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ON A HIGH ORDER ITERATIVE SCHEME FOR A NONLINEAR WAVE EQUATION

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Abstract. In this paper, a high order iterative scheme is established in order to get a convergent sequence, at a rate of order N, to a local unique weak solution of a nonlinear wave equation associated with homogeneous Dirichlet boundary conditions. This scheme shows that the convergence can be obtained with a high rate if the nonlinear term in the original equation is smooth enough.

1. Introduction

In this paper, we shall establish a high order iterative scheme in order to get a convergent sequence, at a rate of order N, to a unique local weak solution

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of the following Dirichlet problem for a nonlinear wave equation

$$u_{tt} - \frac{\partial}{\partial x}\left(\mu(x,t)u_x\right) + \lambda u_t = f(x,t,u),\ 0 < x < 1,\ 0 < t < T, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \tag{1.2}$$

$$u(x,0) = \tilde{u}_0(x),\ u_t(x,0) = \tilde{u}_1(x), \tag{1.3}$$

where μ, f, $\tilde{u}_0$, $\tilde{u}_1$ are given functions and $\lambda \neq 0$ is a given constant.

It is well known that Newton's method and its variants are used to solve nonlinear operator equations $F(x) = 0$. Newton's method arises naturally when replace $F(x)$ by the linear term in the Taylor series, so that with x_0 as a first approximation, by constructing an approximating sequence $\{x_n\}$ and showing its convergence, a zero of F will be obtained. The sequence $\{x_n\}$ can be very rapidly convergent to the zero x, if it is given a sufficiently close first approximation x_0 to x and provided derivatives of the function F behave nicely in a neighbourhood of x. In this case, one speaks of *convergence of order* N if $|u_{n+1} - u| \leq C\,|u_n - u|^N$, for some $C > 0$ and all large n. For the details, it can be found in, for example, [1], [9], [10] and references therein.

Based on the ideas about a high order method for solving the equation $F(x) = 0$ as above and based on Faedo - Galerkin method, recently, in [6], [8] and in some other works, the authors have constructed a high order iterative scheme in order to obtain existence results where recurrent sequences converge at a rate of order N.

In this paper, we consider Prob.(1.1)-(1.3) and associate with Eq.(1.1) a recurrent sequence $\{u_m\}$ defined by

$$\begin{aligned} &\frac{\partial^2 u_m}{\partial t^2} - \frac{\partial}{\partial x}\left(\mu(x,t)\frac{\partial u_m}{\partial x}\right) + \lambda\frac{\partial u_m}{\partial t} \\ &= \sum_{k=0}^{N-1} \frac{1}{k!}\frac{\partial^k f}{\partial u^k}(x,t,u_{m-1})\left(u_m - u_{m-1}\right)^k, \end{aligned} \tag{1.4}$$

$0 < x < 1$, $0 < t < T$, with u_m satisfying (1.2), (1.3). The first term u_0 is chosen as $u_0 \equiv 0$. If $\mu \in C^1([0,1] \times \mathbb{R}_+)$, and $f \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, we prove that the sequence $\{u_m\}$ converges at a rate of order N to a unique weak solution of Prob.(1.1)-(1.3). The main result is given in Theorems 2.1 and 2.3. In our proofs, the fixed point method and Faedo-Galerkin method are employed. This result is a relative generalization of [4]-[8].

2. A high order iterative scheme

First, we put $\Omega = (0,1)$ and denote the usual function spaces used in this paper by the notations $L^p = L^p(\Omega), H^m = H^m(\Omega)$. Let $\langle\cdot,\cdot\rangle$ be either the scalar product in L^2 or the dual pairing of a continuous linear functional and an element of a function space. The notation $\|\cdot\|$ stands for the norm in L^2, $\|\cdot\|_X$ is the norm in the Banach space X, and X' is the dual space of X.

We denote by $L^p(0,T;X), 1 \leq p \leq \infty$ for the Banach space of real functions $u:(0,T) \to X$ measurable, such that

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt\right)^{1/p} < \infty \quad \text{for} \quad 1 \leq p < \infty,$$

and

$$\|u\|_{L^\infty(0,T;X)} = \underset{0<t<T}{ess\sup} \|u(t)\|_X \quad \text{for} \quad p = \infty.$$

Let $u(t)$, $u'(t) = u_t(t) = \dot{u}(t)$, $u''(t) = u_{tt}(t) = \ddot{u}(t)$, $u_x(t) = \bigtriangledown u(t)$, $u_{xx}(t) = \Delta u(t)$, denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$, $\frac{\partial^2 u}{\partial x^2}(x,t)$, respectively.

With $f \in C^k([0,1] \times \mathbb{R}_+ \times \mathbb{R})$, $f = f(x,t,u)$, we put $D_1 f = \frac{\partial f}{\partial x}$, $D_2 f = \frac{\partial f}{\partial t}$, $D_3 f = \frac{\partial f}{\partial u}$ and $D^\alpha f = D_1^{\alpha_1} D_2^{\alpha_2} D_3^{\alpha_3} f$; $\alpha = (\alpha_1,\, \alpha_2,\, \alpha_3) \in \mathbb{Z}^3$, $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = k$, $D^{(0,0,0)} f = D^{(0)} f = f$.

Similarly, with $\mu = \mu(x,t)$, we also put $D_1 \mu = \frac{\partial \mu}{\partial x}$, $D_2 \mu = \frac{\partial \mu}{\partial t}$.

