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CONVERGENCE FOR NEARLY ASYMPTOTICALLY NONEXPANSIVE MAPPINGS BY THREE-STEP ITERATIONS

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Abstract. In this paper, we establish some strong convergence results and a weak convergence result for nearly asymptotically nonexpansive mappings under suitable conditions in the setting of Banach spaces.

1. Introduction

Let E be a real Banach space and C be a nonempty subset of E. Let $T\colon C \to C$ be a mapping, then we denote the set of all fixed points of T by $F(T)$. A self mapping $T\colon C \to C$ is said to be Lipschitzian if for each $n \in \mathbb{N}$, there exists a positive number k_n such that

$$\|T^n x - T^n y\| \quad \leq \quad k_n \|x - y\|$$

for all $x,\, y \in C$.

A Lipschitzian mapping T is said to be uniformly k-Lipschitzian if $k_n = k$ for all $n \in \mathbb{N}$ and asymptotically nonexpansive [4] if $k_n \geq 1$ for all $n \in \mathbb{N}$ with $\lim_{n\to\infty} k_n = 1$.

It is easy to observe that every nonexpansive mapping T (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x,\, y \in C$) is asymptotically nonexpansive with constant sequence $\{1\}$ and every asymptotically nonexpansive mapping is uniformly k-Lipschitzian with $k = \sup_{n\in\mathbb{N}}\{k_n\}$.

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Fix a sequence $\{a_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} a_n = 0$, then according to Agarwal et al. [1], T is said to be nearly Lipschitzian with respect to $\{a_n\}$ if for each $n \in \mathbb{N}$, there exist constants $k_n \geq 0$ such that $\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n)$ for all $x, y \in C$. The infimum of constants k_n for which the above inequality holds is denoted by $\eta(T^n)$ and is called nearly Lipschitz constant.

A nearly Lipschitzian mapping T with sequence $\{a_n, \eta(T^n)\}$ is said to be nearly asymptotically nonexpansive if $\eta(T^n) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \eta(T^n) = 1$ and nearly uniformly k-Lipschitzian if $\eta(T^n) \leq k$ for all $n \in \mathbb{N}$.

In 2007, Agarwal et al. [1] introduced the following iteration process:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, \ n \geq 1 \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. They showed that this process converge at a rate same as that of Picard iteration and faster than Mann for contractions and also they established some weak convergence theorems using suitable conditions in the framework of uniformly convex Banach space.

Recently, Khan et al. [6] studied the modified two-step iteration process for two mappings as follows:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n S^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n x_n, \ n \geq 1 \end{aligned} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. They established weak and strong convergence theorems in the setting of real Banach spaces.

Inspired and motivated by [1, 6] and some others, in this paper we introduce the following three-step iteration scheme as follows:

$$\begin{aligned} x_1 &= x \in C, \\ x_{n+1} &= (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T^n z_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n T^n x_n, \ n \geq 1 \end{aligned} \tag{1.3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$.

If we put $\gamma_n = 0$ for all $n \geq 1$, then scheme (1.3) reduces to the scheme (1.1) and if we put $S = T$, then scheme (1.2) also reduces to the scheme (1.1).

The three-step iterative approximation problems were studied extensively by Noor [7, 8], Glowinsky and Le Tallec [3] and Haubruge et al [5]. It has been shown [3] that three step iterative scheme gives better numerical results than the two step and one step approximate iterations. Thus we conclude that three step scheme plays an important and significant role in solving various problems, which arise in pure and applied sciences.

The aim of this paper is to establish some strong convergence theorems and a weak convergence theorem of newly proposed iteration scheme (1.3) for nearly asymptotically nonexpansive mapping in the framework of real Banach spaces.

2. Preliminaries

For the sake of convenience, we restate the following concepts.

A mapping $T\colon C \to C$ is said to be demiclosed at zero, if for any sequence $\{x_n\}$ in C, the condition x_n converges weakly to $x \in C$ and Tx_n converges strongly to 0 imply $Tx = 0$.

A mapping $T\colon C \to C$ is said to be semi-compact [2] if for any bounded sequence $\{x_n\}$ in C such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x^* \in C$ strongly.

We say that a Banach space E satisfies the *Opial's condition* [9] if for each sequence $\{x_n\}$ in E weakly convergent to a point x and for all $y \neq x$

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

The examples of Banach spaces which satisfy the Opial's condition are Hilbert spaces and all $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy Opial's condition [9].

