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SOME NEW GENERALIZATION OF ENESTRÖM KAKEYA THEOREM

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Abstract. If $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$, then all the zeros of $P(z)$ lie in $|z| \leq 1$. The result is due to Eneström and Kakeya (for reference see [6], [7]). In this paper, we prove several Eneström-Kakeya type results concerning the location of zeros of a polynomial in the complex plane. By relaxing the hypothesis and putting less restrictive conditions on the coefficients of the polynomials, our result generalize some classical results.

1. Introduction

The following result due to Eneström and Kakeya [6] is well known in the theory of distribution of zeros of polynomials.

Theorem A. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0,$$

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then all the zeros of $P(z)$ *lie in* $|z| \leq 1$.

Aziz and Zargar [1] also relaxed the hypothesis of Eneström-Kakeya theorem and proved the following result.

Theorem B. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that for some* $k \geq 1$

$$ka_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0,$$

then all the zeros of $P(z)$ *lie in*

$$|z + k - 1| \leq k.$$

Recently Shah and Liman [8] extended this result to the class of polynomials with complex coefficients also generalizes the result of Govil and Rahman [4] and proved the following two results.

Theorem C. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients. If* $\mathrm{Re}(a_j) = \alpha_j$ *and* $\mathrm{Im}(a_j) = \beta_j$, *for* $j = 0, 1, 2, \cdots, n$, $a_n \neq 0$ *and for some* $k \geq 1$

$$k\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0,$$
$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0,$$

then all the zeros of $P(z)$ *lie in*

$$\left| z - \frac{\alpha_n}{a_n}(k-1) \right| \leq \frac{1}{a_n} \left\{ k\alpha_n - \alpha_0 + |\alpha_0| + \beta_n \right\}.$$

Theorem D. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients such that for some real* β, $|\mathrm{arg} a_j - \beta| \leq \alpha \leq \frac{\pi}{2}$, $j = 0, 1, 2, \cdots, n$ *and for some* $k \geq 1$

$$k|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_0|,$$

then all the zeros of $P(z)$ *lie in*

$$|z + k - 1| \leq \frac{1}{|a_n|} \left\{ (k|a_n| - |a_0|)(\sin\alpha + \cos\alpha) + |a_0| + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| \right\}.$$

In this paper we relax the hypothesis and considerable improvement in the bound, we establish some generalizations and extensions of Eneström Kakeya theorem. We first prove:

2. Theorems and proofs

Theorem 2.1. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that for some* $\lambda \neq 1$, $\mu \geq 1$, $1 \leq k \leq n$ *and* $a_{n-k} \neq 0$

$$\mu\, a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0,$$

if $a_{n-k-1} > a_{n-k}$, *then all the zeros of* $P(z)$ *lie in the disc* $|z| \leq K_1$, *where* K_1 *is the greatest positive root of the equation*

$$K^{k+1} - (\mu - 1 + \delta_1)K^k - |\gamma_1| = 0,$$

where $\gamma_1 = \frac{(\lambda-1)a_{n-k}}{a_n}$ *and* $\delta_1 = \frac{\mu a_n + (\lambda-1)a_{n-k} - a_0 + |a_0|}{|a_n|}$.
If $a_{n-k} > a_{n-k+1}$, *then all the zeros of* $P(z)$ *lie in the disc* $|z| \leq K_2$, *where* K_2 *is the greatest positive root of the equation*

$$K^k - (\mu - 1 + \delta_2)K^{k-1} - |\gamma_2| = 0,$$

where $\gamma_2 = \frac{(1-\lambda)a_{n-k}}{a_n}$ *and* $\delta_2 = \frac{\mu a_n + (1-\lambda)a_{n-k} - a_0 + |a_0|}{|a_n|}$.

Remark 2.2. The result of Choo [2, Theorem 1] is a special case of above result, if we let $\mu = 1$. If we choose all the coefficients to be positive, we obtain the following:

Corollary 2.3. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that for some* $\lambda \neq 1$, $\mu \geq 1$, $1 \leq k \leq n$ *and* $a_{n-k} \neq 0$

$$\mu\, a_n \geq a_{n-1} \geq \cdots \geq \lambda\, a_{n-k} \geq a_{n-k-1} \geq \cdots a_1 \geq a_0 > 0,$$

if $a_{n-k-1} > a_{n-k}$, *then all the zeros of* $P(z)$ *lie in the disk* $|z| \leq K_1$, *where* K_1 *is the greatest positive root of the equation*

$$K^{k+1} - (\mu - 1 + \delta_1)K^k - |\gamma_1| = 0,$$

where $\gamma_1 = \frac{(\lambda-1)a_{n-k}}{a_n}$ *and* $\delta_1 = \frac{\mu a_n + (\lambda-1)a_{n-k}}{|a_n|}$.
If $a_{n-k} > a_{n-k+1}$, *then all the zeros of* $P(z)$ *lie in the disc* $|z| \leq K_2$ *where,* K_2 *is the greatest positive root of the equation*

$$K^k - (\mu - 1 + \delta_2)K^{k-1} - |\gamma_2| = 0,$$

where $\gamma_2 = \frac{(1-\lambda)a_{n-k}}{a_n}$ *and* $\delta_2 = \frac{\mu a_n + (1-\lambda)a_{n-k}}{|a_n|}$.