We shall use the following norm on H^1

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}.$$

It is well known that the imbedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact and for all $v \in H^1$,

$$\|v\|_{C^0(\overline{\Omega})} \leq \sqrt{2}\, \|v\|_{H^1}.$$

Furthermore, on $H_0^1 = \{v \in H^1 : v(0) = v(1) = 0\}$, two norms $v \longmapsto \|v\|_{H^1}$ and $v \longmapsto \|v_x\|$ are equivalent and

$$\|v\|_{C^0(\overline{\Omega})} \leq \|v_x\| \text{ for all } v \in H_0^1. \tag{2.1}$$

We make the following assumptions:

(H_1) $(\tilde{u}_0, \tilde{u}_1) \in \left(H_0^1 \cap H^2\right) \times H_0^1$;

(H_2) $f \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R})$ satisfying $f(0,t,0) = f(1,t,0) = 0$, $\forall\, t \geq 0$;

(H_3) $\mu \in C^2([0,1] \times \mathbb{R}_+)$ and there exists constant $\mu_0 > 0$ such that

$$\mu(x,t) \geq \mu_0 \quad \text{for all } (x,t) \in [0,1] \times \mathbb{R}_+.$$

Fix $T^* > 0$. For each $M > 0$ given, we set the constants $K_0(M,f)$, $K_M(f)$, $\tilde{K}_0(\mu)$, $\tilde{K}(\mu)$ as follows

$$\begin{cases} K_0(M,f) = \sup\{|f(x,t,u)| : 0 \leq x \leq 1,\ 0 \leq t \leq T^*,\ |u| \leq M\}, \\ K_M(f) = \sum_{|\alpha| \leq N} K_0(M, D^\alpha f), \\ \tilde{K}_0(\mu) = \|\mu\|_{C^0([0,1]\times[0,T^*])} = \sup\limits_{(x,t)\in[0,1]\times[0,T^*]} |\mu(x,t)|, \\ \tilde{K}(\mu) = \|\mu\|_{C^2([0,1]\times[0,T^*])} = \sum\limits_{i+j\leq 2} \tilde{K}_0\left(D_1^i D_2^j \mu\right). \end{cases}$$

For every $T \in (0, T^*]$ and $M > 0$, we put

$$\begin{cases} W(M,T) = \{v \in L^\infty(0,T;H_0^1 \cap H^2) : v_t \in L^\infty(0,T;H_0^1),\ v_{tt} \in L^2(Q_T), \\ \qquad\qquad \text{with } \|v\|_{L^\infty(0,T;H_0^1\cap H^2)}, \|v_t\|_{L^\infty(0,T;H_0^1)}, \|v_{tt}\|_{L^2(Q_T)} \le M\}, \\ W_1(M,T) = \{v \in W(M,T) : v_{tt} \in L^\infty(0,T;L^2)\}, \end{cases}$$

in which $Q_T = \Omega \times (0,T)$.

Now, we establish the recurrent sequence $\{u_m\}$. The first term is chosen as $u_0 \equiv 0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{2.2}$$

we associate problem (1.1) - (1.3) with the following problem.

Find $u_m \in W_1(M,T)$ $(m \ge 1)$ satisfying the linear variational problem

$$\begin{cases} \langle u_m''(t), w\rangle + \langle \mu(t) u_{mx}(t), w_x\rangle + \lambda \langle u_m'(t), w\rangle \\ = \langle \Phi_m(t), w\rangle, \ \forall w \in H_0^1, \\ u_m(0) = \tilde{u}_0,\ u_m'(0) = \tilde{u}_1, \end{cases} \tag{2.3}$$

where

$$\Phi_m(x,t) = \textstyle\sum_{i=0}^{N-1} \frac{1}{i!} \frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m - u_{m-1})^i. \tag{2.4}$$

Then we have the following theorem.

Theorem 2.1. *Let* (H_1)-(H_3) *hold. Then there exist a constant* $M > 0$ *depending on* $\tilde{u}_0, \tilde{u}_1, \mu$ *and* $T > 0$ *depending on* $\tilde{u}_0$, $\tilde{u}_1$, μ, f *such that, for* $u_0 \equiv 0$, *there exists a recurrent sequence* $\{u_m\} \subset W_1(M,T)$ *defined by* (2.3) *and* (2.4).

Proof. (i) *Approximating solutions.*

Let $\{w_j\}$ be a basis of H_0^1, formed by eigenfunction w_j of the operator $-\triangle = -\frac{\partial^2}{\partial x^2}$:

$$\begin{aligned} &-\triangle w_j = \lambda_j w_j, \\ &w_j \in H_0^1 \cap H^2, \\ &w_j(x) = \sqrt{2}\sin(j\pi x), \lambda_j = (j\pi)^2,\ j = 1,2,3,\cdots. \end{aligned}$$

Put

$$u_m^{(k)}(t) = \textstyle\sum_{j=1}^{k} c_{mj}^{(k)}(t) w_j, \tag{2.5}$$

where the coefficients $c_{mj}^{(k)}$ satisfy the system of nonlinear differential equations

$$\begin{cases} \left\langle \ddot{u}_m^{(k)}(t), w_j\right\rangle + \left\langle \mu(t) u_{mx}^{(k)}(t), w_{jx}\right\rangle + \lambda \langle \dot{u}_m^{(k)}(t), w_j\rangle = \left\langle \Phi_m^{(k)}(t), w_j\right\rangle, \\ u_m^{(k)}(0) = \tilde{u}_{0k},\ \dot{u}_m^{(k)}(0) = \tilde{u}_{1k},\ \ j = 1,2,...,k, \end{cases} \tag{2.6}$$

in which

$$\begin{cases} \tilde{u}_{0k} = \sum_{j=1}^{k} \alpha_j^{(k)} w_j \longrightarrow \tilde{u}_0 \text{ strongly } H_0^1 \cap H^2, \\ \tilde{u}_{1k} = \sum_{j=1}^{k} \beta_j^{(k)} w_j \longrightarrow \tilde{u}_1 \text{ strongly } H_0^1, \end{cases} \tag{2.7}$$

and

$$\begin{aligned} \Phi_m^{(k)}(x,t) &= \sum_{i=0}^{N-1} \frac{1}{i!} \frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m^{(k)} - u_{m-1})^i \\ &= \sum_{j=0}^{N-1} A_j(x,t,u_{m-1})(u_m^{(k)})^j, \end{aligned} \tag{2.8}$$

with

$$A_j(x,t,u_{m-1}) = \sum_{i=j}^{N-1} \frac{(-1)^{i-j}}{j!(i-j)!} \frac{\partial^i f}{\partial u^i}(x,t,u_{m-1}) u_{m-1}^{i-j}.$$

