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THE OPTIMAL LOWER BOUND FOR A POLYNOMIAL NORM WHICH IS A PRODUCT OF LINEAR AND CONTINUOUS FORMS IN A HILBERT SPACE

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Abstract. The estimation of lower bounds for the norms of homogeneous polynomials which are products of linear forms in a Banach space, was obtained by K. Ball in a very precise description in the case where H is a complex Hilbert space with dimension $\geq n$. He also managed to obtain a better bound estimate for $c_n(H) = n^{-n/2}$. The above result is taken as a corollary of Ball's theorem, which is not valid in the case of real Hilbert spaces. In this paper we studied the reasonable question if the above result is valid in the case of real Hilbert spaces.

1. Introduction

We define polynomials in infinite dimension spaces using multilinear mappings. Let E and F be two vector spaces on $\mathbf{K}$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. We shall call the mapping $L : E^n \to F$ n-linear form if the mapping $x_i \mapsto L(x_1, \ldots, x_i, \ldots, x_n)$, $i = 1, 2, \ldots, n$, is linear. Also we shall call the $L : E^n \to$

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F symmetric if

$$L(x_1, x_2, \dots, x_n) = L(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}),$$

for all $(x_1, x_2, \dots, x_n) \in E^n$ and every permutation of the first n natural numbers. If $L : E^n \to F$ is a n-linear form we put:

$$S(L)(x_1, x_2, \dots, x_n) := \frac{1}{n!} \sum_{\sigma \in S_n} L\left(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(n)}\right),$$

where S_n is the set of all permutations of the first n natural numbers. Obviously $S(L) : E^n \to F$ is a symmetric n-linear form. We put

$$\widehat{L}(x) := L(x, x, \dots, x), \quad \forall\, x \in E.$$

Definition 1.1. We define the mapping $P : E \to F$ as a homogeneous polynomial of n-degree if there exists an n-linear form $L : E^n \to F$ such that $P = \widehat{L}$, i.e.,

$$P(x) = \widehat{L}(x) = L(x, x, \dots, x) .$$

Generally there is no a bijection between the n-linear forms and the homogeneous polynomials of n-degree. Though there exists a bijection between the symmetric n-linear forms and the homogeneous polynomials of n-degree.

The proof of this claim is based on the following Lemma where we use the polarization formulas.

Lemma 1.2. *If $L : E^n \to F$ is a symmetric n-linear form and $P : E \to F$ a homogeneous polynomial of n-degree with $P = \widehat{L}$, then*:

$$L(x_1, x_2, \dots, x_n) = \frac{1}{2^n n!} \sum_{\varepsilon_i = \pm 1} \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n P\left(\sum_{k=1}^{n} \varepsilon_k x_k\right),$$

where the sum is over all $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \in \{-1, 1\}$.

Definition 1.3. We define the mapping $P : E \to F$ as a polynomial of n-degree, if $P = P_n + P_{n-1} + \cdots + P_0$ where P_k, $1 \le k \le n$ is a homogeneous polynomial of k-degree, $P_n \neq 0$ and P_0 is a constant mapping.

2. Optimal lower bound for norms of linear products of polynomials on a Banach space

Let $P_1, \dots, P_n$ be polynomials in a Banach space E. Then the product $P_1 \cdots P_n : E \to \mathbb{K}$ with $(P_1 \cdots P_n)(x) := P_1(x) \cdots P_n(x)$, for all $x \in E$, is also a polynomial. We also have that the following relation holds true:

$$\|P_1 \cdots P_n\| \le \|P_1\| \cdots \|P_n\| .$$

Problem: Find a positive constant M, depending only on the degrees of polynomials, such that

$$\|P_1\|\cdots\|P_n\| \le M\cdot\|P_1\cdots P_n\| \ . \tag{2.1}$$

A number of inequalities of this type has already appeared in the literature. In the classical setting of single complex variable polynomials with the supremum norm, Mahler [5] established relation (2.1) with $M = 2^m$ where $deg(P_1\cdots P_n) = m$. Kroó and Pritsker [4] improved the above constant and proved that $M = 2^{m-1}$. Generally constant 2^{m-1} can't be improved any more. In the general case of complex Banach spaces Benítez, Sarantopoulos and Tonge [3] obtained the optimal constant:

$$M = \frac{(m_1+\cdots+m_n)^{m_1+\cdots+m_n}}{m_1^{m_1}\cdots m_n^{m_n}},$$

where $degP_k = m_k,\ k = 1,2,\ldots,n$. It is easy to see that on the space $E = \ell_1$ the homogenous polynomials:

$$P_j\left((x_a)_{a=1}^{\infty}\right) := x_{m_1+\cdots+m_{j-1}+1}\cdots x_{m_1+\cdots+m_j}$$

with $m_0 := 0,\ j = 1,\ldots,n$ and $degP_j = m_j$ satisfy the following:

$$\|P_1\|\cdots\|P_n\| = \frac{(m_1+\cdots+m_n)^{m_1+\cdots+m_n}}{m_1^{m_1}\cdots m_n^{m_n}}\|P_1\cdots P_n\| \ .$$

Especially, if E is a **complex** Banach space and $L_k \in E^*,\ k = 1,\ldots,n$, then

$$\|L_1\|\cdots\|L_n\| \le n^n\cdot\|L_1\cdots L_n\|$$

and the constant n^n is the best possible.

3. Optimal lower bound for norms of linear products of polynomials on a Hilbert space

3.1. **H is a complex Hilbert space.** In some Banach spaces constant n^n can be improved. Arias-de-Reyna [1] proved that

$$\|L_1\|\cdots\|L_n\| \le n^{\frac{n}{2}}\cdot\|L_1\cdots L_n\|,$$

where $L_k \in H^*,\ k = 1,\ldots,n$ and $(H,\langle\cdot,\cdot\rangle)$ is a **complex** Hilbert space.

