$F(x, y_0) \leq F(x, y_1) \ \forall x \in X.$$

In particular, we get $F(x_1, y_0) \leq F(x_1, y_1)$. Thus $g(x_1) \leq g(x_2)$. Again from (ii) we have,$F(x_1, y_1) \leq F(x_2, y_2)$, that is, $g(x_2) \leq g(x_3)$. Continuing we obtain

$$\begin{aligned} F(x_0, y_0) &\leq F(x_1, y_1) \leq F(x_2, y_2) \leq F(x_3, y_3) \leq \cdots \\ &\leq F(x_n, y_n) \leq F(x_{n+1}, y_{n+1}) \leq \cdots . \end{aligned}$$

That is

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq \cdots \leq g(x_{n+1}) \leq g(x_{n+2}) \cdots .$$

Similarly, since $g(y_0) \geq F(y_0, x_0)$ and $g(y_1) = F(y_0, x_0)$, we have $g(y_1) \leq g(y_0)$. Then from (ii) we have,

$$F(y_1, x_1) \leq F(y_0, x_1).$$

Since $g(x_0) \leq g(x_1)$ from (ii) we have, $F(y_0, x_1) \leq F(y_0, x_0)$. Thus $g(y_2) \leq g(y_1)$. Again from (ii) we have,$F(y_2, x_2) \leq F(y_1, x_1)$, that is, $g(y_3) \leq g(y_2)$. Continuing we obtain

$$\begin{aligned} \cdots &\leq F(y_{n+1}, x_{n+1}) \leq F(y_n, x_n) \\ &\leq \cdots \leq F(y_3, x_3) \leq F(y_2, x_2) \leq F(y_1, x_1) \leq F(y_0, x_0). \end{aligned}$$

That is

$$\cdots \leq g(y_{n+2}) \leq g(y_{n+1}) \leq \cdots g(y_2) \leq g(y_1) \leq g(y_0).$$

Since $g(x_n) \leq g(x_{n+1})$ and $g(y_n) \geq g(y_{n+1})$, that is $(x_n, y_n) \preceq_g (x_{n+1}, y_{n+1})$. From (iii) we have

$$\begin{aligned} S(gx_{n+1}, gx_{n+1}, gx_{n+2}) &= S(F(x_n, y_n), F(x_n, y_n), F(x_{n+1}, y_{n+1})) \\ &\leq \psi(S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1})) \\ &= \psi(\delta_n). \end{aligned}$$

Similarly, since $g(y_{n+1}) \leq g(y_n)$ and $g(x_{n+1}) \geq g(x_n)$, that is $(y_{n+1}, x_{n+1}) \preceq_g (y_n, x_n)$. From (iii) we have,

$$\begin{aligned} S(gy_{n+1}, gy_{n+1}, gy_{n+2}) &\leq \psi(S(gy_n, gy_n, gy_{n+1}) + S(gx_n, gx_n, gx_{n+1})) \\ &= \psi(\delta_n), \end{aligned}$$

where

$$\delta_n = S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1}).$$

Adding the above inequalities we obtain

$$\delta_{n+1} \leq 2\psi(\delta_n) \leq 2k\delta_n.$$

Thus, we have

$$\delta_n \leq l\ \delta_{n-1} \leq \cdots \leq l^n\ \delta_0,$$

where $l = 2k$. That is, we have

$$S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1}) \leq l^n\ [S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)].$$

Hence for every $m > n$ and $0 < l < 1$ we have, by the triangle inequality,

$$\begin{aligned}
&S(gx_n, gx_n, gx_m) + S(gy_n, gy_n, gy_m) \\
&\le 2[S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1})] \\
&\quad + 2[S(gx_{n+1}, gx_{n+1}, gx_{n+2}) + S(gy_{n+1}, gy_{n+1}, gy_{n+2})] \\
&\quad \vdots \\
&\quad + 2[S(gx_{m-1}, gx_{m-1}, gx_m) + S(gy_{m-1}, gy_{m-1}, gy_m)] \\
&\le 2[l^n + l^{n+1} + \cdots + l^{m-1}][S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)] \\
&\le \frac{2l^n}{1-l}\Big[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1)\Big] \\
&\to 0.
\end{aligned}$$

Therefore, for each $\epsilon > 0$ there exits $n_0 \in \mathbb{N}$ such that, for each $n, m \ge n_0$

$$S(gx_n, gx_n, gx_m) + S(gy_n, gy_n, gy_m) < \varepsilon.$$

Hence

$$S(gx_n, gx_n, gx_m) < \varepsilon \text{ and } S(gy_n, gy_n, gy_m) < \varepsilon.$$

These shows that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in X. Since X is complete, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = x \quad\text{and}\quad \lim_{n\to\infty} gy_n = y. \tag{2.2}$$

From (2.2) and continuity of g, we have

$$\lim_{n\to\infty} ggx_n = gx \quad\text{and}\quad \lim_{n\to\infty} ggy_n = gy. \tag{2.3}$$

