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# EXISTENCE OF A UNIQUE SOLUTION OF A NONLINEAR FUNCTIONAL INTEGRAL EQUATION

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**Abstract.** In this paper, we consider a nonlinear functional integral equation with variable delays. Using tools of functional analysis and Banach's fixed point theorem in a Fréchet space, the existence of a unique solution for the above equation is proved. Nontrivial examples are also given to illustrate our result.

## 1. Introduction

In this paper, we consider the following nonlinear functional integral equation with variable delays

$$\begin{aligned} x(t) &= V\left(t, x(t), \int_0^{\mu_1(t)} V_1\left(t, s, x(\theta_1(s)), ..., x(\theta_p(s)), \bar{V}_2[x](s)\right) ds\right), \\ \bar{V}_2[x](s) &= \int_0^{\mu_2(s)} V_2\left(s, r, x(\tilde{\theta}_1(r)), ..., x(\tilde{\theta}_q(r))\right) dr, \end{aligned} \tag{1.1}$$

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$t \in \mathbb{R}_+$, where $E$ is a Banach space, $V : \mathbb{R}_+ \times E^2 \to E$; $V_1 : \Delta_{\mu_1} \times E^{p+1} \to E$; $V_2 : \Delta_{\mu_2} \times E^q \to E$ are supposed to be continuous and $\Delta_{\mu_i} = \{(t,s) \in \mathbb{R}_+^2 : s \leq \mu_i(t)\}$, the functions $\mu_1$, $\mu_2$, $\theta_i$, $\tilde{\theta}_j \in C(\mathbb{R}_+; \mathbb{R}_+)$ are continuous, $\mu_1(t)$, $\mu_2(t)$, $\theta_i(t)$, $\tilde{\theta}_j(t) \in [0,t]$, $i = 1,...,p$; $j = 1,...,q$.

It is well known that integral equations have wide applications in engineering, mechanics, physics, economics, optimization, vehicular traffic, biology, queuing theory and so on. The theory of integral equations is rapidly developing with the help of tools in functional analysis, topology and fixed-point theory (see [1] - [11] and the references given therein).

Applying a fixed point theorems and giving the suitable assumptions, Dhage and Ntouyas [3], Purnaras [11] also obtained some results on the existence of solutions to the following nonlinear functional integral equation

$$x(t) = Q(t) + \int_0^{\mu(t)} k(t,s) f(s, x(\theta(s)))ds + \int_0^{\sigma(t)} v(t,s) g(s, x(\eta(s)))ds, \quad (1.2)$$

$t \in [0,1]$, where $E = \mathbb{R}$, $0 \leq \mu(t) \leq t$; $0 \leq \sigma(t) \leq t$; $0 \leq \theta(t) \leq t$; $0 \leq \eta(t) \leq t$, for all $t \in [0,1]$. Some more general equations than (1.2) were also studied in [11].

Using the technique of the measure of noncompactness and the Darbo fixed point theorem, Z. Liu et al. [6] have proved the existence and asymptotic stability of solutions for the equation

$$x(t) = f\left(t,\ x(t),\ \int_0^t u(t,s,x(a(s)),x(b(s)))\ ds\right),\ \ t \in \mathbb{R}_+.$$

In [2], using a fixed point theorem of Krasnosel'skii, Avramescu and Vladimirescu have proved the existence of asymptotically stable solutions to the equation

$$u(t) = q(t) + \int_0^t K(t,s,u(s))ds + \int_0^\infty G(t,s,u(s))ds, \quad t \in \mathbb{R}_+,$$

where functions given with real values satisfying suitable conditions. In case the Banach space $E$ is arbitrary, the existence of asymptotically stable solutions of equation

$$\begin{aligned} x(t) &= q(t) + f(t, x(t)) + \int_0^t V\left(t, s, x(s), \int_0^s V_1\left(t, s, r, x(r)\right) dr\right) ds \\ &\quad + \int_0^\infty G\left(t, s, x(s), \int_0^s G_1\left(t, s, r, x(r)\right) dr\right) ds, \end{aligned}$$

$t \in \mathbb{R}_+$, have been proved in [8], by using the fixed point theorem of Krasnosel'skii type.