We also have that, if $a_k \in H$ with $\|a_k\| = 1,\ k = 1,\ldots,n$, the above inequality takes the following form

$$\sup_{\|x\|=1} |\langle a_1,x\rangle\cdots\langle a_n,x\rangle| \ge n^{-n/2} \ . \tag{3.1}$$

Recently Ball [2] proved the next theorem:

Theorem 3.1. *Let $(a_j)_1^n$ be a sequence of normed one vectors in a* **complex** *Hilbert space and $(t_j)_1^n$ is a sequence of non-negative numbers with $\sum_{j=1}^{n} t_j^2 = 1$. Then there exist a unit vector x such that*

$$|\langle a_j, x\rangle| \geq t_j \ , \quad j = 1, \ldots, n \ .$$

3.2. **H is a real Hilbert space.** Just like in the complex case, for a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$, with $a_k \in H$ and $\|a_k\| = 1$ $(1 \leq k \leq n)$, we have that

$$\sup_{\|x\|=1} |\langle a_1, x\rangle \cdots \langle a_n, x\rangle| \geq n^{-n/2} \ .$$

The question is if we obtain the same result for a real Hilbert space.

Lemma 3.2. *Let $a_1, a_2, \ldots, a_n$ unit vectors in a Hilbert space. If we have that*

$$\sup_{\|x\|\leq 1} |\langle a_1, x\rangle\langle a_2, x\rangle \cdots \langle a_n, x\rangle| = |\langle a_1, \xi\rangle\langle a_2, \xi\rangle \cdots \langle a_n, \xi\rangle|$$

for $\xi \in H$ with $\|\xi\| = 1$, then we get

$$n\xi = \frac{a_1}{\langle a_1, \xi\rangle} + \cdots + \frac{a_n}{\langle a_n, \xi\rangle} \ . \tag{3.2}$$

Remark 3.3. If $H = \mathbb{R}^n$ or $\mathbb{C}^n$, from the previous Lemma, we get that

$$\xi_1\langle a_1, \xi\rangle = \cdots = \xi_n\langle a_n, \xi\rangle = \frac{1}{n} \ ,$$

where $\xi = \xi_1 a_1 + \cdots + \xi_n a_n$.

Now we are able to prove the next theorem:

Theorem 3.4. *Let $a_1, \ldots, a_n$ n-linear unit vectors which are also independent in the Euclid space $\mathbb{R}^n$. Then the next estimate holds*:

$$\max_{\|x\|_2=1} |\langle a_1, x\rangle\langle a_2, x\rangle \cdots \langle a_n, x\rangle| \geq \left(\frac{t_1}{n}\right)^{\frac{n}{2}} \ ,$$

where t_1 is the lowest eigenvalue of the matrix Gram A of the unit vectors $a_1, \ldots, a_n$, that is, $A = [\langle a_i, a_j\rangle]$, $i, j = 1, 2, \ldots, n$.

Proof. If $\xi = \xi_1 a_1 + \cdots + \xi_n a_n \in \mathbb{R}^n$ with $\|\xi\|_2 = 1$ and

$$\max_{\|x\|_2=1} |\langle a_1, x\rangle\langle a_2, x\rangle \cdots \langle a_n, x\rangle| = |\langle a_1, \xi\rangle\langle a_2, \xi\rangle \cdots \langle a_n, \xi\rangle| \ ,$$

then from (3.2) we get

$$\begin{aligned} |\langle a_1,\xi\rangle\langle a_2,\xi\rangle\cdots\langle a_n,\xi\rangle| &= \frac{1}{n^n}\cdot\frac{1}{|\xi_1\cdots\xi_n|} = \frac{1}{n^n}\cdot\left(\frac{1}{\xi_1^2\cdots\xi_n^2}\right)^{\frac{1}{2}} \\ &\geq \frac{1}{n^n}\cdot\left(\frac{n}{\xi_1^2+\cdots+\xi_n^2}\right)^{\frac{n}{2}} \\ &= \frac{1}{n^{n/2}}\cdot\frac{1}{(\xi_1^2+\cdots+\xi_n^2)^{n/2}}\ . \end{aligned}$$

Thus we have that:

$$|\langle a_1,\xi\rangle\langle a_2,\xi\rangle\cdots\langle a_n,\xi\rangle| \geq \frac{1}{n^{n/2}}\cdot\frac{1}{(\xi_1^2+\cdots+\xi_n^2)^{n/2}}\ . \tag{3.3}$$

If $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$, then the equation

$$q(x) := \|x_1a_1+\cdots+x_na_n\|_2^2 = \sum_{i,j=1}^{n}\langle a_i,a_j\rangle x_ix_j \tag{3.4}$$

is a symmetric and positive quadratic form. We have that $q(\xi_1,\ldots,\xi_n)=1$, so in order to find a lower bound for the inequality (3.3), we must find the extremals points of the function $f(x):=x_1^2+\cdots+x_n^2$ which satisfy the equation $q(x)=1$. That is, we will find the extremals points of f under the assumption that $g(x)=0$, where $g(x):=q(x)-1$. Using the Lagrange multipliers method we assume the vector equation

$$\nabla f(x)+k\nabla q(x)=0\ . \tag{3.5}$$

Since f and q are homogenous polynomials of 2-degree, applying Euler's Theorem we obtain

$$\langle x,\nabla\left(f(x)+kq(x)\right)\rangle = 2\left(f(x)+kq(x)\right)\ .$$

Also, since $\nabla(f(x)+kq(x))=\nabla f(x)+k\nabla q(x)=0$, we have $f(x)+kq(x)=0$. We already have that $q(x)=1$, hence $k=-f(x)$. So, relation (3.5) becomes

$$t\nabla f(x)-\nabla q(x)=0\ ,\quad \text{where}\ \ t=1/f(x)\ . \tag{3.6}$$

The vector equation (3.6) leads to the following n equations with n solutions system

$$\begin{gathered} [\langle a_1,a_1\rangle-t]x_1+\langle a_1,a_2\rangle x_2+\cdots+\langle a_1,a_n\rangle x_n=0, \\ \langle a_2,a_1\rangle x_1+[\langle a_2,a_2\rangle-t]x_2+\cdots+\langle a_2,a_n\rangle x_n=0, \\ \vdots \\ \langle a_n,a_1\rangle x_1+\langle a_n,a_2\rangle x_2+\cdots+[\langle a_n,a_n\rangle-t]x_n=0\ . \end{gathered}$$