For $\mu = 1$, we have

Corollary 2.4. *Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n such that for some $\lambda \neq 1$, $1 \leq\ k \leq\ n$ and $a_{n-k} \neq 0$*

$$a_n \geq a_{n-1} \geq \cdots \geq \lambda\ a_{n-k} \geq\ a_{n-k-1} \geq \cdots a_1 \geq\ a_0 > 0$$

if $a_{n-k-1} > a_{n-k}$, then all the zeros of $P(z)$ lie in the disc $|z| \leq\ K_1$, where K_1 is the greatest positive root of the equation

$$K^{k+1} - \delta_1 K^k - |\gamma_1| = 0,$$

where $\gamma_1 = \frac{(\lambda-1)a_{n-k}}{a_n}$ *and* $\delta_1 = \frac{a_n+(\lambda-1)a_{n-k}}{|a_n|}$.
If $a_{n-k} > a_{n-k+1}$, the all the zeros of $P(z)$ lie in the disc $|z| \leq\ K_2$ where, K_2 is the greatest positive root of the equation

$$K^k - \delta_2 K^{k-1} - |\gamma_2| = 0,$$

where $\gamma_2 = \frac{(1-\lambda)a_{n-k}}{a_n}$ *and* $\delta_2 = \frac{a_n+(1-\lambda)a_{n-k}}{|a_n|}$.

Next, for the complex polynomials we prove the following.

Theorem 2.5. *Let $P(z)=\sum_{j=0}^{n} a_j z^j$ be the nth order complex polynomial with $\mathrm{Re}(a_j)=\alpha_j$ and $\mathrm{Im}(a_j)=\beta_j$, $j=0,1,2,...$ and assume that for some $\lambda \neq 1$, $\mu \geq 1$, $1 \leq\ k \leq\ n$ and $\alpha_{n-k} \neq 0$*

$$\mu\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \lambda\alpha_{n-k} \geq \alpha_{n-k-1} \geq \cdots \geq \alpha_1 \geq \alpha_0$$

and

$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0,$$

if $\alpha_{n-k-1} > \alpha_{n-k}$, then all the zeros of $P(z)$ lie in the disc $|z| \leq\ K_1$, where K_1 is the greatest positive root of the equation

$$K^{k+1} - (\mu - 1 + \delta_1)K^k - |\gamma_1| = 0,$$

where $\gamma_1 = \frac{(\lambda-1)\alpha_{n-k}}{a_n}$ *and* $\delta_1 = \frac{\mu\alpha_n+(\lambda-1)\alpha_{n-k}-\alpha_0+|\alpha_0|+\beta_n-\beta_0+|\beta_0|}{|a_n|}$.
If $\alpha_{n-k} > \alpha_{n-k+1}$, then all the zeros of $P(z)$ lie in the disc $|z| \leq\ K_2$, where K_2 is the greatest positive root of the equation

$$K^k - (\mu - 1 + \delta_2)K^{k-1} - |\gamma_2| = 0,$$

where $\gamma_2 = \frac{(1-\lambda)\alpha_{n-k}}{a_n}$ *and* $\delta_2 = \frac{\mu\alpha_n+(1-\lambda)\alpha_{n-k}-\alpha_0+||\alpha_0|+\beta_n-\beta_0+|\beta_0|}{|a_n|}$.

Remark 2.6. If we take $\mu = 1$, we obtain the result of Choo [2, Theorem 3]. And if all coefficients are taken positive, we obtain the following.

Corollary 2.7. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be the nth order complex polynomial with* $\mathrm{Re}(a_j) = \alpha_j$ *and* $\mathrm{Im}(a_j) = \beta_j$*,* $j = 0, 1, 2, ...$ *and assume that for some* $\lambda \neq 1$*,* $\mu \geq 1$ *and* $\alpha_{n-k} \neq 0$

$$\mu\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \lambda\alpha_{n-k} \geq \alpha_{n-k-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 > 0$$

and

$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0 > 0,$$

if $\alpha_{n-k-1} > \alpha_{n-k}$*, then all the zeros of* $P(z)$ *lie in the disc* $|z| \leq K_1$*, where* K_1 *is the greatest positive root of the equation*

