Nonlinear Functional Analysis and Applications
Vol. 20, No. 1 (2015), pp. 97-107

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RATE OF GROWTH OF POLYNOMIALS NOT VANISHING INSIDE A DISK

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Abstract. A. Aziz and Q. Aliya proved that if $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$ is a polynomial of degree n not vanishing in the disk $|z| < k$ where $k \geq 1$, then for every $R > r \geq 1$, $0 \leq t \leq 1$ and $|z| = 1$,

$$\left|P(Rz) - P(rz)\right| \leq \left(\frac{R^n - r^n}{1 + k^\mu \phi_1(R, r, \mu, k)}\right)\left(\max_{|z|=1}|P(z)| - t \min_{|z|=k}|P(z)|\right)$$

where

$$\phi_1(R, r, \mu, k) := \frac{k + \lambda_1(R, r, \mu, k)}{1 + k\lambda_1(R, r, \mu, k)},$$

and

$$\lambda_1(R, r, \mu, k) := \left(\frac{R^\mu - r^\mu}{R^n - r^n}\right)\left(\frac{|a_\mu| k^n}{|a_0| - mt}\right) \leq 1$$

with $m = \min_{|z|=1}|P(z)|$. In this paper, a refinement of above inequality is obtained.

1. Introduction

Let $P(z)$ be a polynomial of degree n and $P'(z)$ be its derivative. Then concerning the estimate of the maximum of $|P'(z)|$ on the unit circle $|z| = 1$,

[0]Received August 31, 2014. Revised November 21, 2014.
[0]2010 Mathematics Subject Classification: 30A10, 30C10, 30E10.
[0]Keywords: Polynomials, restricted zeros, growth, inequalities.

we have

$$\max_{|z|=1}\left|P'(z)\right| \leq n \max_{|z|=1}|P(z)|. \tag{1.1}$$

The above result is due to S. Bernstien [4] known as Bernstein's inequality. The result is best possible and equality in (1.1) holds for $P(z) = \lambda z^n, \lambda \neq 0$.

Concerning the estimate for the maximum modulus on a larger circle $|z| = R$, where $R > 1$, it is well known and is a simple consequence of the Maximum Modulus Principle(for reference see [13, Vol. I, p.137]) that if $P(z)$ is a polynomial of degree n, then

$$\max_{|z|=R>1}|P(z)| \leq R^n \max_{|z|=1}|P(z)|. \tag{1.2}$$

The result is sharp and the extremal polynomial is $P(z) = \lambda z^n; \lambda \neq 0$.

If we restrict ourselves to the class of polynomials having no zero in $|z| < 1$, then both the inequalities (1.1) and (1.2) can sharpened. In fact it was conjectured by P. Erdös and later verified by P.D. Lax [9] that if $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then

$$\max_{|z|=1}\left|P'(z)\right| \leq \frac{n}{2} \max_{|z|=1}|P(z)|. \tag{1.3}$$

The result is best possible and equality in (1.3) holds for $P(z) = \alpha + \beta z^n, |\alpha| = |\beta|$.

As an extension of (1.3), Malik [11] proved that if $P(z)$ is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < k$, $k \geq 1$, then

$$\max_{|z|=1}\left|P'(z)\right| \leq \frac{n}{1+k} \max_{|z|=1}|P(z)|. \tag{1.4}$$

Ankeny and Rivilin [1] used inequality (1.3) and proved that if $P(z)$ is a polynomial of degree n and $P(z)$ does not vanish in $|z| < 1$, then

$$\max_{|z|=R>1}|P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1}|P(z)|. \tag{1.5}$$

The result is sharp and equality in (1.5) holds for $P(z) = \alpha + \beta z^n, |\alpha| = |\beta|$.

As a compact generalization of the inequalities (1.3) and (1.5), A. Aziz and Rather [3] have proved that if $P(z)$ is a polynomial of degree n which does not vanish in $|z| < 1$, then for $R > 1$,

$$|P(Rz) - P(z)| \leq \frac{R^n - 1}{2} \max_{|z|=1}|P(z)| \quad \text{for} \quad |z| = 1. \tag{1.6}$$

The result is sharp and equality in (1.6) holds for the polynomial $P(z) = \lambda z^n + \mu$, $|\lambda| = |\mu| = 1$.