Since we have that $(x_1, \dots, x_n) = (0, \dots, 0)$ is not a solution for our problem, the determinant of the system must be equal to zero, that is

$$\begin{vmatrix} \langle a_1, a_1\rangle - t & \langle a_1, a_2\rangle & \dots & \langle a_1, a_n\rangle \\ \langle a_2, a_1\rangle & \langle a_2, a_2\rangle - t & \dots & \langle a_2, a_n\rangle \\ \vdots & \vdots & & \vdots \\ \langle a_n, a_1\rangle & \langle a_n, a_2\rangle & \dots & \langle a_n, a_n\rangle - t \end{vmatrix} = 0 . \tag{3.7}$$

This is the characteristic equation of $q(x)$. Since $q(x)$ is symmetric and positive, if $t_1, t_2, \dots, t_n$ are the solutions of (3.7) (with other words the eigenvalues of the matrix $A = [\langle a_i, a_j\rangle], i, j = 1, 2, \dots, n)$, then all the solutions are real and positive. Let t_1 the smallest solution. Thus for the vector $(\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ we have:

$$f(\xi_1, \dots, \xi_n) \leq \frac{1}{t_1} \Leftrightarrow \left(\xi_1^2 + \dots + \xi_n^2\right)^{1/2} \leq \frac{1}{t_1^{1/2}} .$$

Equivalently, we have the following relation

$$\frac{1}{\left(\xi_1^2 + \dots + \xi_n^2\right)^{n/2}} \geq t_1^{n/2} . \tag{3.8}$$

The last inequality combined with relation (3.3) prove our Theorem. □

4. The n-th (linear) polarization constant of a normed space

4.1. **The n-th (linear) polarization constant of a Banach space.** If $f_1, f_2, \dots, f_n$ are bounded linear functionals on a Banach space X, then the product $(f_1 f_2 \cdots f_n)(x) := f_1(x) f_2(x) \cdots f_n(x)$ is a continuous n-homogeneous polynomial on X. Ryan and Turett have shown (see Theorem 9 in [7]), in their study of the strongly exposed points of the predial of the space of continuous 2-homogeneous polynomials, threat there exists $C_n > 0$ such that:

$$\|f_1\| \, \|f_2\| \cdots \|f_n\| \leq C_n \|f_1 f_2 \cdots f_n\| ,$$

where

$$\|f_1 f_2 \cdots f_n\| = \sup_{\|x\|=1} |f_1(x) f_2(x) \cdots f_n(x)| .$$

Definition 4.1. ([3]) The n-th (linear) polarization constant of a normed Banach space X is defined by

$$\begin{aligned} c_n(X) : &= \inf\{M > 0 : \|f_1\| \cdots \|f_n\| \leq M \cdot \|f_1 \cdots f_n\|, \quad \forall\, f_1, \dots, f_n \in X^*\} \\ &= 1 / \inf_{f_1, \dots, f_n \in S_{X^*}} \sup_{\|x\|=1} |f_1(x) \cdots f_n(x)| . \end{aligned} \tag{4.1}$$

On the other hand, the *general* n-th polarization constant $\mathbb{K}(n, X)$ of a Banach space X on a space $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) is defined by:

$$\mathbb{K}(n, X) := \inf\{M : \|L\| \leq M\|P\| , \quad \forall\, P \in \mathcal{P}(^nX)\} ,$$

where L is the symmetric continuous n-linear mapping $P = \widehat{L}$ (in the above definition we have to do with all the continuous n-homogeneous polynomials, not only with the products of linear mappings). In the case of complex $L_p(\mu)$ spaces the constant n^n can be improved.

Proposition 4.2. *If* $X = L_p(\mu)$*, then*

$$c_n(L_p(\mu)) \leq \begin{cases} n^{\frac{n}{p}} , & \text{if } \ 1 \leq p \leq 2, \\ n^{\frac{n}{p'}} , & \text{if } \ 2 \leq p \leq \infty, \end{cases} \tag{4.2}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$*. For* $1 \leq p \leq 2$ *the constant is the best possible.*

Remark 4.3. Let us remark here that for the proof of relation (4.2) we assumed that the dimension of the $L_p(\mu)$ spaces is at least n. In the case where $2 \leq p \leq \infty$ we have that

$$c_n(L_p(\mu)) \leq n^{n/p'} .$$

The constant $n^{n/p'}$, in that case is not the best possible. Using the result from Kroó and Pritsker [4], we see that

$$\|L_1\| \cdots \|L_n\| \leq 2^{n-1}\|L_1 \cdots L_n\|$$

and the constant 2^{n-1} is the best possible. So, for $n = 2$ we have $c_2(\ell^2_\infty) = 2$ although relation (4.2) gives us the following result

$$c_2(\ell^2_\infty) \leq 2^2 .$$

Proposition 4.4. *Let* X *be a Banach space, then*:

$$c_n(X) \leq n^n .$$

4.2. **The n-th (linear) polarization constant of a Hilbert space.** In the special case of a Hilbert space H, using the Riesz Representation Theorem, Definition 4.1 turns to be:

$$c_n(H) := 1/ \inf_{x_1,\ldots,x_n \in S_H} \sup_{\|x\|=1} |\langle x, x_1\rangle\langle x, x_2\rangle \cdots \langle x, x_n\rangle| .$$