Now, we state the following useful lemma to prove our main results.

Lemma 2.1. ([14]) *Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ and $\{r_n\}_{n=1}^{\infty}$ be sequences of non-negative numbers satisfying the inequality*

$$\alpha_{n+1} \leq (1 + \beta_n)\alpha_n + r_n, \ \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$ and $\sum_{n=1}^{\infty} r_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists.

3. Main Results

In this section, we prove some strong convergence theorems and a weak convergence theorem for nearly asymptotically nonexpansive mapping in the framework of real Banach space.

Theorem 3.1. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T\colon C \to C$ be a nearly asymptotically nonexpansive* *mapping with sequence* $\{a_n, \eta(T^n)\}$ *and* $F(T) \neq \emptyset$ *such that* $\sum_{n=1}^{\infty} a_n < \infty$ *and* $\sum_{n=1}^{\infty}\Big(\eta(T^n) - 1\Big) < \infty$. *Let* $\{x_n\}$ *be the three-step iteration defined by* (1.3). *Then* $\{x_n\}$ *converges strongly to a fixed point of the mapping* T *if and only if* $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Proof. The necessity is obvious. Thus we only prove the sufficiency. Let $q \in F(T)$. For the sake of convenience, set

$$B_n x = (1-\gamma_n)x + \gamma_n T^n x,$$

$$A_n x = (1-\beta_n)x + \beta_n T^n B_n x$$

and

$$W_n x = (1-\alpha_n)T^n x + \alpha_n T^n A_n x.$$

Then $z_n = B_n x_n$, $y_n = A_n x_n$ and $x_{n+1} = W_n x_n$. Moreover, it is clear that q is a fixed point of W_n for all n. Let $\eta = \sup_{n\in\mathbb{N}} \eta(T^n)$. Consider

$$\begin{aligned} \|B_n x - B_n y\| &= \|((1-\gamma_n)x+\gamma_n T^n x)-((1-\gamma_n)y+\gamma_n T^n y)\| \\ &= \|(1-\gamma_n)(x-y) + \gamma_n(T^n x - T^n y)\| \\ &\leq (1-\gamma_n)\|x-y\| + \gamma_n \eta(T^n)(\|x-y\| + a_n) \\ &\leq (1-\gamma_n)\eta(T^n)\|x-y\| + \gamma_n\eta(T^n)\|x-y\| \\ &\quad +\gamma_n a_n \eta(T^n) \\ &\leq \eta(T^n)\|x-y\| + a_n\eta(T^n). \end{aligned} \tag{3.1}$$

Choosing $x = x_n$ and $y = q$, we get

$$\|z_n - q\| \leq \eta(T^n)\|x_n - q\| + a_n\eta(T^n). \tag{3.2}$$

Next consider,

$$\begin{aligned} \|A_n x - A_n y\| &= \|((1-\beta_n)x+\beta_n T^n B_n x)-((1-\beta_n)y+\beta_n T^n B_n y)\| \\ &= \|(1-\beta_n)(x-y) + \beta_n(T^n B_n x - T^n B_n y)\| \\ &\leq (1-\beta_n)\|x-y\| + \beta_n\eta(T^n)(\|B_n x - B_n y\| + a_n) \\ &\leq (1-\beta_n)\eta(T^n)\|x-y\| + \beta_n\eta(T^n)\|B_n x - B_n y\| \\ &\quad +\beta_n a_n\eta(T^n) \\ &\leq (1-\beta_n)\eta(T^n)\|x-y\| + \beta_n\eta(T^n)\|B_n x - B_n y\| \\ &\quad +\beta_n a_n\eta(T^n). \end{aligned} \tag{3.3}$$

Now using (3.1) in (3.3), we get

$$\begin{aligned}\|A_nx - A_ny\| &\leq (1-\beta_n)\eta(T^n)\|x-y\| + \beta_n\eta(T^n)[\eta(T^n)\|x-y\| \\ &\quad +a_n\eta(T^n)] + \beta_n a_n\eta(T^n) \\ &\leq (1-\beta_n)(\eta(T^n))^2\|x-y\| + \beta_n(\eta(T^n))^2\|x-y\| \\ &\quad +a_n\beta_n(\eta(T^n))^2 + \beta_n a_n\eta(T^n) \\ &\leq (\eta(T^n))^2\|x-y\| + a_n\eta(T^n)(1+\eta(T^n)). \qquad (3.4)\end{aligned}$$