As a generalization of (1.4), it was shown by Chan and Malik [5] that if $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$ is a polynomial of degree n which does not vanish in the disk $|z| < k$, $k \geq 1$ then

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{n}{1+k^{\mu}} \max_{|z|=1} |P(z)| . \tag{1.7}$$

Inequality (1.7) was independently proved by Qazi [14, Lemma 1], who under the same hypothesis has shown that if $P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$ is a polynomial of degree n which does not vanish in the disk $|z| < k$, $k \geq 1$ then

$$\max_{|z|=1} \left|P'(z)\right| \leq \frac{n}{1+k^{\mu}\phi(\mu,k)} \max_{|z|=1} |P(z)| , \tag{1.8}$$

where

$$\phi(\mu,k) = \frac{k + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^{\mu}}{1 + \frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^{\mu+1}} \tag{1.9}$$

and

$$\frac{\mu}{n}\left|\frac{a_\mu}{a_0}\right| k^{\mu} \leq 1, \quad 1 \leq \mu \leq n. \tag{1.10}$$

Clearly $\phi(\mu,k) \geq 1$ for $k \geq 1$ and $1 \leq \mu \leq n$. Hence, (1.8) is refinement of inequality (1.7). For $\mu = 1$ inequality (1.7) is due to Malik [10] and inequality (1.8) was proposed by Govil, Rahman and Schmeisser [8].

A. Aziz and Q. Aliya [2] considered for a fixed μ, the class of polynomials

$$\mathcal{P}_{n,\mu} := \left(P(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu, \quad 1 \leq \mu \leq n \right)$$

of degree at most n not vanishing in the disk $|z| < k$ where $k \geq 1$ and investigated the dependence of

$$\max_{|z|=1}\left|P(Rz) - P(rz)\right| \quad \text{on} \quad \max_{|z|=1}|P(z)|, \quad \min_{|z|=k}|P(z)|.$$

In this direction, they [2] proved the following more general result which constitute a multi faced generalization of several well known polynomial inequalities.

Theorem 1.1. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z| < k$, where $k \geq 1$, then for every $R > r \geq 1$, $0 \leq t \leq 1$ and $|z| = 1$,*

$$\begin{aligned} &\left|P(Rz) - P(rz)\right| \\ &\leq \left(\frac{R^n - r^n}{1 + k^{\mu}\phi_1(R,r,\mu,k)}\right)\left(\max_{|z|=1}|P(z)| - t\min_{|z|=k}|P(z)|\right), \end{aligned} \tag{1.11}$$

where

$$\phi_1(R,r,\mu,k) := \frac{k + \lambda_1(R,r,\mu,k)}{1 + k\lambda_1(R,r,\mu,k)}, \tag{1.12}$$

and

$$\lambda_1(R,r,\mu,k) := \Big(\frac{R^\mu - r^\mu}{R^n - r^n}\Big)\Big(\frac{|a_\mu|k^n}{|a_0| - mt}\Big) \leq 1 \tag{1.13}$$

with $m = \min\limits_{|z|=1}|P(z)|$.

2. Lemmas

For the proofs of our main results, we need the following Lemmas. The first Lemma is due to Aziz and Aliya [2].

Lemma 2.1. *If $P \in \mathcal{P}_{n,\mu}$ and $P(z)$ does not vanish in the disk $|z| < k$, where $k \geq 1$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, then for $R \geq r \geq 1$ and $|z| = 1$,*

$$k^\mu \phi_1(R,r,\mu,k)\big|P(Rz) - P(rz)\big| \leq \big|Q(Rz) - Q(rz)\big| - \big(R^n - r^n\big)tm, \tag{2.1}$$

where $\phi_1(R,r,\mu,k)$ is given by (1.12) and $m = \min\limits_{|z|=k}|P(z)|$.

We also need the following lemma which is a special case of a result due to Govil and Rahman [7, Lemma 10].

Lemma 2.2. *If $P(z)$ is a polynomial of degree n, then for $|z| = 1$,*

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1}|P(z)|,$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

Next Lemma is due to Frappier et al. [6].