Theorem 4.5. ([1]) *We have that the following result holds*:

$$c_n(\mathbb{C}^n) = n^{\frac{n}{2}} .$$

The optimal lower bound for a normed polynomial which is a product of continuous and linear functionals in a Banach space was studied in the case of a *complex Hilbert* space H with dimension $\geq n$, from Ball [2]. He managed to calculate the optimal lower bound

$$c_n(H) = n^{-n/2} .$$

The above result is a corollary of Ball's Theorem, which (theorem) doesn't holds for real Hilbert spaces. We study the following crucial question:

If the above result holds for *real Hilbert* spaces. For this, we assume that

$$c_n(\mathbb{R}^n) = n^{\frac{n}{2}} . \tag{4.3}$$

For the previous result we refer to Pappas and Revez [6].

Theorem 4.6. *We have that $c_n(\mathbb{R}^n) = n^{n/2}$ for $n = 2, 3, 4$ and 5. Thus, for $n \leq \min\{d, 5\}$, we also obtain that $c_n(\mathbb{R}^d) = n^{n/2}$.*

Lemma 4.7. *Let H be a real Hilbert space and let $a_k \in S_H$, $k = 1, \ldots, n$ be arbitrary unit vectors. Suppose that for some unit vector $\xi \in S_H$ and with some $\delta > 0$ we have*

$$\max_{\|x\|=1, |x-\xi|\leq\delta} |\langle a_1, x\rangle| \cdots |\langle a_n, x\rangle| = |\langle a_1, \xi\rangle \cdots \langle a_n, \xi\rangle| , \tag{4.4}$$

that is, ξ is a local (conditional) maximum of polynomial P on S_H. Then we have

$$n\xi = \frac{a_1}{\langle a_1, \xi\rangle} + \cdots + \frac{a_n}{\langle a_n, \xi\rangle} . \tag{4.5}$$

Proof of Theorem 4.6. Let $n = 2, 3, 4$ or 5 and let the linear functionals be fixed as $a_1, \ldots, a_n \in S_{\mathbb{R}^n}$. Consider the Gram matrix:

$$A := (\langle a_i, a_j\rangle)_{i,j=1,\ldots,n} \in \mathbb{R}^{n\mathrm{x}n} . \tag{4.6}$$

By an appropriate change of signs $\varepsilon_i = \pm 1$ of the vectors a_i, which does not change norm of P, we want to achieve that the row (and thus the collum) sums of the entries of A are all add up at least 1. To get this, select signs ε_i to maximize $\|\sum_{i=1}^n \varepsilon_i a_i\|_2$. Write $a := \sum_{i=1}^n \varepsilon_i a_i$ for this (or, any) maximal vector. If $1 \leq j \leq n$ is an arbitrary index, put

$$b := -2\varepsilon_j a_j + a.$$

Then $\|b\|_2 \leq \|a\|_2$, by assumption. On the other hand, by the parallelogram law

$$\begin{aligned} \| - \varepsilon_j a_j + a\|_2^2 + \|\varepsilon_j a_j\|_2^2 = 1/2(\|a\|_2^2 + \|b\|_2^2) \leq \|a\|_2^2 \\ \Leftrightarrow 2\,\langle \varepsilon_j a_j,\, \varepsilon_j a_j - a\rangle \leq 0\,, \end{aligned}$$

that is $\langle \varepsilon_j a_j, \varepsilon_j a_j - a\rangle \le 0$. Obviously this implies $\langle a, \varepsilon_j a_j\rangle \ge 1$, $j = 1, \dots, n$ as needed. So, without loss of generality we can assume

$$\begin{aligned} y_1 &:= \langle a_1, a_1\rangle + \langle a_1, a_2\rangle + \cdots + \langle a_1, a_n\rangle \ge 1, \\ y_2 &:= \langle a_2, a_1\rangle + \langle a_2, a_2\rangle + \cdots + \langle a_2, a_n\rangle \ge 1, \\ &\vdots \\ y_n &:= \langle a_n, a_1\rangle + \langle a_n, a_2\rangle + \cdots + \langle a_n, a_n\rangle \ge 1. \end{aligned} \tag{4.7}$$

Now let us consider the mean vector

$$x := \frac{a}{\|a\|_2} = \frac{a_1 + \cdots + a_n}{\|a_1 + \cdots + a_n\|_2} \ . \tag{4.8}$$

The theorem will be proved once we show the following lemma.

Lemma 4.8. *Let $n \le 5$. Suppose that the signs of the unit vectors a_i $(i = 1, \dots, n)$ are chosen so that* (4.7) *holds. Then the mean vector* (4.8) *satisfies $|P(x)| \ge n^{-n/2}$.*

Proof. By definition and (4.7), we have $1 \le y_i \le n$ $(i = 1, \dots, n)$. The assertion is equivalent to state that the inequality

$$y_1^2 y_2^2 \cdots y_n^2 \ge \left(\frac{y_1 + y_2 + \cdots + y_n}{n}\right)^n , \tag{4.9}$$

holds true for all the possible vectors $y := (y_1, y_2, \dots, y_n)$ which arise from Gram matrices (4.6) of unit vectors systems satisfying (4.7). However, it is rather difficult to describe the exact set of the arising vectors y, so we settle with the following. □