The system (2.6), (2.8) can be written in the form

$$\begin{cases} \ddot{c}_{mj}^{(k)}(t) + \sum_{i=1}^{k} \langle \mu(t) w_{ix}, w_{jx} \rangle c_{mi}^{(k)}(t) + \lambda \dot{c}_{mj}^{(k)}(t) = \Phi_{mj}^{(k)}(t), \\ c_{mj}^{(k)}(0) = \alpha_j^{(k)},\ \dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)},\ 1 \le j \le k, \end{cases} \tag{2.9}$$

where

$$\Phi_{mj}^{(k)}(t) = \left\langle \Phi_m^{(k)}(t), w_j \right\rangle,\ \lambda_j = \mu_j^2 = (j\pi)^2,\ 1 \le j \le k. \tag{2.10}$$

It can see that, system (2.9) is equivalent to system of intergal equations

$$\begin{aligned} & c_{mj}^{(k)}(t) + \sum_{i=1}^{k} \int_0^t d\tau \int_0^\tau e^{-\lambda(\tau-s)} \langle \mu(s) w_{ix}, w_{jx} \rangle c_{mi}^{(k)}(s) ds \\ & = \alpha_j^{(k)} + \frac{1}{\lambda} \beta_j^{(k)} \left(1 - e^{-\lambda t}\right) + \int_0^t d\tau \int_0^\tau e^{-\lambda(\tau-s)} \Phi_{mj}^{(k)}(s) ds, \quad 1 \le j \le k. \end{aligned} \tag{2.11}$$

Omitting the indexs m, k, it is written as follows

$$c = F[c], \tag{2.12}$$

where $F[c] = (F_1[c], ..., F_k[c])$, $c = (c_1, .., c_k)$, and

$$\begin{cases} F_j[c](t) = q_j(t) - \sum_{i=1}^{k} \int_0^t d\tau \int_0^\tau e^{-\lambda(\tau-s)} \langle \mu(s) w_{ix}, w_{jx} \rangle c_i(s) ds \\ \qquad + \sum_{i=1}^{N-1} \int_0^t d\tau \int_0^\tau e^{-\lambda(\tau-s)} \left\langle A_i(s, u_{m-1})(u(s))^i, w_j \right\rangle ds, \\ q_j(t) = \alpha_j + \frac{1}{\lambda} \beta_j \left(1 - e^{-\lambda t}\right) + \int_0^t d\tau \int_0^\tau e^{-\lambda(\tau-s)} \langle A_0(s, u_{m-1}), w_j \rangle ds, \\ \qquad 1 \le j \le k, \\ u(t) = \sum_{i=1}^{k} c_i(t) w_i. \end{cases} \tag{2.13}$$

Applying the contraction principle, system (2.11) has a unique solution $c_{mj}^{(k)}(t)$ in $[0, T_m^{(k)}]$, with certain $T_m^{(k)} \in (0, T]$. Indeed, for every $T_m^{(k)} \in (0, T]$ and $\rho > 0$ chosen later, we set

$$X = C^0 \left([0, T_m^{(k)}]; \mathbb{R}^k \right),\quad S = \{ c \in X : \|c\|_X \le \rho \},$$

where

$$\|c\|_X = \sup_{0\le t\le T_m^{(k)}} |c(t)|_1\,, \;\; |c(t)|_1 = \sum_{j=1}^k |c_j(t)|\,.$$

Clearly, S is a nonempty closed subset of X and $F : X \to X$. We will choose $\rho > 0$ and $T_m^{(k)} > 0$ such that $F : S \to S$ is contractive as follows.

First we note that, for all $c = (c_1, .., c_k) \in S$,

$$\begin{aligned}&\|u(t)\| \le |c(t)|_1 \le \|c\|_X \le \rho,\\ &\|u(t)\|_{C^0(\overline{\Omega})} \le \sqrt{2}\,|c(t)|_1 \le \sqrt{2}\rho.\end{aligned}$$

Now, by

$$|A_j(x,t,u_{m-1})| \le K_M(f) \sum_{i=j}^{N-1} \tfrac{1}{j!(i-j)!} M^{i-j} \equiv \bar{D}_j(M), \;\; j = \overline{0, N-1},$$

we have

$$\begin{aligned}&\left|\left\langle A_i(s,u_{m-1})(u(s))^i, w_j\right\rangle\right|\\ &\le \|A_i(s,u_{m-1})\|\, \|u(t)\|^i_{C^0(\overline{\Omega})}\, \|w_j\| \le \bar{D}_i(M)(\sqrt{2}\rho)^i, \;\; i = \overline{0, N-1}.\end{aligned}$$

It follows that

$$\begin{aligned}|F_j[c](t)| &\le |q_j(t)| + \lambda_k \tilde{K}(\mu) e^{|\lambda| T} \textstyle\int_0^t d\tau \int_0^\tau |c(s)|_1\, ds\\ &\quad + \frac{1}{2}\left(T_m^{(k)}\right)^2 e^{|\lambda|T} \textstyle\sum_{i=1}^{N-1} \bar{D}_i(M)(\sqrt{2}\rho)^i\\ &\le |q_j(t)| + e^{|\lambda|T}\left(\lambda_k \tilde{K}(\mu)\rho + \frac{1}{2}\textstyle\sum_{i=1}^{N-1} \bar{D}_i(M)(\sqrt{2}\rho)^i\right)\left(T_m^{(k)}\right)^2.\end{aligned}$$

So

$$|F[c](t)|_1 \le \|q\|_T + \overline{D}_\rho \left(T_m^{(k)}\right)^2, \;\; \forall t \in [0, T_m^{(k)}],$$

in which

$$\|q\|_T = \sup_{t\in[0,T]} |q(t)|_1, \;\; \overline{D}_\rho = k e^{|\lambda|T}\left(\lambda_k \tilde{K}(\mu)\rho + \tfrac{1}{2}\textstyle\sum_{i=1}^{N-1} \bar{D}_i(M)(\sqrt{2}\rho)^i\right).$$

Consequently

$$\|F[c]\|_X \le \|q\|_T + \overline{D}_\rho \left(T_m^{(k)}\right)^2. \tag{2.14}$$