Lemma 2.3. *Let $P(z)$ be a polynomial of degree n, where $n \geq 2$. Then for all $R \geq 1$,*

$$\max_{|z|=R}|P(z)| \leq R^n \max_{|z|=1}|P(z)| - (R^n - R^{n-2})|P(0)| \quad for \quad n \geq 2, \tag{2.2}$$

and

$$\max_{|z|=R}|P(z)| \leq R \max_{|z|=1}|P(z)| - (R-1)|P(0)| \quad for \quad n = 1. \tag{2.3}$$

We use Lemma 2.3 to prove the following result which is also of independent interest.

Lemma 2.4. *Let $P(z)$ be a polynomial of degree $n \geq 3$ and $Q(z) = z^n\overline{P(1/\overline{z})}$. Then for every $R > r \geq 1$ and $|z| = 1$,*

$$\begin{aligned}&|P(Rz) - P(rz)| + |Q(Rz) - Q(rz)| \\ &\leq (R^n - r^n)\max_{|z|=1}|P(z)| - \left(\frac{R^n - r^n}{n} - \frac{R^{n-2} - r^{n-2}}{n-2}\right)\big||P'(0)| - |Q'(0)|\big|. \end{aligned} \tag{2.4}$$

Proof. By Lemma 2.2, we have

$$\left|P'(z)+\alpha Q'(z)\right| \leq n \max_{|z|=1}|P(z)| \tag{2.5}$$

for $|z|=1$ and for every $\alpha \in \mathbb{C}$ with $|\alpha|=1$. Applying Lemma 2.3 to the polynomial $P'(z)+\alpha Q'(z)$ and using (2.5), we obtain for $t \geq 1$, $\alpha \in \mathbb{C}$ with $|\alpha|=1$ and $|z|=1$,

$$\begin{aligned}\left|P'(tz)+\alpha Q'(tz)\right| &\leq t^{n-1} \max_{|z|=1}\left|P'(z)+\alpha Q'(z)\right| \\ &\qquad -\left(t^{n-1}-t^{n-3}\right)\left|P'(0)+\alpha Q'(0)\right| \\ &\leq n t^{n-1}\max_{|z|=1}|P(z)|-\left(t^{n-1}-t^{n-3}\right)\left|P'(0)+\alpha Q'(0)\right|. \end{aligned} \tag{2.6}$$

Choosing the argument of α in (2.6) such that

$$\left|P'(tz)+\alpha Q'(tz)\right| = \left|P'(tz)\right| + \left|Q'(tz)\right|$$

for $|z|=1$ and from (2.6) by using traingle inequality, we obtain

$$\begin{aligned}&\left|P'\left(te^{i\theta}\right)\right| + \left|Q'\left(te^{i\theta}\right)\right| \\ &\leq n t^{n-1}\max_{|z|=1}|P(z)| - \left(t^{n-1}-t^{n-3}\right)\left||P'(0)|-|Q'(0)|\right| \end{aligned} \tag{2.7}$$

where $0 \leq \theta < 2\pi$. Hence for $R > r \geq 1$ and $0 \leq \theta \leq 2\pi$, we get with the help of (2.7).

$$\begin{aligned}&\left|P\left(Re^{i\theta}\right)-P\left(re^{i\theta}\right)\right| + \left|Q\left(Re^{i\theta}\right)-Q\left(re^{i\theta}\right)\right| \\ &= \left|\int_r^R e^{i\theta}P'(te^{i\theta})dt\right| + \left|\int_r^R e^{i\theta}Q'(te^{i\theta})dt\right| \\ &\leq \int_r^R \left|P'(te^{i\theta})\right| dt + \int_r^R \left|Q'(te^{i\theta})\right| dt \\ &= \int_r^R \left(\left|P'(te^{i\theta})\right| + \left|Q'(te^{i\theta})\right|\right) dt \\ &\leq \max_{|z|=1}|P(z)| \int_r^R n t^{n-1} dt - \left||P'(0)|-|Q'(0)|\right| \int_r^R \left(t^{n-1}-t^{n-3}\right) dt \\ &= \left(R^n - r^n\right)\max_{|z|=1}|P(z)| - \left(\frac{R^n-r^n}{n} - \frac{R^{n-2}-r^{n-2}}{n-2}\right)\left||P'(0)|-|Q'(0)|\right|. \end{aligned}$$

This completes the proof of Lemma 2.4. □

Next Lemma is also obtained by using Lemma 2.3.