$$K^{k+1} - (\mu - 1 + \delta_1)K^k - |\gamma_1| = 0,$$

where $\gamma_1 = \frac{(\lambda-1)\alpha_{n-k}}{a_n}$ *and* $\delta_1 = \frac{\mu\alpha_n + (\lambda-1)\alpha_{n-k} - \beta_n}{|a_n|}$.
If $\alpha_{n-k} > \alpha_{n-k+1}$*, then all the zeros of* $P(z)$ *lie in the disc* $|z| \leq K_2$*, where* K_2 *is the greatest positive root of the equation*

$$K^k - (\mu - 1 + \delta_2)K^{k-1} - |\gamma_2| = 0,$$

where $\gamma_2 = \frac{(1-\lambda)\alpha_{n-k}}{a_n}$ *and* $\delta_2 = \frac{\mu\alpha_n + (1-\lambda)\alpha_{n-k} + \beta_n}{|a_n|}$.

For $\mu = 1$ we have

Corollary 2.8. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be the nth order complex polynomial with* $\mathrm{Re}(a_j) = \alpha j$ *and* $\mathrm{Im}(a_j) = \beta j$*,* $j = 0, 1, 2, ...$ *and assume that for some* $\lambda \neq 1$*,* $1 \leq k \leq n$ *and* $\alpha_{n-k} \neq 0$

$$\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \lambda\alpha_{n-k} \geq \alpha_{n-k-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 > 0$$

and

$$\beta_n \geq \beta_{n-1} \geq \cdots \geq \beta_1 \geq \beta_0 > 0,$$

if $\alpha_{n-k-1} > \alpha_{n-k}$*, then all the zeros of* $P(z)$ *lie in the disc* $|z| \leq K_1$*, where* K_1 *is the greatest positive root of the equation*

$$K^{k+1} - \delta_1 K^{k-1} - \gamma_1| = 0,$$

where $\gamma_1 = \frac{(\lambda-1)\alpha_{n-k}}{a_n}$ *and* $\delta_1 = \frac{\alpha_n + (\lambda-1)\alpha_{n-k} + \beta_n}{|a_n|}$.
If $\alpha_{n-k} > \alpha_{n-k+1}$*, then all the zeros of* $P(z)$ *lie in the disc* $|z| \leq K_2$*, where* K_2 *is the greatest positive root of the equation*

$$K^k - \delta_2 K^{k-1}| - |\gamma_2| = 0,$$

where $\gamma_2 = \frac{(1-\lambda)\alpha_{n-k}}{a_n}$ *and* $\delta_2 = \frac{\alpha_n + (1-\lambda)\alpha_{n-k} + \beta_n}{|a_n|}$.

Theorem 2.9. *Let* $P(z) = \sum\limits_{j=0}^{n} a_j z^j$ *be the nth order complex polynomial with such that for some real* β*,*

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2}, \quad j = 0, 1, 2, ...$$

and for some $\lambda \neq 1$, $\mu \geq 1$, $1 \leq \ k \leq \ 1$ *and* $a_{n-k} \neq 0$

$$\mu|a_n| \geq \ |a_{n-1}| \geq \cdots \geq \lambda|a_{n-k}| \geq \ |a_{n-k-1}| \geq \cdots \geq \ |a_1| \geq \ |a_0|$$

if $|a_{n-k}| < |a_{n-k-1}|$*, then all the zeros of* $P(z)$ *lie in the disc* $|z| \leq \ K_1$*, where* K_1 *is the greatest positive root of the equation*

$$K^{k+1} - (\mu - 1 + \delta_1)K^k - |\gamma_1| = 0,$$

where $\gamma_1 = \frac{(\lambda-1)a_{n-k}}{a_n}$ *and* $\delta_1 = \frac{\{\mu|a_n|+(\lambda-1)|a_{n-k}|\}(\cos\alpha+\sin\alpha)+2\sin\alpha\sum\limits_{i=0}^{n--1}|a_j|}{|a_n|}$.

If $|a_{n-k}| > |a_{n-k+1}|$*, then all the zeros of* $P(z)$ *lie in the disc* $|z| \leq \ K_2$*, where* K_2 *is the greatest positive root of the equation*

$$K^k - (\mu - 1 + \delta_2)K^{k-1} - |\gamma_2| = 0,$$

where $\gamma_2 = \frac{(1-\lambda)a_{n-k}}{a_n}$ *and* $\delta_2 = \frac{\{\mu|a_n|+(\lambda-1)|a_{n-k}|\}(\cos\alpha+\sin\alpha)+2\sin\alpha\sum\limits_{i=0}^{n}|a_j|}{|a_n|}$.

Remark 2.10. For $\mu = 1$ we obtain the result of Choo [2, Theorem 4].