Next, with $c = (c_1, ..., c_k) \in S, d = (d_1, ..., d_k) \in S$ and $t \in [0, T_m^{(k)}]$, by considering

$$u(t) = \sum_{j=1}^k c_j(t) w_j, \;\;\; v(t) = \sum_{j=1}^k d_j(t) w_j,$$

$$
\begin{aligned}
&|F_j[c](t) - F_j[d](t)| \\
&\le e^{|\lambda|T} \textstyle\sum_{i=1}^{k} \int_0^t d\tau \int_0^\tau \langle \mu(s) w_{ix}, w_{jx} \rangle \, |c_i(s) - d_i(s)| \, ds \\
&\quad + e^{|\lambda|T} \textstyle\sum_{i=1}^{N-1} \int_0^t d\tau \int_0^\tau \left| \left\langle A_i(s, u_{m-1}) \left[u^i(s) - v^i(s) \right], w_j \right\rangle \right| ds \\
&\le 2\lambda_k \tilde{K}(\mu) e^{|\lambda|T} \int_0^t d\tau \int_0^\tau |c(s) - d(s)|_1 \, ds \\
&\quad + e^{|\lambda|T} \textstyle\sum_{i=1}^{N-1} \int_0^t d\tau \int_0^\tau \|A_i(s, u_{m-1})\| \left\| u^i(s) - v^i(s) \right\|_{C^0(\overline{\Omega})} ds.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
\left\| u^i(s) - v^i(s) \right\|_{C^0(\overline{\Omega})} &\le \textstyle\sum_{j=0}^{i-1} \|u(s)\|_{C^0(\overline{\Omega})}^j \|v(s)\|_{C^0(\overline{\Omega})}^{i-j-1} \|u(s) - v(s)\|_{C^0(\overline{\Omega})} \\
&\le \textstyle\sum_{j=0}^{i-1} \left(\sqrt{2}\rho\right)^j \left(\sqrt{2}\rho\right)^{i-j-1} \sqrt{2} |c(s) - d(s)|_1 \\
&\le i \left(\sqrt{2}\rho\right)^{i-1} \sqrt{2} \, \|c - d\|_X \, .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&|F[c](t) - F[d](t)|_1 \\
&\le 2k\lambda_k \tilde{K}(\mu) e^{|\lambda|T} \int_0^t d\tau \int_0^\tau |c(s) - d(s)|_1 \, ds \\
&\quad + k e^{|\lambda|T} \textstyle\sum_{i=1}^{N-1} \bar{D}_i(M) i \left(\sqrt{2}\rho\right)^{i-1} \sqrt{2} \, \|c - d\|_X \int_0^t d\tau \int_0^\tau ds \\
&\le \frac{k}{2} \left(T_m^{(k)}\right)^2 e^{|\lambda|T} \left[2\lambda_k \tilde{K}(\mu) + \textstyle\sum_{i=1}^{N-1} \bar{D}_i(M) \rho^{i-1} \sqrt{2^i} \right] \|c - d\|_X \\
&\le \zeta_\rho \left(T_m^{(k)}\right)^2 \|c - d\|_X \, ,
\end{aligned}
$$

where

$$
\zeta_\rho = \frac{k}{2} e^{|\lambda|T} \left[2\lambda_k \tilde{K}(\mu) + \textstyle\sum_{i=1}^{N-1} \bar{D}_i(M) \rho^{i-1} \sqrt{2^i} \right],
$$

it leads to

$$
\|F[c] - F[d]\|_X \le \zeta_\rho \left(T_m^{(k)}\right)^2 \|c - d\|_X \, . \tag{2.15}
$$

By choosing $\rho > \|q\|_T$ and $T_m^{(k)} \in (0, T]$ with the properties

$$
0 < \overline{D}_\rho \left(T_m^{(k)}\right)^2 \le \rho - \|q\|_T \text{ and } \zeta_\rho \left(T_m^{(k)}\right)^2 < 1, \tag{2.16}
$$

thanks to (2.14), (2.15) and (2.16), it is easy to see that $F : S \longrightarrow S$ is contractive. Then, system (2.11) has a unique solution $c_{mj}^{(k)}(t)$ in $[0, T_m^{(k)}]$. We deduce that system (2.6) has a unique solution $u_m^{(k)}(t)$ in $[0, T_m^{(k)}]$.

The following estimates allow one to take $T_m^{(k)} = T$ independent of m and k.

(ii) *Estimates.*

Multiply $(2.6)_1$ by $\dot{c}_m^{(k)}(t)$ and sum for $j = 1, ..., k$, and then integrating with respect to the time variable from 0 to t yields

$$\begin{aligned} p_m^{(k)}(t) = & p_m^{(k)}(0) - 2\lambda \int_0^t \left\| \dot{u}_m^{(k)}(s) \right\|^2 ds + \int_0^t ds \int_0^1 \mu'(x,s) \left| u_{mx}^{(k)}(x,s) \right|^2 dx \\ & + 2 \int_0^t \langle \Phi_m^{(k)}(s), \dot{u}_m^{(k)}(s) \rangle ds, \end{aligned} \tag{2.17}$$

where

$$p_m^{(k)}(t) = \left\| \dot{u}_m^{(k)}(t) \right\|^2 + \left\| \sqrt{\mu(t)} u_{mx}^{(k)}(t) \right\|^2.$$