Lemma 2.5. *If $P(z)$ is a polynomial of degree n where $n \geq 3$, with $|P(0)| \neq 0$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, then for every $R \geq r \geq 1$ and $|z| = 1$,*

$$\begin{aligned}&|P(Rz) - P(rz)| + |Q(Rz) - Q(rz)|\\ \leq&(R^n - r^n)\max_{|z|=1}|P(z)| - \left(\frac{R^n - r^n}{n} - \frac{R^{n-2} - r^{n-2}}{n-2}\right)\left(|P'(0)| + |Q'(0)|\right),\end{aligned}$$

provided $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z| = 1$.

Proof. Since $P(z)$ is a polynomial of degree n and $P(0) \neq 0$, then $P'(z)$ and $Q'(z)$ are polynomials of degree $n-1$ therefore by Lemma 2.3, we have

$$|P'(te^{i\theta})| \leq t^{n-1}\max_{|z|=1}|P'(z)| - (t^{n-1} - t^{n-3})|P'(0)|, \quad n \geq 3 \tag{2.8}$$

and

$$|Q'(te^{i\theta})| \leq t^{n-1}\max_{|z|=1}|Q'(z)| - (t^{n-1} - t^{n-3})|Q'(0)|, \quad n \geq 3 \tag{2.9}$$

for all $t \geq 1, 0 \leq \theta \leq 2\pi$. Adding (2.8) and (2.9), we get

$$\begin{aligned}&|P'(te^{i\theta})| + |Q'(te^{i\theta})|\\ &\leq t^{n-1}\left(\max_{|z|=1}|P'(z)| + \max_{|z|=1}|Q'(z)|\right) - \left(t^{n-1} - t^{n-3}\right)\left(|P'(0)| + |P'(0)|\right).\end{aligned}$$

If $|P'(z)|$ and $|Q'(z)|$ have maximum at $z_0 = e^{i\theta_0}$, therefore by Lemma 2.2

$$\begin{aligned}&|P'(te^{i\theta})| + |Q'(te^{i\theta})|\\ &\leq t^{n-1}\left(|P'(e^{i\theta_0})| + |Q'(e^{i\theta_0})|\right) - \left(t^{n-1} - t^{n-3}\right)\left(|P'(0)| + |P'(0)|\right) \qquad (2.10)\\ &\leq nt^{n-1}\max_{|z|=1}|P(z)| - \left(t^{n-1} - t^{n-3}\right)\left(|P'(0)| + |P'(0)|\right)\end{aligned}$$

for all $t \geq 1, 0 \leq \theta \leq 2\pi$. Hence for every $R \geq r \geq 1$ and $0 \leq \theta \leq 2\pi$, we have by using (2.10),

$$\begin{aligned}
&|P(Re^{i\theta}) - P(re^{i\theta})| + |Q(Re^{i\theta}) - Q(re^{i\theta})| \\
&= \left| \int_r^R e^{i\theta} P'(te^{i\theta}) dt \right| + \left| \int_r^R e^{i\theta} Q'(te^{i\theta}) dt \right| \\
&\leq \int_r^R \left| P'(te^{i\theta}) \right| dt + \int_r^R \left| Q'(te^{i\theta}) \right| dt \\
&= \int_r^R \left(\left| P'(te^{i\theta}) \right| + \left| Q'(te^{i\theta}) \right| \right) dt \\
&\leq \left\{ \int_r^R nt^{n-1} dt \right\} \max_{|z|=1} |P(z)| - \left(|P'(0)| + |Q'(0)| \right) \int_r^R (t^{n-1} - t^{n-3}) dt \\
&= (R^n - r^n) \max_{|z|=1} |P(z)| - \left(\frac{R^n - r^n}{n} - \frac{R^{n-2} - r^{n-2}}{n-2} \right) \left(|P'(0)| + |Q'(0)| \right)
\end{aligned}$$

which is equivalent to the desired result. □

3. Main Results

In this paper, we first present the following result.