Lemma 4.9. *Let $n \le 5$. Then* (4.9) *holds true for all $y \in [1, n]^n$.*

Proof. First we remark that $n^2 \ge \left(2 - \frac{1}{n}\right)^n$ for $n = 2, 3, 4$ and 5, while it is false for $n > 5$. However, inequality $n^2 \ge (2 - \frac{1}{n})^n$ is just the special case of (4.9) when $y = (1, \dots, 1, n)$, whence in general (4.9) fails at $y = (1, \dots, 1, n)$ for $n > 5$. So let $n \le 5$ and let us exploit the fact that (4.9) holds when $y = (1, \dots, 1, n)$. First let us consider the variable values $y(t) := (1, \dots, 1, t)$ in the interval $1 \le t \le n$. For these special values the left-hand side of (4.9) is t^2 and the right-hand side is $(\frac{n-1+t}{n})^n$, hence (4.9) is equivalent to $q(t) \ge 0$, with $q(t) := 2\log t - n \log\left(\frac{n-1+t}{n}\right)$. By the above we have $q(n) \ge 0$, while $q(1) = 0$, hence it suffices to show that $q(t)$ is, in fact, a concave function on $[1, n]$. This follows from computing

$$q''(t) = \frac{-2}{t^2} + \frac{n}{(n-1+t)^2} = \frac{(n-2)t^2 - 4(n-1)t - 2(n-1)^2}{t^2(n-1+t)^2} < 0 \ ,$$

the last inequality being valid between the two roots $t_1^{(n)}$ and $t_2^{(n)}$ of the quadratic polynomial in the numerator. (Here, again, one has to use the restricted range of n when calculating $[1,n] \subset [t_1^{(n)}, t_2^{(n)}]$). Let now m be the number of indices of coordinates y_j with $1 < y_j \le n$. When $m = 0$, (4.9) degenerates to $1 = 1$, and when $m = 1$, we obtain (4.9) from the above consideration for $y(t)$. So we argue by induction. Let now $1 \le m < n$, suppose that (4.9) holds for the values when at most m of the variables differ from 1, and let us prove (4.9) for the vector $y = (1, \dots, 1, y_k, y_{k+1}, \dots, y_n)$, where $k := n - m$. First let us apply the inductive hypothesis for $y = (1, \dots, 1, y_{k+1}, \dots, y_n)$ to get

$$y_{k+1}^2 \cdots y_n^2 \ge \left(\frac{k + y_{k+1} + \cdots + y_n}{n} \right)^n . \tag{4.10}$$

Now, put $t := 1 + \frac{n(y_k - 1)}{k + y_{k+1} + \cdots + y_n}$. Then obviously $1 \le t \le y_k$, hence by the $m = 1$ case of $y(t)$ we get

$$y_k^2 \ge t^2 \ge \left(1 + \frac{t-1}{n} \right)^n = \left(1 + \frac{y_k - 1}{k + y_{k+1} + \cdots + y_n} \right)^n . \tag{4.11}$$

Multiplying together equations (4.10) and (4.11) gives (4.9). □

We also give another proof for $n = 2, 3, 4$. We must prove that:

$$\sup_{\|\xi\|_2 = 1} |\langle x_1, \xi \rangle \langle x_2, \xi \rangle \cdots \langle x_n, \xi \rangle| \ge \frac{1}{n^{n/2}} ,$$

where $x_1, x_2, \dots, x_n$ are unit vectors in a real Hilbert space. We must show that this holds for

$$\xi := \frac{x_1 + \cdots + x_n}{\|x_1 + \cdots + x_n\|_2} .$$

So, without loss of generality we can assume:

$$\langle x_1, x_2 \rangle + \langle x_1, x_3 \rangle + \cdots + \langle x_1, x_n \rangle \ge 0,$$

$$\langle x_2, x_1 \rangle + \langle x_2, x_3 \rangle + \cdots + \langle x_2, x_n \rangle \ge 0,$$

$$\vdots$$

$$\langle x_n, x_1 \rangle + \langle x_n, x_2 \rangle + \cdots + \langle x_n, x_{n-1} \rangle \ge 0 .$$

The case n=2 : We must show that:

$$\sup_{\|\xi\|_2 = 1} |\langle x_1, \xi \rangle \langle x_2, \xi \rangle| \ge \frac{1}{2} ,$$

where $\xi := \frac{x_1+x_2}{\|x_1+x_2\|_2}$. We obtain that:

$$\begin{aligned}\langle x_1,\xi\rangle\langle x_2,\xi\rangle &= \frac{\langle x_1, x_1+x_2\rangle\langle x_2, x_1+x_2\rangle}{\|x_1+x_2\|_2^2} = \frac{[1+\langle x_1,x_2\rangle]^2}{2+2\langle x_1,x_2\rangle}\\ &= \frac{1+\langle x_1,x_2\rangle}{2} \geq \frac{1}{2}\ ,\end{aligned}$$

as we have that $\langle x_1, x_2\rangle \geq 0$. The equality holds if and only if $\langle x_1, x_2\rangle = 0$.