By replacing w_j in $(2.6)_1$ by $-w_{jxx}$, we obtain that

$$\begin{aligned} & \left\langle \ddot{u}_{mx}^{(k)}(t), w_{jx} \right\rangle + \left\langle \left(\mu(t) u_{mx}^{(k)}(t) \right)_x, w_{jxx} \right\rangle + \lambda \left\langle \dot{u}_{mx}^{(k)}(t), w_{jx} \right\rangle \\ & = \left\langle \Phi_{mx}^{(k)}(t), w_{jx} \right\rangle, \quad 1 \le j \le k, \end{aligned}$$

similar to $(2.6)_1$, it gives

$$\begin{aligned} & q_m^{(k)}(t) \\ & = q_m^{(k)}(0) - 2\lambda \int_0^t \left\| \dot{u}_{mx}^{(k)}(s) \right\|^2 ds + \int_0^t ds \int_0^1 \mu'(x,s) \left| u_{mxx}^{(k)}(x,s) \right|^2 dx \\ & \quad -2 \int_0^t \left\langle \mu_x(s) u_{mx}^{(k)}(s), \dot{u}_{mxx}^{(k)}(s) \right\rangle ds + 2 \int_0^t \left\langle \Phi_{mx}^{(k)}(s), \dot{u}_{mx}^{(k)}(s) \right\rangle ds, \end{aligned} \tag{2.18}$$

where

$$q_m^{(k)}(t) = \left\| \dot{u}_{mx}^{(k)}(t) \right\|^2 + \left\| \sqrt{\mu(t)} u_{mxx}^{(k)}(t) \right\|^2.$$

Eq. (2.6) can be rewritten as follows

$$\begin{aligned} & \left\langle \ddot{u}_m^{(k)}(t), w_j \right\rangle - \left\langle \frac{\partial}{\partial x} \left(\mu(t) u_{mx}^{(k)}(t) \right), w_j \right\rangle + \lambda \left\langle \dot{u}_m^{(k)}(t), w_j \right\rangle \\ & = \left\langle \Phi_m^{(k)}(t), w_j \right\rangle, \quad 1 \le j \le k. \end{aligned} \tag{2.19}$$

Hence, it follows after replacing w_j with $\ddot{u}_m^{(k)}(t)$ and integrating that

$$\begin{aligned} & \int_0^t ||\ddot{u}_m^{(k)}(s)||^2 ds \\ & \le 3 \int_0^t \left\| \frac{\partial}{\partial x} \left(\mu_m(s) u_{mx}^{(k)}(s) \right) \right\|^2 ds + 3\lambda^2 \int_0^t ||\dot{u}_m^{(k)}(s)||^2 ds \\ & \quad +3 \int_0^t ||\Phi_m^{(k)}(s)||^2 ds. \end{aligned} \tag{2.20}$$

Combining (2.17), (2.18) and (2.20) lead to

$$\begin{aligned}
&S_m^{(k)}(t) \\
&= p_m^{(k)}(t) + q_m^{(k)}(t) + \int_0^t ||\ddot{u}_m^{(k)}(s)||^2 ds \\
&= S_m^{(k)}(0) + \int_0^t ds \int_0^1 \mu'(x,s) \left(\left| u_{mx}^{(k)}(x,s) \right|^2 + \left| u_{mxx}^{(k)}(x,s) \right|^2 \right) dx \\
&\quad +3\lambda^2 \int_0^t \left\| \dot{u}_m^{(k)}(s) \right\|^2 ds - 2\lambda \int_0^t \left(\left\| \dot{u}_m^{(k)}(s) \right\|^2 + \left\| \dot{u}_{mx}^{(k)}(s) \right\|^2 \right) ds \\
&\quad -2 \int_0^t \left\langle \mu_x(s) u_{mx}^{(k)}(s), \dot{u}_{mxx}^{(k)}(s) \right\rangle ds + 3 \int_0^t \left\| \frac{\partial}{\partial x} \left(\mu_m(s) u_{mx}^{(k)}(s) \right) \right\|^2 ds \\
&\quad +3 \int_0^t ||\Phi_m^{(k)}(s)||^2 ds + 2 \int_0^t \langle \Phi_m^{(k)}(s), \dot{u}_m^{(k)}(s) \rangle ds \\
&\quad +2 \int_0^t \langle \Phi_{mx}^{(k)}(s), \dot{u}_{mx}^{(k)}(s) \rangle ds \\
&\equiv S_m^{(k)}(0) + \sum_{j=1}^8 I_j .
\end{aligned} \tag{2.21}$$

We shall estimate, respectively, the following integrals and $S_m^{(k)}(0)$ on the right-hand side of (2.21).

First integral I_1 :

$$S_m^{(k)}(t) \geq \mu_0 \left(\left\| u_{mx}^{(k)}(t) \right\|^2 + \left\| u_{mxx}^{(k)}(t) \right\|^2 \right). \tag{2.22}$$

By (2.22), we have

$$\begin{aligned}
I_1 &= \int_0^t ds \int_0^1 \mu'(x,s) \left(\left| u_{mx}^{(k)}(x,s) \right|^2 + \left| u_{mxx}^{(k)}(x,s) \right|^2 \right) dx \\
&\leq \frac{1}{\mu_0} \tilde{K}(\mu) \int_0^t \left(\left\| \sqrt{\mu(s)} u_{mx}^{(k)}(s) \right\|^2 + \left\| \sqrt{\mu(s)} u_{mxx}^{(k)}(s) \right\|^2 \right) ds \\
&\leq \frac{1}{\mu_0} \tilde{K}(\mu) \int_0^t S_m^{(k)}(s) ds.
\end{aligned} \tag{2.23}$$

Second integral I_2 :

$$I_2 = 3\lambda^2 \int_0^t \left\| \dot{u}_m^{(k)}(s) \right\|^2 ds \leq 3\lambda^2 \int_0^t S_m^{(k)}(s) ds. \tag{2.24}$$

Third integral I_3 :

$$I_3 = -2\lambda \int_0^t \left(\left\| \dot{u}_m^{(k)}(s) \right\|^2 + \left\| \dot{u}_{mx}^{(k)}(s) \right\|^2 \right) ds \leq 2 \left| \lambda \right| \int_0^t S_m^{(k)}(s) ds. \tag{2.25}$$

Fourth integral I_4 :

$$\begin{aligned} I_4 &= -2\int_0^t \left\langle \mu_x(s)u_{mx}^{(k)}(s), \dot{u}_{mxx}^{(k)}(s)\right\rangle ds \\ &= 2\left\langle \mu_x(0)u_{mx}^{(k)}(0), u_{mxx}^{(k)}(0)\right\rangle - 2\left\langle \mu_x(t)u_{mx}^{(k)}(t), u_{mxx}^{(k)}(t)\right\rangle \\ &\quad +2\int_0^t \left\langle \frac{\partial}{\partial s}\left(\mu_x(s)u_{mx}^{(k)}(s)\right), u_{mxx}^{(k)}(s)\right\rangle ds \\ &= 2\left\langle \mu_x(0)\tilde{u}_{0kx}, \tilde{u}_{0kxx}\right\rangle + I_4^{(1)} + I_4^{(2)}. \end{aligned} \tag{2.26}$$