3. Proofs of the theorems

Proof of Theorem 2.1. Consider a polynomial

$$\begin{aligned}
\phi(z) &= (1-z)P(z) \\
&= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \cdots \ + a_1 z + a_0) \\
&= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} \\
&\qquad + \cdots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0 \\
&= -a_n z^{n+1} + (\mu a_n + a_n - \mu a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} \\
&\qquad + \cdots + (a_{n-k} - \lambda a_{n-k} + \lambda a_{n-k} - a_{n-k-1})z^{n-k} \\
&\qquad + \cdots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0.
\end{aligned}$$

If $a_{n-k-1} > a_{n-k}$, then $a_{n-k+1} > a_{n-k}$ and $\phi(z)$ can be written as

$$\begin{aligned}
\phi(z) = -a_n z^{n+1} &- (\mu - 1)a_n z^n - (\lambda - 1)a_{n-k}z^{n-k} + (\mu a_n - a_{n-1})z^n \\
&+ (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{n-k+1} - a_{n-k})z^{n-k+1} \\
&+ (\lambda a_{n-k} - a_{n-k-1})z^{n-k} + (a_{n-k-1} - a_{n-k-2})z^{n-k} \\
&+ \cdots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0.
\end{aligned}$$

For $|z| > 1$

$$\begin{aligned}|\phi(z)| \geq\ & |a_n z^{n+1} + (\mu-1)a_n z^n + (\lambda-1)a_{n-k}z^{n-k}| \\ & - |z|^n \Big\{ (\mu a_n - a_{n-1}) + \frac{(a_{n-1}-a_{n-2})}{|z|} + \cdots + \frac{(a_{n-k+1}-a_{n-k})}{|z|^{k-1}} \\ & + \frac{(\lambda a_{n-k} - a_{n-k-1})}{|z|^k} + \cdots + \frac{(a_1 - a_0)}{|z|^{n-1}} + \frac{a_0}{|z|^n} \Big\} \\ \geq\ & |z|^{n-k}|a_n z^{k+1} + (\mu-1)a_n z^k + (\lambda-1)a_{n-k}| \\ & - |z|^n \Big\{ \mu a_n + (\lambda-1)a_{n-k} - a_0 + |a_0| \Big\} \\ >\ & 0.\end{aligned}$$

If

$$|z|^{n-k}|a_n z^{k+1} + (\mu-1)a_n z^k + (\lambda-1)a_{n-k}| > |z|^n\{\mu a_n + (\lambda-1)a_{n-k} - a_0 + |a_0|\}$$

that is if

$$|z^{k+1} + (\mu-1)z^k + \gamma_1| > |z|^k \delta_1.$$

This inequality holds if

$$|z|^{k+1} - (\mu - 1 + \delta_1)|z|^k - |\gamma_1| > |z|^k \delta_1$$

that is

$$|z|^{k+1} - (\mu - 1 + \delta_1)|z|^k - |\gamma_1| > 0.$$

Hence all the zeros of $P(z)$ with modulus greater than one lie in the disc $|z| \leq\ K_1$, where K_1 is the greatest positive root of the equation

$$K^{k+1} - (\mu - 1 + \delta_1)K^k - |\gamma_1| = 0.$$

But the zeros of $P(z)$ with modulus less than or equal to one are already contained in the disc $|z| \leq\ K_1$, since $K_1 > 1$ [2, Remark 1].

The second part can be proved similarly. If $a_{n-k} > a_{n-k+1}$, then $a_{n-k} > a_{n-k-1}$ and $\phi(z)$ can be written as

$$\begin{aligned}\phi(z) = & -a_n z^{n+1} - (\mu-1)a_n z^n - (1-\lambda)a_{n-k}z^{n-k+1} + (\mu a_n - a_{n-1})z_n \\ & + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{n-k+1} - \lambda a_{n-k})z^{n-k+1} \\ & + (a_{n-k} - a_{n-k-1})z^{n-k} + (a_{n-k-1} - a_{n-k-2})z^{n-k-1} \\ & + \cdots + (a_1 - a_0)z + a_0.\end{aligned}$$

For $|z| > 1$

$$\begin{aligned}|\phi(z)| &= |a_n z^{n+1} + (\mu - 1)a_n z^n + (1-\lambda)a_{n-k}z^{n-k+1}| \\ &\quad - |z|^n\Big\{(\mu a_n - a_{n-1}) + \frac{(a_{n-1}-a_{n-2})}{|z|} + \cdots + \frac{(a_{n-k+1}-\lambda a_{n-k})}{|z|^{k-1}} \\ &\quad + \frac{(a_{n-k}-a_{n-k-1})}{|z|^k} + \cdots + \frac{(a_1 - a_0)}{|z|^{n-1}} + \frac{a_0}{|z|^n}\Big\} \\ &\geq |z|^{n-k+1}|a_n z^k + (\mu-1)a_n z^{k-1} + (1-\lambda)a_{n-k}| \\ &\quad - |z|^n\{\mu a_n - (1-\lambda)a_{n-k} - a_0 + |a_0|\} \\ &> 0.\end{aligned}$$