The case n=3: We must show that:

$$\sup_{\|\xi\|_2=1} |\langle x_1,\xi\rangle\langle x_2,\xi\rangle\langle x_3,\xi\rangle| \geq \frac{1}{3^{\frac{3}{2}}}\ ,$$

where $\xi := \frac{x_1+x_2+x_3}{\|x_1+x_2+x_3\|_2}$. We also obtain the following relation

$$\begin{aligned}&\langle x_1,\xi\rangle\langle x_2,\xi\rangle\langle x_3,\xi\rangle\\ =\ & \frac{\langle x_1, x_1+x_2+x_3\rangle\langle x_2, x_1+x_2+x_3\rangle\langle x_3, x_1+x_2+x_3\rangle}{\|x_1+x_2+x_3\|_2^3}\\ =\ & \frac{[1+\langle x_1,x_2\rangle+\langle x_1,x_3\rangle]\cdot[1+\langle x_1,x_2\rangle+\langle x_2,x_3\rangle]\cdot[1+\langle x_1,x_3\rangle+\langle x_2,x_3\rangle]}{\{3+2\,[\langle x_1,x_2\rangle+\langle x_1,x_3\rangle+\langle x_2,x_3\rangle]\}^{\frac{3}{2}}}\ .\end{aligned}$$

If we take that

$$\begin{aligned}a &= 1+\langle x_1,x_2\rangle+\langle x_1,x_3\rangle \geq 1,\\ b &= 1+\langle x_1,x_2\rangle+\langle x_2,x_3\rangle \geq 1,\\ c &= 1+\langle x_1,x_3\rangle+\langle x_2,x_3\rangle \geq 1\ ,\end{aligned}$$

then we must prove the following estimate:

$$\begin{aligned}\langle x_1,\xi\rangle\langle x_2,\xi\rangle\langle x_3,\xi\rangle = \frac{a\cdot b\cdot c}{(a+b+c)^{\frac{3}{2}}} &\geq \frac{1}{3^{\frac{3}{2}}}\\ \Leftrightarrow \frac{(a\cdot b\cdot c)^2}{(a+b+c)^3} &\geq \frac{1}{3^3}\\ \Leftrightarrow (a\cdot b\cdot c)^2 &\geq \left(\frac{a+b+c}{3}\right)^3\ .\end{aligned}$$

We assume that

$$f(a,b,c) = (a\cdot b\cdot c)^2 - \left(\frac{a+b+c}{3}\right)^3,$$

then we must prove that $f(a,b,c) \geq 0\ for\ 1 \leq a,b,c \leq 3$. We are looking for local extremals

$$\begin{cases} f_a = 2ab^2c^2 - \left(\frac{a+b+c}{3}\right)^2 = 0,\\ f_b = 2a^2bc^2 - \left(\frac{a+b+c}{3}\right)^2 = 0,\\ f_c = 2a^2b^2c - \left(\frac{a+b+c}{3}\right)^2 = 0\ .\end{cases}$$

So, inside the interval we must have $a = b = c$. Then equality $f_a = 0$ becomes:

$$\begin{aligned} 2a^5 - a^2 = 0 \quad &\Rightarrow \quad a^2\left(2a^3 - 1\right) = 0 \\ &\Rightarrow \quad a = 0 \ \ or \ \ a^3 = \frac{1}{2} \\ &\Rightarrow \quad a = \frac{1}{\sqrt[3]{2}} < 1, \end{aligned}$$

outside the interval. Similarly we must examine the smallest value:

$$1 \leq a, b, c \leq 3.$$

- $a = 1$ or $a = 3$. We have that the following relations hold true

$$f(1, b, c) = b^2c^2 - \left(\frac{1+b+c}{3}\right)^3 \quad \text{and} \quad f(3, b, c) = 9b^2c^2 - \left(\frac{3+b+c}{3}\right)^3 .$$

Now, since $\frac{1+b+c}{3} \geq 1$, we obtain

$$\begin{aligned} f(3, b, c) &= 9b^2c^2 - \left[\frac{2}{3} + \left(\frac{1+b+c}{3}\right)\right]^3 \\ &\geq 9b^2c^2 - \left(\frac{1+b+c}{3}\right)^3 \cdot \left(\frac{2}{3} + 1\right)^3 \\ &= 9b^2c^2 - \frac{125}{27}\left(\frac{1+b+c}{3}\right)^3 \\ &\geq \frac{125}{27} f(1, b, c) . \end{aligned}$$

The above result shows that we have equality only for $a = b = c = 1$. Thus, we must show that $f(1, b, c) \geq 0$ for $1 \leq b, c \leq 3$. We define:

$$g(b, c) \equiv f(1, b, c) \quad \Leftrightarrow \quad g(b, c) = b^2c^2 - \left(\frac{1+b+c}{3}\right)^3 .$$

We are looking for local extremals:

$$\begin{cases} g_b = 2bc^2 - \left(\frac{1+b+c}{3}\right)^2 = 0, \\ g_c = 2b^2c - \left(\frac{1+b+c}{3}\right)^2 = 0. \end{cases}$$

Inside the interval we get $b = c$, since $b = \frac{b+2b}{3} \geq \frac{1+2b}{3}$ and $b \geq 1$, hence we have:

$$g(b, b) = b^4 - \left(\frac{1+2b}{3}\right)^3 \geq b^4 - b^3 \geq 0.$$

Thus, we must check the results for the boundary values $b = 1$ or $b = 3$. We have the following relations

$$g(1,c) = c^2 - \left(\frac{2+c}{3}\right)^3 \quad \text{and} \quad g(3,c) = 9c^2 - \left(\frac{4+c}{3}\right)^3 .$$

Now, since $\frac{2+c}{3} \geq 1$, we have

$$\begin{aligned} g(3,c) &= 9c^2 - \left[\frac{2}{3} + \left(\frac{2+c}{3}\right)\right]^3 \\ &\geq 9c^2 - \left(\frac{2+c}{3}\right)^3 \cdot \left(\frac{2}{3}+1\right)^3 \\ &= 9c^2 - \frac{125}{27}\left(\frac{2+c}{3}\right)^3 \ \geq \ \frac{125}{27}\, g(1,c). \end{aligned}$$

Thus, it is enough for us to prove that: $g(1,c) \geq 0$ for $1 \leq c \leq 3$.
Indeed, for

$$\begin{aligned} & g(1,c) = c^2 - \left(\frac{2+c}{3}\right)^3 \\ \Rightarrow \quad & g'(1,c) = 2c - \left(\frac{2+c}{3}\right)^2 = -\frac{1}{9}\left(c^2 - 14c + 4\right). \end{aligned}$$

So, we have that $g'(1,c) = 0$ for $c = \frac{14 \pm \sqrt{14^2 - 16}}{2} = 7 \pm \sqrt{7^2 - 4}$. We also get that $g'(1,c) \overset{c=1}{=} 1 > 0$ and $g'(1,c) \overset{c=3}{=} 6 - \left(\frac{5}{3}\right)^2 > 1 > 0$. The biggest number is > 3, and the smallest is < 1, since it holds that

$$7 - \sqrt{7^2 - 4} < 1 \quad \Leftrightarrow \quad c \leq \sqrt{45}\,.$$

Finally we have the next relations $g'(1,0) > 0$ and $g(1,1) = 0$.