Estimate $I_4^{(1)}$:

$$\begin{aligned} I_4^{(1)} &= -2\left\langle \mu_x(t)u_{mx}^{(k)}(t), u_{mxx}^{(k)}(t)\right\rangle \leq 2\tilde{K}(\mu)\left\|u_{mx}^{(k)}(t)\right\|\left\|u_{mxx}^{(k)}(t)\right\| \\ &\leq \frac{1}{2}\left\|u_{mxx}^{(k)}(t)\right\|^2 + 2\tilde{K}^2(\mu)\left\|u_{mx}^{(k)}(t)\right\|^2 \\ &\leq \frac{1}{2}\left\|u_{mxx}^{(k)}(t)\right\|^2 + 2\tilde{K}^2(\mu)\left[\left\|\tilde{u}_{0kx}\right\| + \int_0^t \left\|\dot{u}_{mx}^{(k)}(s)\right\| ds\right]^2 \\ &\leq \frac{1}{2}\left\|u_{mxx}^{(k)}(t)\right\|^2 + 2\tilde{K}^2(\mu)\left[2\left\|\tilde{u}_{0kx}\right\|^2 + 2t\int_0^t \left\|\dot{u}_{mx}^{(k)}(s)\right\|^2 ds\right] \\ &\leq \frac{1}{2}S_m^{(k)}(t) + 2\tilde{K}^2(\mu)\left[2\left\|\tilde{u}_{0kx}\right\|^2 + 2t\int_0^t S_m^{(k)}(s)ds\right] \\ &\leq 4\tilde{K}^2(\mu)\left\|\tilde{u}_{0kx}\right\|^2 + \frac{1}{2}S_m^{(k)}(t) + 4T^*\tilde{K}^2(\mu)\int_0^t S_m^{(k)}(s)ds. \end{aligned} \tag{2.27}$$

Estimate $I_4^{(2)}$:

$$\begin{aligned} I_4^{(2)} &= 2\int_0^t \left\langle \frac{\partial}{\partial s}\left(\mu_x(s)u_{mx}^{(k)}(s)\right), u_{mxx}^{(k)}(s)\right\rangle ds \\ &= 2\int_0^t \left\langle \dot{\mu}_x(s)u_{mx}^{(k)}(s) + \mu_x(s)\dot{u}_{mx}^{(k)}(s), u_{mxx}^{(k)}(s)\right\rangle ds \\ &\leq 2\tilde{K}(\mu)\int_0^t \left(\left\|u_{mx}^{(k)}(s)\right\| + \left\|\dot{u}_{mx}^{(k)}(s)\right\|\right)\left\|u_{mxx}^{(k)}(s)\right\| ds \\ &\leq 2\tilde{K}(\mu)\int_0^t \left(\sqrt{\frac{S_m^{(k)}(s)}{\mu_0}} + \sqrt{S_m^{(k)}(s)}\right)\sqrt{\frac{S_m^{(k)}(s)}{\mu_0}} ds \\ &= 2\tilde{K}(\mu)\frac{1+\sqrt{\mu_0}}{\mu_0}\int_0^t S_m^{(k)}(s)ds. \end{aligned} \tag{2.28}$$

Hence, we deduce from (2.26)-(2.28) that

$$\begin{aligned} I_4 &= 2\left\langle \mu_x(0)\tilde{u}_{0kx}, \tilde{u}_{0kxx}\right\rangle + I_4^{(1)} + I_4^{(2)} \\ &\leq 2\left\langle \mu_x(0)\tilde{u}_{0kx}, \tilde{u}_{0kxx}\right\rangle + 4\tilde{K}^2(\mu)\left\|\tilde{u}_{0kx}\right\|^2 + \frac{1}{2}S_m^{(k)}(t) \\ &\quad +2\left(2T^*\tilde{K}(\mu) + \frac{1+\sqrt{\mu_0}}{\mu_0}\right)\tilde{K}(\mu)\int_0^t S_m^{(k)}(s)ds. \end{aligned} \tag{2.29}$$

Fifth integral I_5 :

$$
\begin{aligned}
I_5 &= 3\int_0^t \left\| \frac{\partial}{\partial x}\left(\mu_m(s)u_{mx}^{(k)}(s)\right)\right\|^2 ds \\
&\le 3\tilde{K}^2(\mu)\int_0^t \left[\left\|u_{mx}^{(k)}(s)\right\| + \left\|u_{mxx}^{(k)}(s)\right\|\right]^2 ds \\
&\le \frac{6}{\mu_0}\tilde{K}^2(\mu)\int_0^t S_m^{(k)}(s)ds.
\end{aligned}
\tag{2.30}
$$

The following properties of $\Phi_m^{(k)}(t), \Phi_{mx}^{(k)}(t)$ are useful to continue estimates

$$
\begin{aligned}
&\text{(i)}\ \left\|\Phi_m^{(k)}(t)\right\| \le \bar{c}_M\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right], \\
&\text{(ii)}\ \left\|\Phi_{mx}^{(k)}(t)\right\| \le \bar{c}_M\left[1+\left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right],
\end{aligned}
\tag{2.31}
$$

where $\bar{c}_M = \sum_{i=0}^{N-1}\tilde{c}_i$, with

$$
\tilde{c}_i = \begin{cases} \left(1+M+(M+N)\sum_{i=1}^{N-1}\frac{1}{i!}2^{i-1}M^i\right)K_M(f), & i=0, \\ (M+N)K_M(f)\frac{1}{i!}\frac{2^{i-1}}{\sqrt{\mu_0^i}}, & i=1,2,...,N-1. \end{cases}
\tag{2.32}
$$