If

$$|z|^{n-k+1}|a_n z^k+(\mu-1)a_n z^{k-1}+(1-\lambda)a_{n-k}| > |z|^n\{\mu a_n-(1-\lambda)a_{n-k}-a_0+|a_0|\}$$

that is if

$$|z^k + (\mu-1)z^{k-1} + \gamma_2| > |z|^{k-1}\delta_2.$$

This inequality holds if

$$|z|^k - (\mu-1)|z|^{k-1} - |\gamma_2| > |z|^{k-1}\delta_2$$

that is

$$|z|^k - (\mu - 1 + \delta_2)|z|^{k-1} - |\gamma_2| > 0.$$

Hence all the zeros of $P(z)$ with modulus greater than one lie in the disc $|z| \leq K_2$, where K_2 is the greatest positive root of the equation

$$K^k - (\mu - 1 + \delta_2)K^{k-1} - |\gamma_2| = 0.$$

But the zeros of $P(z)$ with modulus less than or equal to one are already contained in the disk $|z| \leq K_2$, since $K_2 > 1$ [2, Remark 2]. □

Proof of Theorem 2.5. Consider a polynomial

$$\begin{aligned}\phi(z) &= (1-z)P(z) \\ &= (1-z)(a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0) \\ &= a_n z^{n+1} + (\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} + \cdots + (\alpha_2 - \alpha_1)z^2 \\ &\quad + (\alpha_1 - \alpha_0)z + \alpha_0 + i\{(\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} \\ &\quad + \cdots + (\beta_2 - \beta_1)z^2 + (\beta_1 - \beta_0)z + \beta_0\} \\ &= -a_n z^{n+1} + (\mu\alpha_n + \alpha_n - \mu\alpha_n - \alpha_{n-1})z^n + (\alpha_{n-1} - \alpha_{n-2})z^{n-1} \\ &\quad + \cdots + (\alpha_{n-k} - \lambda\alpha_{n-k} + (\lambda\alpha_{n-k} - \alpha_{n-k-1})z^{n-k} + \cdots + (\alpha_2 - \alpha_1)z^2 \\ &\quad + (\alpha_1 - \alpha_0)z + \alpha_0 + i\{(\beta_n - \beta_{n-1})z^n + (\beta_{n-1} - \beta_{n-2})z^{n-1} \\ &\quad + \cdots + (\beta_2 - \beta_1)z^2 + (\beta_1 - \beta_0)z + \beta_0\}.\end{aligned}$$

If $\alpha_{n-k-1} > \alpha_{n-k}$, then $\alpha_{n-k+1} > \alpha_{n-k}$ and $\phi(z)$ can be written as

$$\begin{aligned}\phi(z) = {} & -a_n z^{n+1} - (\mu - 1)\alpha_n z^n + (\lambda - 1)\alpha_{n-k} z^{n-k} + (\mu\alpha_n - \alpha_n) z^{n-1} \\ & + \cdots - (\lambda\alpha_{n-k} - \alpha_{n-k-1}) z^{n-k} + \cdots + (\alpha_2 - \alpha_1) z^2 + (\alpha_1 - \alpha_0) z \\ & + \alpha_0 + i\{(\beta_n - \beta_{n-1}) z^n + (\beta_{n-1} - \beta_{n-2}) z^{n-1} \\ & + \cdots + (\beta_2 - \beta_1) z^2 + (\beta_1 - \beta_0) z + \beta_0\}.\end{aligned}$$

If $|z| > 1$ then

$$\begin{aligned}|\phi(z)| \geq {} & |a_n z^{n+1} + (\mu - 1)\alpha_n z^n + (\lambda - 1)\alpha_{n-k} z^{n-k}| - |z|^n \{(\mu\alpha_n - \alpha_{n-1}) \\ & + \frac{(\alpha_{n-1} - \alpha_{n-2})}{|z|} + \cdots + \frac{(\alpha_{n-k+1} - \alpha_{n-k})}{|z|^{k-1}} + \frac{(\lambda\alpha_{n-k} - \alpha_{n-k-1})}{|z|^k} \\ & + \cdots + \frac{(\alpha_1 - \alpha_0)}{|z|^{n-1}} + \frac{\alpha_0}{|z|^n} \\ & + i|z|^n \left\{ (\beta_n - \beta_{n-1}) + \frac{(\beta_{n-1} - \beta_{n-2})}{|z|} \right. \\ & \left. + \cdots + \frac{(\beta_{n-k+1} - \beta_{n-k})}{|z|^{k-1}} + \cdots + \frac{(\beta_1 - \beta_0)}{|z|^{n-1}} + \frac{\beta_0}{|z|^n} \right\} \\ \geq {} & |z|^{n-k} |a_n z^{k+1} + (\mu - 1)\alpha_n z^k + (\lambda - 1)\alpha_{n-k}| \\ & - |z|^n \{\mu a_n + (\lambda - 1)\alpha_{n-k} - \alpha_0 + |\alpha_0| + \beta_n - \beta_0 + |\beta_0|\} \\ > {} & 0\end{aligned}$$