Choosing $x = x_n$ and $y = q$, we get

$$\|y_n - q\| \leq (\eta(T^n))^2\|x_n - q\| + a_n\eta(T^n)(1+\eta(T^n)). \qquad (3.5)$$

Finally, consider

$$\begin{aligned}&\|W_nx - W_ny\| \\ =\ &\|((1-\alpha_n)T^nx + \alpha_nT^nA_nx) - ((1-\alpha_n)T^ny + \alpha_nT^nA_ny)\| \\ =\ &\|(1-\alpha_n)(T^nx - T^ny) + \alpha_n(T^nA_nx - T^nA_ny)\| \\ \leq\ &(1-\alpha_n)\eta(T^n)(\|x-y\| + a_n) + \alpha_n\eta(T^n)(\|A_nx - A_ny\| + a_n) \\ =\ &(1-\alpha_n)\eta(T^n)\|x-y\| + \alpha_n\eta(T^n)\|A_nx - A_ny\| \\ &+a_n\eta(T^n). \qquad (3.6)\end{aligned}$$

Now using (3.4) in (3.6), we get

$$\begin{aligned}\|W_nx - W_ny\| &\leq (1-\alpha_n)\eta(T^n)\|x-y\| + \alpha_n\eta(T^n)[(\eta(T^n))^2\|x-y\| \\ &\quad +a_n\eta(T^n)(1+\eta(T^n))] + a_n\eta(T^n) \\ &\leq (1-\alpha_n)(\eta(T^n))^3\|x-y\| + \alpha_n(\eta(T^n))^3\|x-y\| \\ &\quad +\alpha_n a_n(\eta(T^n))^2(1+\eta(T^n)) + a_n\eta(T^n) \\ &\leq (\eta(T^n))^3\|x-y\| + a_n\eta(T^n)[1+\eta(T^n)+(\eta(T^n))^2] \\ &\leq (\eta(T^n))^3\|x-y\| + a_n\eta(1+\eta+\eta^2) \\ &= [1+(\mu_n - 1)]\|x-y\| + \nu_n, \qquad (3.7)\end{aligned}$$

where $\mu_n = (\eta(T^n))^3$ and $\nu_n = a_n\eta(1+\eta+\eta^2)$. Moreover,

$$\begin{aligned}\sum_{n=1}^{\infty}(\mu_n - 1) &= \sum_{n=1}^{\infty}\Big((\eta(T^n))^3 - 1\Big) \\ &= \sum_{n=1}^{\infty}\Big((\eta(T^n))^2 + \eta(T^n) + 1\Big)(\eta(T^n) - 1) \\ &\leq (\eta^2+\eta+1)\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty,\end{aligned}$$

and $\sum_{n=1}^{\infty} \nu_n < \infty$ since by hypothesis $\sum_{n=1}^{\infty} a_n < \infty$. Choosing $x = x_n$ and $y = q$ in (3.7), we get

$$\|x_{n+1} - q\| \;=\; \|W_n x_n - q\| \leq [1 + (\mu_n - 1)]\|x_n - q\| + \nu_n. \tag{3.8}$$

Applying Lemma 2.1 in (3.8), we have $\lim_{n\to\infty} \|x_n - q\|$ exists.

Next, we shall prove that $\{x_n\}$ is a Cauchy sequence. Since $1 + x \leq e^x$ for $x \geq 0$, therefore, for any $m, n \geq 1$ and for given $q \in F(T)$, from (3.8) with taking $\mu_n - 1 = t_n$, we have

$$\begin{aligned}
\|x_{n+m} - q\| &\leq (1 + t_{n+m-1})\|x_{n+m-1} - q\| + \nu_{n+m-1} \\
&\leq e^{t_{n+m-1}}\|x_{n+m-1} - q\| + \nu_{n+m-1} \\
&\leq e^{t_{n+m-1}}[e^{t_{n+m-2}}\|x_{n+m-2} - q\| + \nu_{n+m-2}] + \nu_{n+m-1} \\
&\leq e^{(t_{n+m-1}+t_{n+m-2})}\|x_{n+m-2} - q\| \\
&\quad + e^{(t_{n+m-1}+t_{n+m-2})}[\nu_{n+m-2} + \nu_{n+m-1}] \\
&\leq \dots \\
&\leq e^{\left(\sum_{k=n}^{n+m-1} t_k\right)}\|x_n - q\| + e^{\left(\sum_{k=n}^{n+m-1} t_k\right)} \sum_{k=n}^{n+m-1} \nu_k \\
&\leq e^{\left(\sum_{n=1}^{\infty} t_k\right)}\|x_n - q\| + e^{\left(\sum_{n=1}^{\infty} t_k\right)} \sum_{k=n}^{n+m-1} \nu_k \\
&= R\,\|x_n - q\| + R \sum_{k=n}^{n+m-1} \nu_k
\end{aligned} \tag{3.9}$$