Theorem 3.1. *If $P \in \mathcal{P}_{n,\mu}$, $n > 2$ and $P(z)$ does not vanish in the disk $|z| < k$, where $k \geq 1$, then for every $R > r \geq 1$, $0 \leq t \leq 1$ and $|z| = 1$,*

$$\begin{aligned}
\left| P(Rz) - P(rz) \right| \leq &\left(\frac{R^n - r^n}{1 + k^\mu} \right) \left\{ \max_{|z|=1} |P(z)| - t \min_{|z|=k} |P(z)| \right\} \\
&- \left(\frac{R^n - r^n}{n} - \frac{R^{n-2} - r^{n-2}}{n-2} \right) \left(\frac{\left| |P'(0)| - |Q'(0)| \right|}{1 + k^\mu} \right),
\end{aligned}$$

where $m = \min_{|z|=1} |P(z)|$ and $Q(z) = z^n \overline{P(1/\overline{z})}$.

Instead of proving Theorem 3.1, we obtain a more improved result which among other things provide a refinement of Theorem 1.1. More precisely, we prove:

Theorem 3.2. *If $P \in \mathcal{P}_{n,\mu}$, $n > 2$ and $P(z)$ does not vanish in the disk $|z| < k$, where $k \geq 1$, then for every $R > r \geq 1, 0 \leq t \leq 1$ and $|z| = 1$,*

$$\begin{aligned}
&\big|P(Rz) - P(rz)\big| \\
&\leq \left(\frac{R^n - r^n}{1 + k^\mu \phi_1(R, r, \mu, k)}\right) \left\{\max_{|z|=1}|P(z)| - t\min_{|z|=k}|P(z)|\right\} \\
&\quad - \left(\frac{R^n - r^n}{n} - \frac{R^{n-2} - r^{n-2}}{n-2}\right)\left(\frac{\Big||P'(0)| - |Q'(0)|\Big|}{1 + k^\mu \phi_1(R, r, \mu, k)}\right),
\end{aligned} \tag{3.1}$$

where $\phi_1(R, r, \mu, k)$ is given by (1.12), $\lambda_1(R, r, \mu, k)$ by (1.13) with $m = \min_{|z|=1}|P(z)|$ and $Q(z) = z^n\overline{P(1/\overline{z})}$.

Proof. By hypothesis $P \in \mathcal{P}_{n,\mu}$ and $P(z) \neq 0$ for $|z| < k$, where $k \geq 1$, therefore by Lemma 2.1, for every $R \geq r \geq 1$,$0 \leq t \leq 1$ and $|z| = 1$, we have

$$\begin{aligned}
&k^\mu \phi_1(R, r, \mu, k)\big|P(Rz) - P(rz)\big| \\
&\leq \big|Q(Rz) - Q(rz)\big| - (R^n - r^n)t\min_{|z|=k}|P(z)|,
\end{aligned} \tag{3.2}$$

where $\phi_1(R, r, \mu, k)$ is defined by (1.12). Also by Lemma 2.4, we get

$$\begin{aligned}
&|P(Rz) - P(rz)| + |Q(Rz) - Q(rz)| \\
&\leq (R^n - r^n)\max_{|z|=1}|P(z)| - \left(\frac{R^n - r^n}{n} - \frac{R^{n-2} - r^{n-2}}{n-2}\right)\Big||P'(0)| - |Q'(0)|\Big|,
\end{aligned} \tag{3.3}$$

for $|z| = 1$ and for every $R \geq r \geq 1$. Inequality (3.2) with the help of inequality (3.3) yields

$$\begin{aligned}
&\big\{1 + k^\mu \phi_1(R, r, \mu, k)\big\}\big|P(Rz) - P(rz)\big| \\
&\leq (R^n - r^n)\left\{\max_{|z|=1}|P(z)| - t\min_{|z|=k}|P(z)|\right\} \\
&\quad - \left(\frac{R^n - r^n}{n} \frac{R^{n-2} - r^{n-2}}{n-2}\right)\Big|P'(0)| - |Q'(0)|\Big|,
\end{aligned}$$

for every $R \geq r \geq 1$,$0 \leq t \leq 1$ and $|z| = 1$, which is equivalent to the inequality (3.1). The proof of Theorem 3.2 is complete. □