From (2.1) and commutativity of F and g, we have

$$ggx_{n+1} = gF(x_n, y_n) = F(gx_n, gy_n) \tag{2.4}$$

and

$$ggy_{n+1} = gF(y_n, x_n) = F(gy_n, gx_n). \tag{2.5}$$

On the other hand, since $gx_n \to x$, and $gy_n \to y$ as $n \to \infty$, then by (2.3), (2.4) we get

$$\begin{aligned}
S(ggx_{n+1}, ggx_{n+1}, F(x,y)) &= S(gF(x_n, y_n), gF(x_n, y_n), F(x,y)) \\
&= S(F(gx_n, gy_n), F(gx_n, gy_n), F(x,y)) \\
&\le \psi(S(ggx_n, ggx_n, gx) + S(ggy_n, ggy_n, gy)).
\end{aligned}$$

So letting $n \to \infty$ by Lemma1.9 yields $S(gx, gx, F(x,y)) \le 0$. Hence $gx = F(x,y)$. Similarly one can show that $g(y) = F(y,x)$. □

Corollary 2.10. *Let $(X, \leq)$ be a partially ordered set and (X, S) be a complete S-metric space. Let there exists function $F : X \times X \to X$ satisfying:*

(i) *F has the mixed I-monotone property, where I is identity map,*

(ii)
$$S(F(x,y), F(x,y), F(u,v)) \leq \psi(S(x,x,u) + S(y,y,v))$$
for all $x, y, u, v \in X$ and $\psi \in \Psi$, where $(x,y) \preceq_I (u,v)$,

(iii) *if $\{x_n\}$ and $\{y_n\}$ in X are two sequences such that $(x_n, y_n) \preceq_I (x_{n+1}, y_{n+1})$ with $x_n \longrightarrow x$ and $y_n \longrightarrow y$ implies that $(x_n, y_n) \preceq_I (x, y)$, for all $n \in \mathbb{N}$,*

(iv) *if there exists $(x_0, y_0) \in X \times X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$.*

Then there exist two points $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$.

Proof. If set $g = I$ identity map in Theorem 2.9, then the proof is complete. □

Example 2.11. Let $(X, \leq)$ be a partially ordered set and (X, S) be a complete S-metric space, where $X = [-3, 3]$. For any $x, y, z \in X$, define $S(x,y,z) = |x - z| + |y - z|$ and a mapping $F : X \times X \to X$ by
$$F(x,y) = 2\sin\left(\frac{\pi}{12}(x - y)\right) + 2.$$
It is easy to see that F is a I-non-decreasing and non-increasing mapping and, for some $k \in (\frac{\pi}{8}, \frac{1}{2})$,
$$S(F(x,y), F(x,y), F(u,v)) \leq k[S(x,x,u) + S(y,y,v)]$$
for all $x, y, u, v \in X$, where $(x,y) \preceq_I (u,v)$.

If $\{x_n\}$ and $\{y_n\}$ in X are two sequences such that $(x_n, y_n) \preceq_I (x_{n+1}, y_{n+1})$ and $x_n \to x$ and $y_n \to y$, then $(x_n, y_n) \preceq_I (x, y)$ for all $n \in \mathbb{N}$. If $x_0 = 2, y_0 = 2 \in X$, then $2 \leq F(2,2)$ and $2 \geq F(2,2)$. Therefore, all the conditions of Corollary 2.10 hold and so there exists $x = 3, y = 1 \in X$ such that $3 = F(3,1)$ and $1 = F(1,3)$.

Theorem 2.12. *Let (X, S) be a bounded, complete S-metric space, P and Q be one to one continuous mapping from X into X, $A : X \to PX \bigcap QX$ be continuous and P and Q be commutative with A. If for every $x \in X$ there exists $n(x) \in \mathbb{N}$ so that for every $y \in X$:*
$$S(A^{n(x)}x, A^{n(x)}x, A^{n(x)}y) \leq q \min\{S(Qx, Qx, Py),\ S(Px, Px, Qy)\}$$
where $q \in (0,1)$, then there exists unique element $z \in X$ such that $z = Az = Pz = Qz$.

Proof. The proof is similar to the proof of Theorem 1 from Hažić [4]. Let $x_0 \in X$. Since $AX \subseteq PX \bigcap QX$ we can define the sequence $\{x_n\}_{n\in\mathbb{N}}$ from X in the following way:

$$Qx_{2k-1} = A^{n(x_{2k-2})}x_{2k-2}, \quad k \in \mathbb{N},$$

$$Px_{2k} = A^{n(x_{2k-1})}x_{2k-1}, \quad k \in \mathbb{N}.$$

Let

$$y_n = \begin{cases} Qx_{2k-1}, & n = 2k-1, \\ Px_{2k}, & n = 2k. \end{cases}$$

Let $n = 2k$. Then

$$\begin{aligned} S(y_n, y_n, y_{n+1}) &= S(y_{2k}, y_{2k}, y_{2k+1}) = S(Px_{2k}, Px_{2k}, Qx_{2k+1}) \\ &= S(A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k})}x_{2k}) \\ &= S(A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k})}P^{-1}Px_{2k}) \\ &= S(A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k})}P^{-1}A^{n(x_{2k-1})}x_{2k-1}) \\ &= S(A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k-1})}S^{-1}A^{n(x_{2k})}x_{2k-1}) \\ &\leq q\, S(Qx_{2k-1}, Qx_{2k-1}, A^{n(x_{2k})}x_{2k-1}) \\ &= q\, S(A^{n_(x_{2k-2})}x_{2k-2}, A^{n_(x_{2k-2})}x_{2k-2}, A^{n(x_{2k})}Q^{-1}Qx_{2k-1}) \\ &\;\vdots \\ &\leq q^{2k}S(Px_0, Px_0, A^{n(x_{2k})}x_0). \end{aligned}$$