Recently, in [10], the existence and uniqueness of a solution of the following system is proved

$$f_i(x) = \sum_{k=1}^{m}\sum_{j=1}^{n}\left[a_{ijk}\Psi\left(x, f_j(R_{ijk}(x)), \int_0^{X_{ijk}(x)} f_j(t)dt\right) + b_{ijk}f_j(S_{ijk}(x))\right] + g_i(x),$$

$i = 1, ..., n$, $x \in \Omega = [-b, b]$, where $a_{ijk}$, $b_{ijk}$ are the given real constants; $R_{ijk}$, $S_{ijk}$, $X_{ijk} : \Omega \to \Omega$, $g_i : \Omega \to \mathbb{R}$, $\Psi : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ are the given continuous functions and $f_i : \Omega \to \mathbb{R}$ are unknown functions. The main tool used here is Banach's fixed point theorem, it is applied in a suitable space, that is Banach space $X = C(\Omega; \mathbb{R}^n)$ of continuous functions $f : \Omega \to \mathbb{R}^n$, with respect to the norm $\|f\|_X = \sup_{x\in\Omega} \sum_{i=1}^{n} |f_i(x)|$, $\quad f = (f_1, ..., f_n) \in X$.

Motivated by the problems in the above mentioned works, we study the existence and uniqueness of a solution for (1.1). This paper consists of three sections. In section 2, we present the main result. Finally, the illustrated examples are given in section 3. The main tool employed here is Banach's fixed point theorem in Fréchet space, with a suitable choice of a numerable family of seminorms.

## 2. Existence of solutions

Let $X = C(\mathbb{R}_+; E)$ be the space of all continuous functions on $\mathbb{R}_+$ to $E$ which equipped with the numerable family of seminorms $|x|_n = \sup_{t\in[0,n]} |x(t)|$, $n \geq 1$. Then $(X, |\cdot|_n)$ is complete in the metric

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|x - y|_n}{1 + |x - y|_n},$$

and $X$ is the Fréchet space. In $X$ we also consider the family of seminorms defined by

$$\|x\|_n = \sup_{0\leq t\leq n} e^{-h_n t} |x(t)|, \ n \geq 1,$$

where $h_n > 0$ is arbitrary number, which is equivalent to $|\cdot|_n$, since

$$e^{-nh_n} |x|_n \leq \|x\|_n \leq |x|_n, \ \forall x \in X, \ \ \forall n \geq 1.$$

Based on the construct of such $(X, |\cdot|_n)$, the following lemma is valid, it is useful to prove existence of a unique solution for (1.1).

**Lemma 2.1.** ([1]) *Let $(X, |\cdot|_n)$ be a Fréchet space and let $\Phi : X \to X$ be an $L_n-$contraction on $X$ with respect to a family of seminorms $\|\cdot\|_n$ equivalent with $|\cdot|_n$. Then $\Phi$ has a unique fixed point in $X$.*

The details of the proof can be found in Appendix of [9].

In order to establish the existence result for (1.1), we need the following assumptions.

$(A_1)$ The functions $\mu_1,\ \mu_2,\ \theta_i,\ \tilde{\theta}_j \in C(\mathbb{R}_+;\mathbb{R}_+)$ are continuous such that $\mu_1(t),\ \mu_2(t),\ \theta_i(t),\ \tilde{\theta}_j(t) \in [0,t]$, for all $t \in \mathbb{R}_+,\ i=1,...,p,\ j=1,....q.$

$(A_2)$ There exist a constant $L \in [0,1)$ and a continuous function $\omega_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\left|V\left(t;x,y\right)-V\left(t;\bar{x},\bar{y}\right)\right| \le L\left|x-\bar{x}\right| + \omega_0(t)\left|y-\bar{y}\right|,$$

for all $(t;x,y)\,,\ \ (t;\bar{x},\bar{y}) \in \mathbb{R}_+ \times E^2$.