where $R = e^{\left(\sum_{n=1}^{\infty} t_k\right)} < \infty$. Since

$$\lim_{n\to\infty} d(x_n, F(T)) = 0, \qquad \sum_{n=1}^{\infty} \nu_n < \infty \tag{3.10}$$

for any given $\varepsilon > 0$, there exists a positive integer n_1 such that

$$d(x_n, F(T)) < \frac{\varepsilon}{4(R+1)}, \qquad \sum_{k=n}^{n+m-1} \nu_k < \frac{\varepsilon}{2R}, \quad \forall\, n \geq n_1. \tag{3.11}$$

Hence, there exists $q_1 \in F(T)$ such that

$$\|x_n - q_1\| < \frac{\varepsilon}{2(R+1)}, \quad \forall\, n \geq n_1. \tag{3.12}$$

Consequently, for any $n \geq n_1$ and $m \geq 1$, from (3.9), we have

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - q_1\| + \|x_n - q_1\| \\
&\leq R\|x_n - q_1\| + R\sum_{k=n}^{n+m-1} \nu_k + \|x_n - q_1\| \\
&= (R+1)\|x_n - q_1\| + R\sum_{k=n}^{n+m-1} \nu_k \\
&< (R+1)\frac{\varepsilon}{2(R+1)} + R\frac{\varepsilon}{2R} = \varepsilon. \qquad (3.13)
\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in E and so is convergent since E is complete. Let $\lim_{n\to\infty} x_n = q^*$. Then $q^* \in C$. It remains to show that $q^* \in F(T)$. Let $\varepsilon_1 > 0$ be given. Then there exists a natural number n_2 such that $\|x_n - q^*\| < \frac{\varepsilon_1}{4}$ for all $n \geq n_2$. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists a natural number $n_3 \geq n_2$ such that for all $n \geq n_3$ we have $d(x_n, F(T)) < \frac{\varepsilon_1}{5}$ and in particular we have $d(x_{n_3}, F(T)) \leq \frac{\varepsilon_1}{5}$. Therefore, there exists $w^* \in F(T)$ such that $\|x_{n_3} - w^*\| < \frac{\varepsilon_1}{4}$. For any $n \geq n_3$, we have

$$\begin{aligned}
\|Tq^* - q^*\| &\leq \|Tq^* - w^*\| + \|w^* - q^*\| \\
&\leq 2\|q^* - w^*\| \\
&\leq 2\Big(\|q^* - x_{n_3}\| + \|x_{n_3} - w^*\|\Big) \\
&< 2\Big(\frac{\varepsilon_1}{4} + \frac{\varepsilon_1}{4}\Big) \\
&< \varepsilon_1.
\end{aligned}$$

This implies that $Tq^* = q^*$ and hence $q^* \in F(T)$. This shows that q^* is a fixed point of T. Thus $\{x_n\}$ converges strongly to a fixed point of the mapping T. This completes the proof. □

Theorem 3.2. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T\colon C \to C$ be a nearly asymptotically nonexpansive mapping with sequence $\{a_n, \eta(T^n)\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}\Big(\eta(T^n) - 1\Big) < \infty$. Let $\{x_n\}$ be the three-step iteration defined by* (1.3). *If T satisfies the following conditions:*

(i) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

(ii) *If the sequence $\{z_n\}$ in C satisfies $\lim_{n\to\infty} \|z_n - Tz_n\| = 0$, then* $\liminf_{n\to\infty} d(z_n, F(T)) = 0$ *or* $\limsup_{n\to\infty} d(z_n, F(T)) = 0$.

Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. It follows from the hypothesis that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. From (ii), $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ or $\limsup_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, the sequence $\{x_n\}$ must converges to a fixed point of T by Theorem 3.1. This completes the proof. $\square$

Theorem 3.3. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T\colon C \to C$ be a nearly asymptotically nonexpansive mapping with sequence $\{a_n, \eta(T^n)\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}\Big(\eta(T^n) - 1\Big) < \infty$. Let $\{x_n\}$ be the three-step iteration defined by* (1.3). *If T satisfies the following conditions:*

- (A_1) $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$,
- (A_2) *there exists a constant $K > 0$ such that $\|x_n - Tx_n\| \geq K\, d(x_n, F(T))$ for all $n \geq 1$.*

Then $\{x_n\}$ converges strongly to a fixed point of the mapping T.