Remark 3.3. For $R \geq r \geq 1$ and $n > 2$

$$\frac{R^n - r^n}{n} - \frac{R^{n-2} - r^{n-2}}{n-2}$$

is always non-negative, therefore (3.1) provides a refinement of Theorem 1.1 provided $|P'(0)| \neq |Q'(0)|$.

Theorems 3.8, as stated above, has various interesting consequences. Here we mention few of these. Dividing the two sides of the inequality (3.1) by $R - r$ and making $R \to r$, so that

$$\lambda(r,\mu,k) := \frac{\mu r^{\mu}}{nr^{n}} \frac{|a_{\mu}|k^{\mu}}{|a_0| - mt} \leq 1,$$

we immediately obtain the following interesting result which is a refinement as well as a generalization of inequality (1.8).

Corollary 3.4. *If $P \in P_{n,\mu}$, $n > 2$ and $P(z)$ does not vanish in the disk $|z| \leq k$, where $k \geq 1$, then $0 \leq t \leq 1$ and $|z| = 1$,*

$$\begin{aligned} \left|P'(rz)\right| \leq & \left(\frac{nr^{n-2}}{1 + k^{\mu}\psi(r,\mu,k)}\right)\left(\max_{|z|=1}|P(z)| - t\min_{|z|=k}|P(z)|\right) \\ & - \left(r^{n-2} - r^{n-4}\right) \frac{\Big||P'(0)| - |Q'(0)|\Big|}{1 + k^{\mu}\psi(r,\mu,k)} \end{aligned} \tag{3.4}$$

where

$$\psi(r,\mu,k) := \frac{k + \dfrac{\mu r^{\mu}}{nr^{n}} \dfrac{|a_{\mu}|k^{\mu}}{|a_0| - mt}}{1 + \dfrac{\mu r^{\mu}}{nr^{n}} \dfrac{|a_{\mu}|k^{\mu+1}}{|a_0| - mt}}, \tag{3.5}$$

$$m = \min_{|z|=1}|P(z)| \quad \text{and} \quad Q(z) = z^{n}\overline{P(1/\overline{z})}.$$

Remark 3.5. For $r = 1$ and $t = 0$, Corollary 3.4 reduces to (1.8).

Taking $t = r = 1$ and using the obvious inequality

$$\left|P(Rz)\right| \leq \left|P(Rz) - P(z)\right| + |P(z)|,$$

in Theorem 3.8, we get the following interesting result.

Corollary 3.6. *If $P \in P_{n,\mu}$, $n > 2$ and $P(z)$ does not vanish in the disk $|z| < k$, where $k \geq 1$, then for every $R \geq 1$,*

$$\begin{aligned} \max_{|z|=R}\left|P(z)\right| \leq & \frac{\left(R^{n} + k^{\mu}\phi_1(R,1,\mu,k)\right)\max_{|z|=1}\left|P(z)\right| - (R^{n} - 1)\min_{|z|=k}\left|P(z)\right|}{1 + k^{\mu}\phi_1(R,1,\mu,k)} \\ & - \left(\frac{R^{n} - 1}{n} - \frac{R^{n-2} - 1}{n-2}\right) \frac{\Big||P'(0)| - |Q'(0)|\Big|}{1 + k^{\mu}\phi_1(R,1,\mu,k)}, \end{aligned} \tag{3.6}$$

where $\phi_1(R,r,\mu,k)$ is defined by (1.12) and $Q(z) = z^{n}\overline{P(1/\overline{z})}$.