$(A_3)$ There exists a continuous function $\omega_1 : \Delta_{\mu_1} \to \mathbb{R}_+$ such that

$$\begin{aligned}&|V_1(t,s,x_1,...,x_p,y) - V_1(t,s,\bar{x}_1,...,\bar{x}_p,\bar{y})|\\ &\le \omega_1(t,s)\left(\sum\nolimits_{i=1}^{p}|x_i-\bar{x}_i| + |y-\bar{y}|\right),\end{aligned}$$

for all $(t,s,x_1,...,x_p,y),\ (t,s,\bar{x}_1,...,\bar{x}_p,\bar{y}) \in \Delta_{\mu_1} \times E^{p+1}$.

$(A_4)$ There exists a continuous function $\omega_2 : \Delta_{\mu_1} \to \mathbb{R}_+$ such that

$$|V_2(s,r,x_1,...,x_q) - V_2(s,r,\bar{x}_1,...,\bar{x}_q)| \le \omega_2(s,r)\sum\nolimits_{j=1}^{q}|x_j-\bar{x}_j|,$$

for all $(s,r,x_1,...,x_q),\ (s,r,\bar{x}_1,...,\bar{x}_q) \in \Delta_{\mu_2} \times E^q$.

Then we have the following theorem.

**Theorem 2.2.** *Let* $(A_1)$-$(A_4)$ *hold. Then* (1.1) *has a unique solution* $x_*$ *on* $\mathbb{R}_+$. *Moreover, given* $x_0 \in X = C(\mathbb{R}_+;E)$, *consider the sequence* $\{x_k\}$ *be defined by*

$$\begin{cases} x_k(t) \\ = V\Big(t, x_{k-1}(t), \int_0^{\mu_1(t)} V_1\left(t,s,x_{k-1}(\theta_1(s)),...,x_{k-1}(\theta_p(s)),\bar{V}_2[x_{k-1}](s)\right)ds\Big), \\ \bar{V}_2[x_{k-1}](s) \\ = \int_0^{\mu_2(s)} V_2\left(s,r,x_{k-1}(\tilde{\theta}_1(r)),...,x_{k-1}(\tilde{\theta}_q(r))\right)dr,\ t\in\mathbb{R}_+,\ k=1,2,... \end{cases} \tag{2.1}$$

*Then sequence* $\{x_k\}$ *converges in* $X$ *to the solution* $x_*$ *with error estimation*

$$\|x_k - x_*\|_n \le \frac{\|x_1-x_0\|_n}{1-L_n}L_n^k,\ \ \forall k,\ \ n \in \mathbb{N}, \tag{2.2}$$

*where* $L_n$, $0 < L_n < 1$ *is a constanst depending only on* $n$.

*Proof.* First, we rewrite the equation (1.1) as follows

$$x(t) = \Phi x(t),\ \ t \in \mathbb{R}_+, \tag{2.3}$$

where

$$\begin{array}{l} \Phi x(t)=V\Big(t,x(t),\int_0^{\mu_1(t)}V_1\left(t,s,x(\theta_1(s)),...,x(\theta_p(s)),\bar{V}_2[x](s)\right)ds\Big), \\ \bar{V}_2[x](s)=\int_0^{\mu_2(s)}V_2\left(s,r,x(\tilde{\theta}_1(r)),...,x(\tilde{\theta}_q(r))\right)dr,\ (t,x)\in\mathbb{R}_+\times X. \end{array} \tag{2.4}$$

By the assumptions $(A_2)$-$(A_4)$, for all $x$, $\bar{x}\in X$, for all $t\in\mathbb{R}_+$, put $y=x-\bar{x}$, we have

$$\begin{array}{l} |\Phi x(t)-\Phi\bar{x}(t)| \\ \leq L\,|y(t)|+\omega_0(t)\sum_{i=1}^p\int_0^{\mu_1(t)}\omega_1(t,s)\,|y(\theta_i(s))|\,ds \\ \quad +\omega_0(t)\sum_{j=1}^q\int_0^{\mu_1(t)}\omega_1(t,s)ds\int_0^{\mu_2(s)}\omega_2(s,r)\left|y(\tilde{\theta}_j(r))\right|dr. \end{array} \tag{2.5}$$