Indeed, use inequalities $(a+b)^p \le 2^{p-1}(a^p+b^p)$, for all $a,b>0, p\ge 1$, and

$$
s^i \le 1+s^q, \ \ \forall\, s\ge 0, \ \ \forall\, i,q, \ \ 0\le i\le q,
\tag{2.33}
$$

we have

$$
\begin{aligned}
\left|\Phi_m^{(k)}(x,t)\right| &\le \sum_{i=0}^{N-1}\left|\frac{1}{i!}\frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m^{(k)}-u_{m-1})^i\right| \\
&\le K_M(f)\left[1+\sum_{i=1}^{N-1}\frac{1}{i!}\left(\left|u_m^{(k)}\right| + |u_{m-1}|\right)^i\right] \\
&\le K_M(f)\left[1+\sum_{i=1}^{N-1}\frac{1}{i!}\left(\left\|u_{mx}^{(k)}(t)\right\| + M\right)^i\right] \\
&\le K_M(f)\left[1+\sum_{i=1}^{N-1}\frac{1}{i!}\left(\sqrt{\frac{S_m^{(k)}(t)}{\mu_0}} + M\right)^i\right] \\
&\le K_M(f)\left[1+\sum_{i=1}^{N-1}\frac{2^{i-1}}{i!}\left(\left(\sqrt{\frac{S_m^{(k)}(t)}{\mu_0}}\right)^i + M^i\right)\right] \\
&= K_M(f)\left[1+\sum_{i=1}^{N-1}\frac{2^{i-1}}{i!}M^i + \sum_{i=1}^{N-1}\frac{1}{i!}\frac{2^{i-1}}{\sqrt{\mu_0^i}}\left(\sqrt{S_m^{(k)}(t)}\right)^i\right]
\end{aligned}
$$

$$\begin{aligned} &\leq \sum_{i=0}^{N-1} \tilde{c}_i \left(\sqrt{S_m^{(k)}(t)} \right)^i \leq \sum_{i=0}^{N-1} \tilde{c}_i \left[1 + \left(\sqrt{S_m^{(k)}(t)} \right)^{N-1} \right] \\ &= \bar{c}_M \left[1 + \left(\sqrt{S_m^{(k)}(t)} \right)^{N-1} \right]. \end{aligned} \tag{2.34}$$

Hence, (2.31(i)) follows. We also have

$$\begin{aligned} &\left| \Phi_{mx}^{(k)}(x,t) \right| \\ &\leq \left| \frac{\partial f}{\partial x}(x,t,u_{m-1}) + \frac{\partial f}{\partial u}(x,t,u_{m-1}) \nabla u_{m-1} \right| \\ &\quad + \sum_{i=1}^{N-1} \left| \left[\frac{1}{i!} \frac{\partial^{i+1} f}{\partial u^i \partial x}(x,t,u_{m-1}) + \frac{1}{i!} \frac{\partial^{i+1} f}{\partial u^{i+1}}(x,t,u_{m-1}) \nabla u_{m-1} \right] (u_m^{(k)} - u_{m-1})^i \right| \\ &\quad + \sum_{i=1}^{N-1} \left| \frac{1}{i!} \frac{\partial^i f}{\partial u^i}(x,t,u_{m-1}) i (u_m^{(k)} - u_{m-1})^{i-1} (\nabla u_m^{(k)} - \nabla u_{m-1}) \right| \\ &\leq (1+M) K_M(f) + (1+M) K_M(f) \sum_{i=1}^{N-1} \frac{1}{i!} \left(\sqrt{\frac{S_m^{(k)}(t)}{\mu_0}} + M \right)^i \\ &\quad + K_M(f) \sum_{i=1}^{N-1} \frac{i}{i!} \left(\sqrt{\frac{S_m^{(k)}(t)}{\mu_0}} + M \right)^{i-1} \left(\left| \nabla u_m^{(k)} \right| + M \right). \end{aligned} \tag{2.35}$$

Hence

$$\begin{aligned} &\left\| \Phi_{mx}^{(k)}(t) \right\| \\ &\leq (1+M) K_M(f) + (1+M) K_M(f) \sum_{i=1}^{N-1} \frac{1}{i!} \left(\sqrt{\frac{S_m^{(k)}(t)}{\mu_0}} + M \right)^i \\ &\quad + K_M(f) \sum_{i=1}^{N-1} \frac{i}{i!} \left(\sqrt{\frac{S_m^{(k)}(t)}{\mu_0}} + M \right)^i \\ &\leq (1+M) K_M(f) + (1+M+N-1) K_M(f) \sum_{i=1}^{N-1} \frac{1}{i!} \left(\sqrt{\frac{S_m^{(k)}(t)}{\mu_0}} + M \right)^i \\ &\leq (1+M) K_M(f) + (M+N) K_M(f) \sum_{i=1}^{N-1} \frac{1}{i!} 2^{i-1} \left[\left(\sqrt{\frac{S_m^{(k)}(t)}{\mu_0}} \right)^i + M^i \right] \\ &\leq \left(1 + M + (M+N) \sum_{i=1}^{N-1} \frac{1}{i!} 2^{i-1} M^i \right) K_M(f) \\ &\quad + (M+N) K_M(f) \sum_{i=1}^{N-1} \frac{1}{i!} \frac{2^{i-1}}{\sqrt{\mu_0^i}} \left(\sqrt{S_m^{(k)}(t)} \right)^i \end{aligned}$$

$$= \sum_{i=0}^{N-1} \tilde{c}_i \left(\sqrt{S_m^{(k)}(t)}\right)^i \leq \sum_{i=0}^{N-1} \tilde{c}_i \left[1 + \left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right]$$
$$= \bar{c}_M \left[1 + \left(\sqrt{S_m^{(k)}(t)}\right)^{N-1}\right]. \tag{2.36}$$