Theorem 3.1 can be improved, if $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z| = 1$. More precisely, we prove:

Theorem 3.7. *Let $P \in \mathcal{P}_{n,\mu}$, $n > 2$ and $P(z)$ does not vanish in the disk $|z| < k$, where $k \geq 1$. If $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z| = 1$, where $Q(z) = z^n\overline{P(1/\overline{z})}$, then for every $R > r \geq 1, 0 \leq t \leq 1$ and $|z| = 1$,*

$$\big|P(Rz) - P(rz)\big| \leq \left(\frac{R^n - r^n}{1 + k^\mu}\right)\left\{\max_{|z|=1}|P(z)| - t\min_{|z|=k}|P(z)|\right\} - \left(\frac{R^n - r^n}{n} - \frac{R^{n-2} - r^{n-2}}{n-2}\right)\left(\frac{|P'(0)| + |Q'(0)|}{1 + k^\mu}\right),$$

where $m = \min_{|z|=1}|P(z)|$.

Instead of proving Theorem 3.7, we obtain a more improved result which among other things provide a refinement of Theorem 3.7. We prove:

Theorem 3.8. *Let $P \in \mathcal{P}_{n,\mu}$, $n > 2$ and $P(z)$ does not vanish in the disk $|z| < k$, where $k \geq 1$. If $|P'(z)|$ and $|Q'(z)|$ become maximum at the same point on $|z| = 1$, where $Q(z) = z^n\overline{P(1/\overline{z})}$, then for every $R > r \geq 1, 0 \leq t \leq 1$ and $|z| = 1$,*

$$\begin{aligned}&\big|P(Rz) - P(rz)\big| \\ &\leq \left(\frac{R^n - r^n}{1 + k^\mu\phi_1(R,r,\mu,k)}\right)\left\{\max_{|z|=1}|P(z)| - t\min_{|z|=k}|P(z)|\right\} \\ &\quad - \left(\frac{R^n - r^n}{n} - \frac{R^{n-2} - r^{n-2}}{n-2}\right)\left(\frac{|P'(0)| + |Q'(0)|}{1 + k^\mu\phi_1(R,r,\mu,k)}\right),\end{aligned} \tag{3.7}$$

where $\phi_1(R,r,\mu,k)$ is given by (1.12), $\lambda_1(R,r,\mu,k)$ by (1.13) with $m = \min_{|z|=1}|P(z)|$ and $Q(z) = z^n\overline{P(1/\overline{z})}$.

Proof. By hypothesis $P \in \mathcal{P}_{n,\mu}$ and $P(z) \neq 0$ for $|z| < k$, where $k \geq 1$, therefore by Lemma 2.1, for every $R \geq r \geq 1, 0 \leq t \leq 1$ and $|z| = 1$, we have

$$\begin{aligned}&k^\mu\phi_1(R,r,\mu,k)\big|P(Rz) - P(rz)\big| \\ &\leq \big|Q(Rz) - Q(rz)\big| - (R^n - r^n)t\min_{|z|=k}|P(z)|,\end{aligned} \tag{3.8}$$

where $\phi_1(R,r,\mu,k)$ is defined by (1.12). Also by Lemma 2.5, we get

$$\begin{aligned}&|P(Rz) - P(rz)| + |Q(Rz) - Q(rz)| \\ &\leq (R^n - r^n)\max_{|z|=1}|P(z)| - \left(\frac{R^n - r^n}{n} - \frac{R^{n-2} - r^{n-2}}{n-2}\right)\big(|P'(0)| + |Q'(0)|\big),\end{aligned} \tag{3.9}$$

for $|z| = 1$ and for every $R \geq r \geq 1$. Inequality (3.8) with the help of inequality (3.9) yields

$$\begin{aligned}&\left\{1 + k^{\mu}\phi_1(R, r, \mu, k)\right\}\left|P(Rz) - P(rz)\right|\\&\leq (R^n - r^n)\left\{\max_{|z|=1}|P(z)| - t\min_{|z|=k}|P(z)|\right\}\\&\quad - \left(\frac{R^n - r^n}{n}\frac{R^{n-2} - r^{n-2}}{n-2}\right)\left(|P'(0)| + |Q'(0)|\right),\end{aligned}$$

for every $R \geq r \geq 1$, $0 \leq t \leq 1$ and $|z| = 1$, which is equivalent to the inequality (3.1). The proof of Theorem 3.8 is complete. □

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