Let $n\in\mathbb{N}$ be fixed. For all $t\in[0,n]$, with $h_n>0$ to be chosen later, $\Phi$ has the following property

$$\begin{array}{l} |\Phi x(t)-\Phi\bar{x}(t)|\,e^{-h_nt} \\ \leq L\,|y(t)|\,e^{-h_nt}+\tilde{\omega}_{0n}\tilde{\omega}_{1n}\sum_{i=1}^p e^{-h_nt}\int_0^t|y(\theta_i(s))|\,ds \\ \quad +\tilde{\omega}_{0n}\tilde{\omega}_{1n}\tilde{\omega}_{2n}\sum_{j=1}^q e^{-h_nt}\int_0^t ds\int_0^s\left|y(\tilde{\theta}_j(r))\right|dr \\ \leq L\,\|y\|_n+\tilde{\omega}_{0n}\tilde{\omega}_{1n}\sum_{i=1}^p I_i^{(1)}+\tilde{\omega}_{0n}\tilde{\omega}_{1n}\tilde{\omega}_{2n}\sum_{j=1}^q I_j^{(2)}, \end{array} \tag{2.6}$$

where

$$\begin{cases} \tilde{\omega}_{0n}=\sup\,\{\omega_0(t):0\leq t\leq n\}, \\ \tilde{\omega}_{1n}=\sup\,\{\omega_1(t,s):(t,s)\in\Delta_{1n}\}, \\ \tilde{\omega}_{2n}=\sup\,\{\omega_2(s,r):(s,r)\in\Delta_{2n}\}, \\ \Delta_{1n}=\{(t,s):0\leq s\leq\mu_1(t),\,0\leq t\leq n\}, \\ \Delta_{2n}=\{(s,r):0\leq r\leq\mu_2(s),\,0\leq s\leq n\}, \end{cases}$$

and

$$\begin{cases} I_i^{(1)}=e^{-h_nt}\int_0^t|y(\theta_i(s))|\,ds,\ i=1,...,p, \\ I_j^{(2)}=e^{-h_nt}\int_0^t ds\int_0^s\left|y(\tilde{\theta}_j(r))\right|dr,\ j=1,...,q. \end{cases} \tag{2.7}$$

Estimating $I_i^{(1)}=e^{-h_nt}\int_0^t|y(\theta_i(s))|\,ds$.

$$\begin{array}{l} I_i^{(1)}=e^{-h_nt}\int_0^t|y(\theta_i(s))|\,ds=e^{-h_nt}\int_0^t e^{h_n\theta_i(s)}e^{-h_n\theta_i(s)}\,|y(\theta_i(s))|\,ds \\ \quad\leq e^{-h_nt}\int_0^t e^{h_n\theta_i(s)}\,\|y\|_n\,ds\leq e^{-h_nt}\int_0^t e^{h_ns}\,\|y\|_n\,ds \\ \quad=e^{-h_nt}\frac{1}{h_n}\left(e^{h_nt}-1\right)\|y\|_n=\frac{1}{h_n}\left(1-e^{-h_nt}\right)\|y\|_n\leq\frac{1}{h_n}\,\|y\|_n\,. \end{array} \tag{2.8}$$

Estimating $I_j^{(2)}=e^{-h_nt}\int_0^t ds\int_0^s\left|y(\tilde{\theta}_j(r))\right|dr$. We have

$$\begin{array}{l} I_j^{(2)}=e^{-h_nt}\int_0^t ds\int_0^s\left|y(\tilde{\theta}_j(r))\right|dr \\ \quad\leq e^{-h_nt}\int_0^t ds\int_0^t\left|y(\tilde{\theta}_j(r))\right|dr\leq ne^{-h_nt}\int_0^t\left|y(\tilde{\theta}_j(r))\right|dr. \end{array} \tag{2.9}$$