if

$$\begin{aligned}& |z|^{n-k} |a_n z^{k+1} + (\mu - 1)\alpha_n z^k + (\lambda - 1)\alpha_{n-k}| \\ & > |z|^n \{\mu a_n + (\lambda - 1)\alpha_{n-k} - \alpha_0 + |\alpha_0| + \beta_n - \beta_0 + |\beta_0|\}\end{aligned}$$

that is if

$$|z^{k+1} + (\mu - 1) z^k + \gamma_1| > |z|^k \delta_1.$$

This inequality holds if

$$|z|^{k+1} - (\mu - 1)|z|^k - |\gamma_1| > |z|^k \delta_1$$

that is

$$|z|^{k+1} - (\mu - 1 + \delta_1)|z|^k - |\gamma_1| > 0.$$

Hence all the zeros of $P(z)$ with modulus greater than one lie in the disk $|z| \leq K_1$, where K_1 is the greatest positive root of the equation

$$K^{k+1} - (\mu - 1 + \delta_1) K^k - |\gamma_1| = 0.$$

But the zeros of $P(z)$ with modulus less than or equal to one are already contained in the disk $|z| \leq K_1$, since $K_1 > 1$ [Theorem 1].

Now assume, $\alpha_{n-k} > \alpha_{n-k+1}$, then $\alpha_{n-k} > \alpha_{n-k-1}$ and $\phi(z)$ can be written as

$$\begin{aligned}\phi(z) = &-a_n z^{n+1} - (\mu-1)\alpha_n z^n - (1-\lambda)\alpha_{n-k} z^{n-k+1} + (\mu\alpha_n - \alpha_{n-1})z^n \\ &+ (\alpha_{n-1}-\alpha_{n-2})z^{n-1} + \cdots + (\alpha_{n-k+1} - \lambda\alpha_{n-k})z^{n-k+1} \\ &+ (\alpha_{n-k}-\alpha_{n-k-1})z^{n-k} + (\alpha_{n-k-1}-\alpha_{n-k-2})z^{n-k-1} + \cdots + (\alpha_1-\alpha_0)z \\ &+ \alpha_0 + i\{(\beta_n-\beta_{n-1})z^n + (\beta_{n-1}-\beta_{n-2})z^{n-1} + \cdots + (\beta_1-\beta_0)z + \beta_0\}.\end{aligned}$$

If $|z| > 1$ then

$$\begin{aligned}|\phi(z)| = &|a_n z^{n+1} + (\mu-1)\alpha_n z^n + (1-\lambda)\alpha_{n-k} z^{n-k+1}| - |z|^n\Bigg\{(\mu\alpha_n - \alpha_{n-1}) \\ &+ \frac{(\alpha_{n-1}-\alpha_{n-2})}{|z|} + \cdots + \frac{(\alpha_{n-k+1}-\lambda\alpha_{n-k})}{|z|^{k-1}} + \frac{(\alpha_{n-k}-\alpha_{n-k-1})}{|z|^k} \\ &+ \cdots + \frac{(\alpha_1-\alpha_0)}{|z|^{n-1}} + \frac{\alpha_0}{|z|^n}\Bigg\} - |z|^n\Bigg\{(\beta_n-\beta_{n-1}) + \frac{(\beta_{n-1}-\beta_{n-2})}{|z|} \\ &+ \cdots + \frac{(\beta_1-\beta_0)}{|z|^{n-1}} + \frac{\beta_0}{|z|^n}\Bigg\} \\ \geq &|z|^{n-k+1}|a_n z^k + (\mu-1)\alpha_n z^{k-1} + (\lambda-1)\alpha_{n-k}| \\ &- |z|^n\Bigg\{\mu\alpha_n + (1-\lambda)\alpha_{n-k} - \alpha_0 + |\alpha_0| + \beta_n - \beta_0 + |\beta_0|\Bigg\} \\ > &0.\end{aligned}$$