Similarly, for $n = 2k+1$

$$S(y_{2k+1}, y_{2k+1}, y_{2k+2}) \quad \leq \quad q^{2k+1}S(Px_0, Px_0, A^{n(x_{2k+1})}x_0).$$

We shall prove that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

$$\begin{aligned} S(y_n, y_n, y_{n+m}) &\leq 2S(y_n, y_n, y_{n+1}) + S(y_{n+1}, y_{n+1}, y_{n+m}) \\ &\leq 2(S(y_n, y_n, y_{n+1}) + S(y_{n+1}, y_{n+1}, y_{n+2}) \\ &\quad + \cdots + S(y_{n+m-1}, y_{n+m-1}, y_{n+m})) \end{aligned}$$

for every $n, m \in \mathbb{N}$ and $q \in (0,1)$, it follows that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. So there exists $z \in X$ such that $\lim\limits_{n\to\infty} y_n = z$, i.e., $\lim\limits_{n\to\infty} S(y_n, y_n, z) = 0$. Since $\{Px_{2k}\}_{k\in\mathbb{N}}$ and $\{Qx_{2k-1}\}_{k\in\mathbb{N}}$ are subsequence of the sequence $\{y_n\}_{n\in\mathbb{N}}$ it follows that

$$\lim_{k\to\infty} S(Px_{2k}, Px_{2k}, z) = 0 \quad \text{and} \quad \lim_{k\to\infty} S(Qx_{2k-1}, Qx_{2k-1}, z) = 0.$$

Since

$$\begin{aligned}
&S(Px_{2k}, Px_{2k}, Ax_{2k})\\
&= S(Px_{2k}, Px_{2k}, AP^{-1}Px_{2k})\\
&= S(Px_{2k}, Px_{2k}, AP^{-1}A^{n(x_{2k-1})}x_{2k-1})\\
&= S(A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k-1})}P^{-1}Ax_{2k-1})\\
&\leq qS(Qx_{2k-1}, Qx_{2k-1}, Ax_{2k-1})\\
&\vdots\\
&\leq q^{2k}S(Px_0, Px_0, Ax_0).
\end{aligned}$$

Taking the limit as $k \to \infty$ we obtain $\lim\limits_{n\to\infty} Ax_{2k} = z$, i.e.,

$$\lim_{n\to\infty} S(z, z, Ax_{2k}) = 0.$$

Then we have the following

$$\begin{aligned}
S(Az, Az, Pz) &\leq 2S(Az, Az, APx_{2k}) + S(Pz, Pz, APx_{2k})\\
&= 2S(Az, Az, APx_{2k}) + S(Pz, Pz, PAx_{2k}).
\end{aligned}$$

Taking the limit as $k \to \infty$ we obtain $S(Az, Az, Pz) = 0$ which implies $Az = Pz$. Similarly we can prove that $Az = Qz$. Let us prove that $\lim\limits_{k\to\infty} A^2x_{2k} = z$.

$$\begin{aligned}
&S(Px_{2k}, Px_{2k}, A^2x_{2k})\\
&= S(A^{(x_{2k-1})}x_{2k-1}, A^{n(x_{2k-1})}x_{2k-1}, A^2P^{-1}Px_{2k})\\
&= S(A^{n(x_{2k-1})}x_{2k-1}, A^{n(x_{2k-1})}x_{2k-1}, A^2P^{-1}A^{n(x_{2k-1})}x_{2k-1})\\
&\leq qS(Qx_{2k-1}, Qx_{2k-1}, A^2x_{2k-1})\\
&\vdots\\
&\leq q^{2k}S(Px_0, Px_0, A^2x_0)
\end{aligned}$$

and letting $k \to \infty$ we have $\lim\limits_{k\to\infty} A^2x_{2k} = z$. Using continuity of A we have

$$Az = A(\lim_{n\to\infty} Ax_{2k}) = \lim_{n\to\infty} A^2x_{2k} = z.$$

Now, it remains to be prove the uniqueness of the common fixed point. Suppose that there exists another common fixed point p, $p \neq z$. As $z = Az = A^2z = \cdots = A^{n(z)}z$, we have

$$\begin{aligned}
S(z, z, p) &= S(A^{n(z)}z, A^{n(z)}z, A^{n(z)}p)\\
&\leq q\min\{S(Qz, Qz, Pp),\ S(Pz, Pz, Qp)\}\\
&= q\min\{S(z, z, p),\ S(z, z, p)\}
\end{aligned}$$

and final result $z = p$. □

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