Similarly

$$e^{-h_n t}\int_0^t \left|y(\tilde{\theta}_j(r))\right| dr \le \frac{1}{h_n}\left\|y\right\|_n. \tag{2.10}$$

Thus

$$I_j^{(2)} \le \frac{n}{h_n}\left\|y\right\|_n.$$

Consequently

$$\left|\Phi x(t)-\Phi\bar{x}(t)\right| e^{-h_n t} \le \left[L+\frac{1}{h_n}\tilde{\omega}_{0n}\tilde{\omega}_{1n}\left(p+nq\tilde{\omega}_{2n}\right)\right]\left\|y\right\|_n.$$

This implies that

$$\left\|\Phi x-\Phi\bar{x}\right\|_n \le \left[L+\frac{1}{h_n}\tilde{\omega}_{0n}\tilde{\omega}_{1n}\left(p+nq\tilde{\omega}_{2n}\right)\right]\left\|y\right\|_n = L_n\left\|x-\bar{x}\right\|_n, \tag{2.11}$$

where $L_n = L+\frac{1}{h_n}\tilde{\omega}_{0n}\tilde{\omega}_{1n}\left(p+nq\tilde{\omega}_{2n}\right)$. Choosing $h_n$ such that

$$L_n = L+\frac{1}{h_n}\tilde{\omega}_{0n}\tilde{\omega}_{1n}\left(p+nq\tilde{\omega}_{2n}\right) < 1, \tag{2.12}$$

then we have $0<L_n<1$, so $\Phi$ is a $L_n-$ contraction operator on the Fréchet space $(X,\left\|\cdot\right\|_n)$, applying Lemma 2.1, (2.3) has a unique solution $x=x_*$.

On the other hand, by the operator $\Phi$ is a $L_n-$contraction, we obtain

$$\left\|x_{k+p}-x_k\right\|_n \le \frac{\left\|x_1-x_0\right\|_n}{1-L_n}L_n^k, \quad \forall\, k,p,n\in\mathbb{N}. \tag{2.13}$$

This implies that for all $p$, $n\in\mathbb{N}$, we have

$$\lim_{k\to+\infty}\left\|x_{k+p}-x_k\right\|_n = 0,$$

which means that $\{x_k\}$ is a Cauchy sequence, by the fact that $\left\|\cdot\right\|_n$ is equivalent with $\left|\cdot\right|_n$. The space $(X,\left|\cdot\right|_n)$ is complete, so $\{x_k\}$ converges to a point $x_*$ of $X$. It is obviously that $x_*$ is a unique fixed point of $\Phi$. Passing to the limit in (2.13) as $p\to+\infty$ for fixed $k$, (2.2) follows. Theorem 2.2 is proved. □

## 3. The examples

Let us illustrate the results obtained by means of the examples.

**Example 3.1.** Let $E=C([0,1];\mathbb{R})$ be the Banach space of all continuous functions $u:[0,1]\to\mathbb{R}$ with the norm

$$\left|u\right|_E = \left\|u\right\| = \sup_{0\le\eta\le1}\left|u(\eta)\right|,\ u\in E.$$

Then, for all $x\in X=C(\mathbb{R}_+;E)$, for any $t\in\mathbb{R}_+,\ x(t)$ is an element of $E$ and we denote

$$x(t)(\eta)=x(t,\eta),\ 0\le\eta\le1.$$

Consider (1.1) in form

$$\begin{cases} x(t) = V\Big(t, x(t), \int_0^{\mu_1(t)} V_1\left(t, s, x(\theta_1(s)), ..., x(\theta_p(s)), \bar{V}_2[x](s)\right) ds\Big), \\ \bar{V}_2[x](s) = \int_0^{\mu_2(s)} V_2\left(s, r, x(\tilde{\theta}_1(r)), ..., x(\tilde{\theta}_q(r))\right) dr, \; t \in \mathbb{R}_+, \end{cases} \tag{3.1}$$

where

$$\begin{cases} \mu_i(t) = \bar{\mu}_i t, \; 0 < \bar{\mu}_i \leq 1, \; i = 1, 2; \\ \theta_i(t) = \bar{\theta}_i t, \; 0 < \bar{\theta}_i \leq 1, \; i = 1, ..., p; \\ \tilde{\theta}_j(t) = \hat{\theta}_j t, \; 0 < \hat{\theta}_i \leq 1, \; j = 1, ..., q. \end{cases}$$

Giving the continuous functions $V$, $V_1$, $V_2$ as follows.