If

$$\begin{aligned}&|z|^{n-k+1}|a_n z^k + (\mu-1)\alpha_n z^{k-1} + (\lambda-1)\alpha_{n-k}| \\ &> |z|^n\Bigg\{\mu\alpha_n - (1-\lambda)\alpha_{n-k} - \alpha_0 + |\alpha_0| + \beta_n - \beta_0 + |\beta_0|\Bigg\}\end{aligned}$$

that is if

$$|z^k + (\mu-1)z^{k-1} + \gamma_2| > |z|^{k-1}\delta_2.$$

This inequality holds if

$$|z|^k - (\mu-1)|z|^{k-1} - |\gamma_2| > |z|^{k-1}\delta_2.$$

That is

$$|z|^k - (\mu-1+\delta_2)|z|^{k-1} - |\gamma_2| > 0.$$

Hence all the zeros of $P(z)$ with modulus greater than one lie in the disc $|z| \leq K_2$, where K_2 is the greatest positive root of the equation

$$K^k - (\mu-1+\delta_2)K^{k-1} - |\gamma_2| = 0.$$

Again it can be easily seen that $K_2 > 1$ and the zeros of $P(z)$ with modulus less than or equal to one are already contained in the disc $|z| \leq \ K_2$. This completes the proof of the Theorem. $\square$

Proof of Theorem 2.9. Consider a polynomial

$$\begin{aligned}\phi(z) &= (1-z)P(z)\\ &= (1-z)(a_nz^n + a_{n-1}z^{n-1} + \cdots \ + a_1z + a_0)\\ &= -a_nz^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_2 - a_1)z^2\\ &\qquad + (a_1 - a_0)z + a_0\\ &= -a_nz^{n+1} + (\mu a_n + a_n - \mu a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1}\\ &\qquad + \cdots + (a_{n-k} - \lambda a_{n-k} + \lambda a_{n-k} - a_{n-k-1})z^{n-k} + \cdots + (a_2 - a_1)z^2\\ &\qquad + (a_1 - a_0)z + a_0.\end{aligned}$$

If $|a_{n-k-1}| > |a_{n-k}|$, then $|a_{n-k+1}| > |a_{n-k}|$ and $\phi(z)$ can be written as

$$\begin{aligned}\phi(z) = -a_nz^{n+1} &+ (\mu - 1)a_nz^n - (\lambda - 1)a_{n-k}z^{n-k} + (\mu a_n - a_{n-1})z^n\\ &+ (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (\lambda a_{n-k} - a_{n-k-1})z^{n-k}\\ &+ (a_{n-k-1} - a_{n-k-2})z^{n-k-1} + \cdots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0.\end{aligned}$$

For $|z| > 1$

$$\begin{aligned}|\phi(z)| \geq \ & |a_nz^{n+1} + (\mu - 1)a_nz^n + (\lambda - 1)a_{n-k}z^{n-k}| - |z|^n\{|\mu a_n - a_{n-1}|\\ &+ \frac{|a_{n-1} - a_{n-2}|}{|z|} + \cdots + \frac{|a_{n-k+1} - a_{n-k}|}{|z|^{k-1}} + \frac{|\lambda a_{n-k} - a_{n-k-1}|}{|z|^k}\\ &+ \cdots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\Big\}\\ \geq\ & |a_nz|^{n+1} + (\mu - 1)a_nz^k + (\lambda - 1)a_{n-k}z^{n-k}| - |z|^n\Big\{|\mu a_n - a_{n-1}|\\ &+ \cdots + |a_{n-k+1} - a_{n-k}| + |\lambda a_{n-k} - a_{n-k-1}| + \cdots + |a_1 - a_0| + |a_0|\Big\}.\end{aligned}$$

It was shown in [4] that for two complex numbers b_0 and b_1, if $|b_0| \geq |b_1|$ and $|\arg b_j - \beta| \leq \alpha \leq \frac{\pi}{2},\ j = 0, 1, 2, ...$ for some β, then

$$|b_0 - b_1| \leq \ (|b_0| - |b_1|)\cos\alpha + (|b_0| + |b_1|)\sin\alpha.$$

Using this we have

$$\begin{aligned}|\phi(z)| &\geq |a_n z^{n+1} + (\mu - 1)a_n z^n + (\lambda - 1)a_{n-k} z^{n-k}| \\ &\quad - |z|^n \Big\{ \{\mu|a_n| + (\lambda - 1)|a_{n-k}|\}(\cos\alpha + \sin\alpha) \\ &\quad - |a_0|(\cos\alpha + \sin\alpha - 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| \Big\} \\ &\geq |a_n z^{n+1} + (\mu - 1)a_n z^n + (\lambda - 1)a_{n-k} z^{n-k}| \\ &\quad - |z|^n \Big\{ \{\mu|a_n| + (\lambda - 1)|a_{n-k}|\}(\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| \Big\} \\ &> 0.\end{aligned}$$

If

$$|z^{k+1} + (\mu - 1)z^k + \gamma_1| > |z|^k \delta_1.$$

This inequality holds is

$$|z|^{k+1} - (\mu - 1)|z|^k - |\gamma_1| > |z|^k \delta_1$$

that is

$$|z|^{k+1} - (\mu - 1 + \delta_1)|z|^k - |\gamma_1| > 0.$$

Hence all the zeros of $P(z)$ with modulus greater than one lie in the disc $|z| \leq K_1$, where K_1 is the greatest positive root of the equation

$$K^{k+1} - (\mu - 1 + \delta_1)K^k - |\gamma_1| = 0.$$

It is easily seen that $K_1 > 1$ and all the zeros of $P(z)$ with modulus less than or equal to one are already contained in the disc $|z| \leq K_1$.