The case n=4: We must show that:

$$\sup_{\|\xi\|_2 = 1} |\langle x_1, \xi\rangle\langle x_2, \xi\rangle\langle x_3, \xi\rangle\langle x_4, \xi\rangle| \geq \frac{1}{4^{\frac{4}{2}}} = \frac{1}{4^2}\ ,$$

where $\xi := \frac{x_1+x_2+x_3+x_4}{\|x_1+x_2+x_3+x_4\|_2}$. We have the following relation

$$\begin{aligned}
&\langle x_1,\xi\rangle\langle x_2,\xi\rangle\langle x_3,\xi\rangle\langle x_4,\xi\rangle \\
= \; & \frac{\langle x_1, x_1+x_2+x_3+x_4\rangle\langle x_2, x_1+x_2+x_3+x_4\rangle\langle x_3, x_1+x_2+x_3+x_4\rangle}{\|x_1+x_2+x_3+x_4\|_2^4} \\
&\times\langle x_4, x_1+x_2+x_3+x_4\rangle \\
= \; & \frac{[1+\langle x_1,x_2\rangle+\langle x_1,x_3\rangle+\langle x_1,x_4\rangle]\cdot[1+\langle x_1,x_2\rangle+\langle x_2,x_3\rangle+\langle x_2,x_4\rangle]}{\{4+2\left[\langle x_1,x_2\rangle+\langle x_1,x_3\rangle+\langle x_1,x_4\rangle+\langle x_2,x_3\rangle+\langle x_2,x_4\rangle+\langle x_3,x_4\rangle\right]\}^{\frac{4}{2}}} \\
&\times\left[1+\langle x_1,x_3\rangle+\langle x_2,x_3\rangle+\langle x_3,x_4\rangle\right]\left[1+\langle x_1,x_4\rangle+\langle x_2,x_4\rangle+\langle x_3,x_4\rangle\right] .
\end{aligned}$$

If

$$a = 1+\langle x_1,x_2\rangle+\langle x_1,x_3\rangle+\langle x_1,x_4\rangle \geq 1,$$

$$b = 1+\langle x_1,x_2\rangle+\langle x_2,x_3\rangle+\langle x_2,x_4\rangle \geq 1,$$

$$c = 1+\langle x_1,x_3\rangle+\langle x_2,x_3\rangle+\langle x_3,x_4\rangle \geq 1,$$

$$d = 1+\langle x_1,x_4\rangle+\langle x_2,x_4\rangle+\langle x_3,x_4\rangle \geq 1 ,$$

then we must show that:

$$\begin{aligned}
\langle x_1,\xi\rangle\langle x_2,\xi\rangle\langle x_3,\xi\rangle\langle x_4,\xi\rangle = \frac{a\cdot b\cdot c\cdot d}{(a+b+c+d)^{\frac{4}{2}}} &\geq \frac{1}{4^{\frac{4}{2}}} \\
\Leftrightarrow \frac{(a\cdot b\cdot c\cdot d)^2}{(a+b+c+d)^4} &\geq \frac{1}{4^4} \\
\Leftrightarrow (a\cdot b\cdot c\cdot d)^2 &\geq \left(\frac{a+b+c+d}{4}\right)^4 .
\end{aligned}$$

Suppose we have

$$f(a,b,c,d) = (a\cdot b\cdot c\cdot d)^2 - \left(\frac{a+b+c+d}{4}\right)^4 ,$$

then we prove that $f(a,b,c,d) \geq 0$ for $1 \leq a,b,c,d \leq 4$. We are looking for local extremals:

$$\begin{cases}
f_a = 2ab^2c^2d^2 - \left(\frac{a+b+c+d}{4}\right)^3 = 0, \\
f_b = 2a^2bc^2d^2 - \left(\frac{a+b+c+d}{4}\right)^3 = 0, \\
f_c = 2a^2b^2cd^2 - \left(\frac{a+b+c+d}{4}\right)^3 = 0, \\
f_d = 2a^2b^2c^2d - \left(\frac{a+b+c+d}{4}\right)^3 = 0 .
\end{cases}$$

So, inside the interval we must have: $a = b = c = d$. Then, equality $f_a = 0$ becomes:

$$\begin{aligned} 2a^7 - a^3 = 0 \quad &\Rightarrow \quad a^3\left(2a^4 - 1\right) = 0 \\ &\Rightarrow \quad a = 0 \quad \text{or} \quad a^4 = \frac{1}{2} \\ &\Rightarrow \quad a = \frac{1}{\sqrt[4]{2}} < 1, \end{aligned}$$

outside the interval. Similarly we must examine the smallest value:

$$1 \leq a, b, c, d \leq 4 \ .$$

- $a = 1$ or $a = 4$. We have the following results

$$f(1, b, c, d) = b^2c^2d^2 - \left(\frac{1 + b + c + d}{4}\right)^4$$

and

$$f(4, b, c, d) = 16b^2c^2d^2 - \left(\frac{4 + b + c + d}{4}\right)^4 \ .$$

Since $\frac{1+b+c+d}{4} \geq 1$, thus

$$\begin{aligned} f(4, b, c, d) &= 16b^2c^2d^2 - \left[\frac{3}{4} + \left(\frac{1 + b + c + d}{4}\right)\right]^4 \\ &\geq 16b^2c^2d^2 - \left(\frac{1 + b + c + d}{4}\right)^4 \cdot \left(\frac{3}{4} + 1\right)^4 \\ &\geq 16b^2c^2d^2 - \left(\frac{3}{4} + 1\right)^4 \left(\frac{1 + b + c + d}{4}\right)^4 \\ &\geq \left(\frac{3}{4} + 1\right)^4 f(1, b, c, d) \ . \end{aligned}$$