(i) Function $V : \mathbb{R}_+ \times E^2 \to E$,

$$V(t, x, y)(\eta) = (1 - k_1) Z_*(t, \eta) + k_1 |x(\eta)| + e^{-t} |y(\eta)|,$$

for all $0 \leq \eta \leq 1$, $(t, x, y) \in \mathbb{R}_+ \times E^2$ with $Z_*(t, \eta) = \frac{1}{\eta + e^t}$ and $k_1$ is given constant such that $0 < k_1 < 1$.

(ii) Function $V_1 : \Delta_1 \times E^{p+1} \to E$, $\Delta_1 = \{(t, s) \in \mathbb{R}_+^2 : s \leq \bar{\mu}_1 t\}$,

$$\begin{aligned} &V_1(t, s, x_1, ..., x_p, y)(\eta) \\ &= e^{-2s} Z_*(t, \eta) \left[ \sum\nolimits_{i=1}^{p} \sin\left( \frac{\pi}{Z_*(\theta_i(s), \eta)} x_i(\eta) \right) + e^{-t} |y(\eta)| \; \right], \end{aligned}$$

for all $0 \leq \eta \leq 1$, $(t, s, x_1, ..., x_p, y) \in \Delta_1 \times E^{p+1}$.

(iii) Function $V_2 : \Delta_2 \times E^q \to E$, $\Delta_2 = \{(s, r) \in \mathbb{R}_+^2 : r \leq \bar{\mu}_2 s\}$,

$$V_2(s, r, x_1, ..., x_q)(\eta) = e^{-2r} Z_*(s, \eta) \sum\nolimits_{j=1}^{q} \sin\left( \frac{2\pi}{Z_*(\tilde{\theta}_j(r), \eta)} x_j(\eta) \right),$$

for all $0 \leq \eta \leq 1$, $(s, r, x_1, ..., x_q) \in \Delta_2 \times E^q$.

We can prove that $(A_1)$-$(A_4)$ hold. It is easy to see that $(A_1)$ holds. Assumption $(A_2)$ holds, for all $(t; x, y)$, $(t; \bar{x}, \bar{y}) \in \mathbb{R}_+ \times E^2$,

$$\| V(t; x, y) - V(t; \bar{x}, \bar{y}) \| \leq k_1 \| x - \bar{x} \| + \omega_0(t) \| y - \bar{y} \|,$$

with $\omega_0(t) = e^{-t}$, $L = k_1$.

Assumption $(A_3)$ holds, for all $(t,s,x_1,...,x_p,y)$, $(t,s,\bar{x}_1,...,\bar{x}_p,\bar{y}) \in \Delta_1 \times E^{p+1}$, $\Delta_1 = \{(t,s) \in \mathbb{R}_+^2 : s \leq \bar{\mu}_1 t\}$, $\forall \eta \in [0,1]$,

$$\begin{aligned}
&|V_1(t,s,x_1,...,x_p,y)(\eta) - V_1(t,s,\bar{x}_1,...,\bar{x}_p,\bar{y})(\eta)| \\
&\leq e^{-2s}\frac{1}{\eta+e^t}\left[\sum\nolimits_{i=1}^{p} \pi(e^{\bar{\theta}_i s}+\eta)\,|x_i(\eta)-\bar{x}_i(\eta)| + e^{-t}\,|y(\eta)-\bar{y}(\eta)|\right] \\
&\leq 2\pi e^{-t-s}\left[\sum\nolimits_{i=1}^{p} \|x_i-\bar{x}_i\| + \|y-\bar{y}\|\right] \\
&= \omega_1(t,s)\left[\sum\nolimits_{i=1}^{p} \|x_i-\bar{x}_i\| + \|y-\bar{y}\|\right],
\end{aligned}$$