Now consider the case if $|a_{n-k}| > |a_{n-k+1}|$, then $|a_{n-k}| > |a_{n-k-1}|$, and $\phi(z)$ can be written as

$$\begin{aligned}\phi(z) &= -a_n z^{n+1} - (\mu - 1)a_n z^n - (1 - \lambda)a_{n-k} z^{n-k+1} + (\mu a_n - a_{n-1})z^n \\ &\quad + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{n-k+1} - \lambda a_{n-k})z^{n-k+1} \\ &\quad + (a_{n-k} - a_{n-k-1})z^{n-k} + \cdots + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_0.\end{aligned}$$

If $|z| > 1$ then

$$
\begin{aligned}
|\phi(z)| &\geq |a_n z^{n+1} + (\mu - 1)a_n z^n + (1-\lambda)a_{n-k} z^{n-k+1}| - |z|^n \bigg\{ |\mu a_n - a_{n-1}| \\
&\quad + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \cdots + \frac{|a_{n-k+1} - \lambda a_{n-k}|}{|z|^{k-1}} + \frac{|a_{n-k} - a_{n-k-1}|}{|z|^k} \\
&\quad + \cdots + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \bigg\} \\
&\geq |a_n z^{n+1} + (\mu - 1)a_n z^n + (1-\lambda)a_{n-k} z^{n-k+1}| - |z|^n \bigg\{ |\mu a_n - a_{n-1}| \\
&\quad + |a_{n-1} - a_{n-2}| + \cdots + |a_{n-k+1} - \lambda a_{n-k}| + |a_{n-k} - a_{n-k-1}| \\
&\quad + \cdots + |a_1 - a_0| + |a_0| \bigg\} \\
&\geq |a_n z^{n+1} + (\mu - 1)a_n z^n + (1-\lambda)a_{n-k} z^{n-k+1}| \\
&\quad - |z|^n \bigg\{ \{\mu |a_n| + (1-\lambda)|a_{n-1}|\}(\cos\alpha + \sin\alpha) \\
&\quad - |a_0|(\cos\alpha + \sin\alpha - 1) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| \bigg\} \\
&\geq |a_n z^{n+1} + (\mu - 1)a_n z^n + (1-\lambda)a_{n-k} z^{n-k+1}| \\
&\quad - |z|^n \bigg\{ \{\mu |a_n| + (1-\lambda)|a_{n-1}|\}(\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n-1} |a_j| \bigg\} \\
&> 0.
\end{aligned}
$$

If

$$|z^k + (\mu - 1)z^{k-1} + \gamma_2| > |z|^{k-1}\delta_2.$$

This inequality holds if

$$|z|^k - (\mu - 1)|z|^{k-1} - |\gamma_2| > |z|^{k-1}\delta_2$$

that is

$$|z|^k - (\mu - 1 + \delta_2)|z|^{k-1} - |\gamma_2| > 0.$$

Hence all the zeros of $P(z)$ with modulus greater than one lie in the disc $|z| \leq K_2$, where K_2 is the greatest positive root of the equation

$$K^k - (\mu - 1 + \delta_2)K^{k-1} - |\gamma_2| = 0.$$

Again it can be shown that $K_2 > 1$ and all the zeros $P(z)$ lie in the disc $|z| \leq K_2$. □

References

[1] A. Aziz and B.A. Zargar, *Some extensions of Eneström-Kakeya theorem,* Glasnik Mathematicki, 31(51) (1996).

[2] Y. Choo, *Further Generalization of Eneström-Kakeya theorem,* Int.Journal of Math. Analysis, **5(20)** (2011), 983-995.

[3] G.H. Hardy, J.E. Littlewood and G. Polya, *Inequalities* Cambridge University Press 1934.

[4] N.K. Govil and Q.I. Rahman, *On the Eneström-Kakeya theorem II,* Tohokv Math. J. 20 (1968), 126-136.

[5] A. Joyal, G. Labelle and Q.I. Rahman, *On the location of zeros of polynomials,* Canad. Math. Bull. 10 (1967), 53-63.

[6] M. Marden, *Geometry of Polynomials,* IInd Ed. Math, Surveys 3, Amer. Math. Soc., Providence, R.I., (1966).

[7] Q.I. Rahman and G. Schmeisser, *Analytic Theory of polynomials,* Oxford University Press, Oxford, 2002.

[8] W.M. Shah and A. Liman, *On the Eneström-Kakeya theorem and related analytic functions,* Proc. Indian Acad. Sci.(Math Sci.) 117(3) 2007, 359-370.