Proof. From conditions (A_1) and (A_2), we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$, it follows as in the proof of Theorem 3.1, that $\{x_n\}$ must converges strongly to a fixed point of the mapping T. This completes the proof. $\square$

Theorem 3.4. *Let E be a real Banach space and C be a nonempty closed convex subset of E. Let $T\colon C \to C$ be a nearly asymptotically nonexpansive mapping with sequence $\{a_n, \eta(T^n)\}$ and $F(T) \neq \emptyset$ such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\sum_{n=1}^{\infty}\Big(\eta(T^n) - 1\Big) < \infty$. Let $\{x_n\}$ be the three-step iteration defined by* (1.3). *If T is semi-compact and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then the sequence $\{x_n\}$ converges to a fixed point of T.*

Proof. From the hypothesis, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Also, since T is semi-compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to q \in C$ and we make use of the fact that every nearly asymptotically nonexpansive mapping is nearly k-Lipschitzian. Hence, we have

$$\begin{aligned} \|q - Tq\| &\leq \|q - x_{n_j}\| + \|x_{n_j} - Tx_{n_j}\| + \|Tx_{n_j} - Tq\| \\ &\leq (1 + k_1)\|q - x_{n_j}\| + \|x_{n_j} - Tx_{n_j}\| \to 0. \end{aligned}$$

Thus $q \in F(T)$. By (3.8),

$$\|x_{n+1} - q\| \leq [1 + (\mu_n - 1)]\|x_n - q\| + \nu_n.$$

Since $\sum_{n=1}^{\infty}(\mu_n - 1) < \infty$ and $\sum_{n=1}^{\infty} \nu_n < \infty$, by Lemma 2.1, $\lim_{n\to\infty} \|x_n - q\|$ exists and $x_{n_j} \to q \in F(T)$ gives that $x_n \to q \in F(T)$. This shows that $\{x_n\}$ converges to a fixed point of T. This completes the proof. $\square$

Theorem 3.5. *Let E be a real Banach space satisfying Opial's condition and C be a nonempty closed convex subset of E. Let $T\colon C \to C$ be a nearly asymptotically nonexpansive mapping with sequence $\{a_n, \eta(T^n)\}$ and $F(T) \neq \emptyset$*

such that $\sum_{n=1}^{\infty} a_n < \infty$ *and* $\sum_{n=1}^{\infty} \Big(\eta(T^n) - 1\Big) < \infty$. *Let* $\{x_n\}$ *be the three-step iteration defined by* (1.3). *Suppose that* T *has a fixed point,* $I - T$ *is demiclosed at zero and* $\{x_n\}$ *is an approximating fixed point sequence for* T, *that is,* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. *Then* $\{x_n\}$ *converges weakly to a fixed point of* T.

Proof. Let p be a fixed point of T. Then $\lim_{n\to\infty} \|x_n - p\|$ exists as proved in Theorem 3.1. We prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. For, let u and v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By hypothesis of the theorem, we know that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $I - T$ is demiclosed at zero, therefore we obtain $Tu = u$. Thus $u \in F(T)$. Again in the same fashion, we can prove that $v \in F(T)$. Next, we prove the uniqueness. To this end, if u and v are distinct then by Opial's condition,

$$\begin{aligned} \lim_{n\to\infty} \|x_n - u\| &= \lim_{n_i\to\infty} \|x_{n_i} - u\| \\ &< \lim_{n_i\to\infty} \|x_{n_i} - v\| = \lim_{n\to\infty} \|x_n - v\| = \lim_{n_j\to\infty} \|x_{n_j} - v\| \\ &< \lim_{n_j\to\infty} \|x_{n_j} - u\| = \lim_{n\to\infty} \|x_n - u\|. \end{aligned}$$

This is a contradiction. Hence $u = v \in F(T)$. Thus $\{x_n\}$ converges weakly to a fixed point of T. This completes the proof. □

Remark 3.6. Our results extend and generalize the corresponding results of [10]-[15] and many others from the existing literature to the case of three-step iteration scheme and more general class of nonexpansive and asymptotically nonexpansive mappings considered in this paper.

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