in which

$$\omega_1(t,s) = 2\pi e^{-t-s}.$$

Assumption $(A_4)$ holds, for all $(s,r,x_1,...,x_q)$, $(s,r,\bar{x}_1,...,\bar{x}_q) \in \Delta_2 \times E^q$, $\Delta_2 = \{(s,r) \in \mathbb{R}_+^2 : r \leq \bar{\mu}_2 s\}$, $\forall \eta \in [0,1]$,

$$\begin{aligned}
&|V_2(s,r,x_1,...,x_q)(\eta) - V_2(s,r,\bar{x}_1,...,\bar{x}_q)(\eta)| \\
&\leq e^{-2r}\frac{1}{e^s+\eta}\sum\nolimits_{j=1}^{q} 2\pi(\eta+e^{\hat{\theta}_j r})\,|x_j(\eta)-\bar{x}_j(\eta)| \\
&\leq 4\pi e^{-s-r}\sum\nolimits_{j=1}^{q} \|x_j-\bar{x}_j\| \\
&= \omega_2(s,r)\sum\nolimits_{j=1}^{q} \|x_j-\bar{x}_j\|,
\end{aligned}$$

with $\omega_2(s,r) = 4\pi e^{-s-r}$. Then, Theorems 2.2 holds for (3.1). For more details, it is not difficult to show that (3.1) has a unique solution $x_* = Z_*$.

**Example 3.2.** Let $E = \mathbb{R}^N$, consider the following system of equations

$$\begin{aligned}
x_i(t) = U_i\Big(&t, x_1(t),...,x_N(t), \textstyle\int_0^{\mu_1(t)} W_1\left(t,s,x_1(\theta_1(s)),...,x_N(\theta_1(s))\right) ds, \\
&..., \textstyle\int_0^{\mu_1(t)} W_N\left(t,s,x_1(\theta_1(s)),...,x_N(\theta_1(s))\right) ds\Big),
\end{aligned} \tag{3.2}$$

$i = 1,...,N$, $t \in \mathbb{R}_+$, where the continuous functions $U_i$, $W_i$ are defined by

$$\begin{cases}
U_i : \mathbb{R}_+ \times \mathbb{R}^{N+1} \to \mathbb{R},\ i = 1,...,N; \\
W_i : \Delta_1 \times \mathbb{R}^N \to \mathbb{R},\ i = 1,...,N; \\
\Delta_1 = \{(t,s) \in \mathbb{R}_+^2 : s \leq \mu_1(t)\}.
\end{cases}$$

Suppose that

$(\tilde{A}_1)$ The functions $\mu_1$, $\theta_1 \in C(\mathbb{R}_+;\mathbb{R}_+)$ are continuous such that $\mu_1(t) \leq t$, $\theta_1(t) \leq t$, for all $t \in \mathbb{R}_+$, we rewrite (3.2) as follows

$$\begin{aligned} & x_i(t) \\ & = U_i\left(t, x(t), \int_0^{\mu_1(t)} W_1\left(t,s,x(\theta_1(s))\right) ds, ..., \int_0^{\mu_1(t)} W_N\left(t,s,x(\theta_1(s))\right) ds\right), \end{aligned} \tag{3.3}$$

$i = 1, ..., N$, $t \in \mathbb{R}_+$, where $x = (x_1, ..., x_N) \in X = C(\mathbb{R}_+;\mathbb{R}^N)$.