The above result shows that we have equality only for $a = b = c = d = 1$. Thus, we must show that $f(1, b, c, d) \geq 0 \ \ for \ \ 1 \leq b, c, d \leq 4$. We define

$$\begin{aligned} &g(b, c, d) \equiv f(1, b, c, d) \\ &\Leftrightarrow g(b, c, d) = b^2c^2d^2 - \left(\frac{1 + b + c + d}{4}\right)^4 , \end{aligned}$$

for $1 \leq b, c, d \leq 4$. Also we have

$$\begin{aligned} &g(b, b, b) = b^6 - \left(\frac{1 + 3b}{4}\right)^4 \quad \text{and} \quad b = \frac{b + 3b}{4} \geq \frac{1 + 3b}{4} \\ &\Rightarrow g(b, b, b) \geq b^6 - b^4 \quad \text{and} \quad b \geq 1 \Rightarrow g(b, b, b) \geq 0 \ . \end{aligned}$$

Hence, we must check the results for the boundary values $b = 1$ or $b = 4$. We have the following relations

$$g(1,c,d) = c^2d^2 - \left(\frac{2+c+d}{4}\right)^4 \quad \text{and} \quad g(4,c,d) = 16c^2d^2 - \left(\frac{5+c+d}{4}\right)^4 .$$

Hence, since $\frac{2+c+d}{4} \geq 1$, we obtain

$$\begin{aligned} g(4,c,d) &= 16c^2d^2 - \left[\frac{3}{4} + \left(\frac{2+c+d}{4}\right)\right]^4 \\ &\geq 16c^2d^2 - \left(\frac{2+c+d}{4}\right)^4 \cdot \left(\frac{3}{4}+1\right)^4 \\ &\geq \left(\frac{3}{4}+1\right)^4 g(1,c,d) . \end{aligned}$$

We define the next relation:

$$\begin{aligned} & h(c,d) \equiv g(1,c,d) \ \text{ for } \ 1 \leq c,d \leq 4 \\ \Rightarrow \quad & h(c,d) = c^2d^2 - \left(\frac{2+2c}{4}\right)^4 \ \text{ and } c \geq 1 \\ \Rightarrow \quad & h(c,c) \geq c^4 - \left(\frac{1+c}{2}\right)^4 \\ \Rightarrow \quad & h(c,c) \geq 0 . \end{aligned}$$

Hence, we must check the results for the boundary values $c = 1$ or $c = 4$. We have that

$$h(1,d) = d^2 - \left(\frac{3+d}{4}\right)^4$$

and

$$h(4,d) = 16d^2 - \left(\frac{2+4+d}{4}\right)^4 = 16d^2 - \left(\frac{6+d}{4}\right)^4 .$$

Now, since $\frac{3+d}{4} \geq 1$, we get that

$$\begin{aligned} h(4,d) &= 16d^2 - \left(\frac{6+d}{4}\right)^4 = 16d^2 - \left[\frac{3}{4} + \left(\frac{3+d}{4}\right)\right]^4 \\ &\geq 16d^2 - \left(\frac{3+d}{4}\right)^4 \cdot \left(\frac{3}{4}+1\right)^4 \ \geq \ \left(\frac{3}{4}+1\right)^4 h(1,d) . \end{aligned}$$

Thus, we must prove that

$$h(1,d) \geq 0 \ \text{ for } \ 1 \leq d \leq 4 \quad \Rightarrow \quad h(1,d) = d^2 - \left(\frac{3+d}{4}\right)^4$$

$$\Rightarrow \quad h(1,d) = 0 \quad \Rightarrow \quad \left(\frac{3+d}{4}\right)^4 = d^2 \quad \Rightarrow \quad \left(\frac{3+d}{4}\right)^2 = \pm d$$

$$\Rightarrow \quad (3+d)^2 = \pm 16d \quad \Rightarrow \quad 9 + 6d + d^2 = \pm 16d$$

$$\Rightarrow \quad d^2 + 22d + 9 = 0 \ \text{ and } \ d^2 - 10d + 9 = 0 \ .$$

- $d^2 + 22d + 9 = 0 \quad \Rightarrow \quad d = -11 \pm \sqrt{112} < 1$, outside the interval.
- $d^2 - 10d + 9 = 0 \quad \Rightarrow \quad d = 1$ and $d = 9$.

The $h(1,d)$ is positive for our interval $(1 \leq d \leq 4)$.

The case n: We must prove the following result:

$$\sup_{\|\xi\|_2 = 1} |\langle x_1, \xi\rangle \langle x_2, \xi\rangle \cdots \langle x_n, \xi\rangle| \geq \frac{1}{n^{n/2}} \ .$$

We have that

$$f(a_1, a_2, \ldots, a_n) = (a_1 \cdot a_2 \cdot \ldots \cdot a_n)^2 - \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n \geq 0 \ ,$$

where $1 \leq a_1, a_2, ..., a_n \leq n$.

We proved the cases $n = 2, 3, 4$ in a different way. Those proofs can be generalized for n. We hope that in the future, we will be able to get results for this special generalized case.

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