We define two vector functions $V : \mathbb{R}_+ \times \mathbb{R}^{2N} \to \mathbb{R}^N$, $V_1 : \Delta_1 \times \mathbb{R}^N \to \mathbb{R}^N$ as follows:

$$\begin{aligned} V(t,x,y) &= \left(U_1(t,x,y), ..., U_N(t,x,y)\right), (t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^{2N}, \\ V_1(t,s,x) &= \left(W_1(t,s,x), ..., W_N(t,s,x)\right), (t,s,x) \in \Delta_1 \times \mathbb{R}^N. \end{aligned}$$

Then, system (3.3) becomes

$$x(t) = V\left(t, x(t), \int_0^{\mu_1(t)} V_1\left(t,s,x(\theta_1(s))\ \right) ds\right) \equiv \Phi x(t),\ t \geq 0. \tag{3.4}$$

Suppose that

$(\tilde{A}_2)$ There exist a constant $L \in [0,1)$ and a continuous function $\omega_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\left|U_i\left(t;x,y\right) - U_i\left(t;\bar{x},\bar{y}\right)\right| \leq L\left|x-\bar{x}\right|_\infty + \omega_0(t)\left|y-\bar{y}\right|_\infty,$$

for all $(t;x,y)$, $(t;\bar{x},\bar{y}) \in \mathbb{R}_+ \times \mathbb{R}^{2N}$, for all $i = 1, ..., N$; and $|\cdot|_\infty$ is a norm in $\mathbb{R}^N$ defined by

$$|x|_\infty = \max_{1 \leq i \leq N} |x_i|, x = (x_1, ..., x_N) \in \mathbb{R}^N.$$

$(\tilde{A}_3)$ There exists a continuous function $\omega_1 : \Delta_{\mu_1} \to \mathbb{R}_+$ such that

$$|W_i(t,s,x) - W_i(t,s,\bar{x})| \leq \omega_1(t,s)\left|x-\bar{x}\right|_\infty,$$

for all $(t,s,x)$, $(t,s,\bar{x}) \in \Delta_1 \times \mathbb{R}^N$.

Note that $C(\mathbb{R}_+;\mathbb{R}^N)$ is the Fréchet space which equipped with the numerable family of seminorms

$$|x|_n = \sup_{t \in [0,n]} |x(t)|_\infty,\ n \geq 1.$$

In $C(\mathbb{R}_+;\mathbb{R}^N)$ we also consider the family of seminorms defined by

$$\|x\|_n = \sup_{0 \leq t \leq n} e^{-h_n t}\left|x(t)\right|_\infty,\ n \geq 1,$$

and $h_n > 0$ is arbitrary number, which is equivalent to $|\cdot|_n$, since

$$e^{-nh_n}|x|_n \leq \|x\|_n \leq |x|_n,\ \ \forall x \in C(\mathbb{R}_+;\mathbb{R}^N),\ \ \forall n \geq 1.$$

With the choose of the suitable parameter $h_n > 0$ ($h_n > 0$ sufficiently large), we will get $L_n \in [0, 1)$ such that, the operator $\Phi$ is a $L_n-$contraction on $C(\mathbb{R}_+; \mathbb{R}^N)$.

Similarly, then we have the following result.

**Proposition 3.3.** *Let* $(\tilde{A}_1)$-$(\tilde{A}_3)$ *hold. Then* (3.2) *has a unique solution* $y_* \in C(\mathbb{R}_+; \mathbb{R}^N)$.

*Furthermore, given* $y^{(0)} \in C(\mathbb{R}_+; \mathbb{R}^N)$, *consider the sequence* $\{y^{(k)}\}$ *be defined by*

$$\begin{aligned} y^{(k)}(t) &= V\left(t, y^{(k-1)}(t), \int_0^{\mu_1(t)} V_1\left(t, s, y^{(k-1)}(\theta_1(s))\ \right) ds\right) \\ &\equiv \Phi y^{(k-1)}(t), \end{aligned} \tag{3.5}$$

$t \in \mathbb{R}_+$, $k = 1, 2, \cdots$. *Then sequence* $\{y^{(k)}\}$ *converges to* $y_*$ *in* $C(\mathbb{R}_+; \mathbb{R}^N)$ *with error estimation*

$$\left\| y^{(k)} - y_* \right\|_n \leq \frac{\left\| y^{(1)} - y^{(0)} \right\|_n}{1 - L_n} L_n^k, \quad \forall\, k, n \in \mathbb{N}, \tag{3.6}$$

*where* $L_n$, $0 < L_n < 1$ *is a constanst depending only on* $n$.

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