Now, we estimate all intergal I_6, I_7, I_8. Using the properties of $\Phi_m^{(k)}(t)$, $\Phi_{mx}^{(k)}(t)$ as in (2.31), we obtain

$$I_6 = 3\int_0^t \left\|\Phi_m^{(k)}(s)\right\|^2 ds \leq 3\int_0^t \bar{c}_M^2 \left[1 + \left(\sqrt{S_m^{(k)}(s)}\right)^{N-1}\right]^2 ds$$
$$\leq 6\bar{c}_M^2 \int_0^t \left[1 + \left(\sqrt{S_m^{(k)}(s)}\right)^{2N-2}\right] ds$$
$$\leq 12\bar{c}_M^2 \left[T + \int_0^t \left(S_m^{(k)}(s)\right)^N ds\right]. \tag{2.37}$$

Similarly,

$$I_7 \leq 4\bar{c}_M \left[T + \int_0^t \left(S_m^{(k)}(s)\right)^N ds\right], \tag{2.38}$$

and

$$I_8 \leq 4\bar{c}_M \left[T + \int_0^t \left(S_m^{(k)}(s)\right)^N ds\right]. \tag{2.39}$$

Combining (2.21), (2.23)-(2.25), (2.29), (2.30), (2.37)-(2.39), we obtain that

$$S_m^{(k)}(t) \leq 2S_m^{(k)}(0) + 4\langle \mu_x(0)\tilde{u}_{0kx}, \tilde{u}_{0kxx}\rangle + 8\tilde{K}^2(\mu)\|\tilde{u}_{0kx}\|^2$$
$$+C_1(M)T + C_1(M)\int_0^t \left(S_m^{(k)}(s)\right)^N ds, \tag{2.40}$$

where

$$C_1(M) = 2\left(\frac{6}{\mu_0} + 4T^*\right)\tilde{K}^2(\mu) + \frac{2}{\mu_0}\left(3 + 2\sqrt{\mu_0}\right)\tilde{K}(\mu)$$
$$+6\lambda^2 + 4|\lambda| + 8\left(3\bar{c}_M^2 + 2\bar{c}_M\right). \tag{2.41}$$

By means of the convergences (2.7) we can deduce the existence of a constant $M > 0$ independent of k and m such that

$$2S_m^{(k)}(0) + 4\langle \mu_x(0)\tilde{u}_{0kx}, \tilde{u}_{0kxx}\rangle + 8\tilde{K}^2(\mu)\|\tilde{u}_{0kx}\|^2 \leq \frac{M^2}{4}, \quad \forall\, m, k \in \mathbb{N}. \tag{2.42}$$

Finally, it follows from (2.40), (2.42) that

$$S_m^{(k)}(t) \leq \frac{M^2}{4} + C_1(M)T + C_1(M)\int_0^t \left(S_m^{(k)}(s)\right)^N ds,$$
$$0 \leq t \leq T_m^{(k)} \leq T. \tag{2.43}$$

Then, by solving a nonlinear Volterra integral inequality (2.43) (based on the methods in [3]), the following lemma is proved.

Lemma 2.2. *There exists a constant $T>0$ independent of k and m such that*

$$S_m^{(k)}(t) \le M^2, \;\; \forall\, t\in[0,T], \;\; \forall\, k,m\in\mathbb{N}. \tag{2.44}$$

By Lemma 2.2, we can take constant $T_m^{(k)}=T$ for all m and k. Therefore, we have

$$u_m^{(k)} \in W(M,T), \text{ for all } m \text{ and } k\in\mathbb{N}. \tag{2.45}$$

(iii) *Convergence.*

Thanks to (2.45), there exist a subsequence $\{u_m^{(k_j)}\}$ of $\{u_m^{(k)}\}$ such that

$$\begin{cases} u_m^{(k)} \to u_m & \text{in } \; L^\infty(0,T;H_0^1\cap H^2) \text{ weakly*}, \\ \dot{u}_m^{(k)} \to u_m' & \text{in } \; L^\infty(0,T;H_0^1) \text{ weakly*}, \\ \ddot{u}_m^{(k)} \to u_m'' & \text{in } \; L^2(0,T;L^2) \text{ weakly}, \\ u_m \in W(M,T). \end{cases} \tag{2.46}$$

By the compactness lemma of Lions ([2], p. 57) and applying the theorem's Fischer - Riesz, from (2.46), one has a subsequence of $\{u_m^{(k)}\}$, denoted by the same symbol satisfying

$$\begin{cases} u_m^{(k)} \to u_m & \text{strongly in} \;\; L^2(0,T;H_0^1) \;\; \text{and a.e. in } Q_T, \\ \dot{u}_m^{(k)} \to u_m' & \text{strongly in} \;\; L^2(0,T;L^2) \;\; \text{and a.e. in } Q_T. \end{cases} \tag{2.47}$$

On the other hand, by $L^\infty(0,T;H_0^1\cap H^2)\hookrightarrow L^\infty(Q_T)$ and using the inequality

$$\left|a^j-b^j\right| \le jM^{j-1}\left|a-b\right|, \;\; \forall\, a,b\in[-M,M], \; \forall\, M>0, \; \forall\, j\in\mathbb{N}, \tag{2.48}$$

we deduce from (2.45) that

$$\left|(u_m^{(k)})^j-(u_m)^j\right| \le jM^{j-1}\left|u_m^{(k)}-u_m\right|, \; j=\overline{0,N-1}. \tag{2.49}$$

Therefore, (2.47) and (2.49) give

$$(u_m^{(k)})^j \to (u_m)^j \quad \text{strongly in} \quad L^2(Q_T). \tag{2.50}$$

We note that

$$\begin{aligned} &\left\|\Phi_m^{(k)}-\Phi_m\right\|_{L^2(Q_T)} \\ &\le \textstyle\sum_{j=0}^{N-1}\left\|A_j(\cdot,\cdot,u_{m-1})\right\|_{L^\infty(Q_T)}\left\|(u_m^{(k)})^j-(u_m)^j\right\|_{L^2(Q_T)} \\ &\le K_M(f)\textstyle\sum_{j=0}^{N-1}\sum_{i=j}^{N-1}\frac{M^{i-j}}{j!(i-j)!}\left\|(u_m^{(k)})^j-(u_m)^j\right\|_{L^2(Q_T)}, \end{aligned} \tag{2.51}$$

so (2.50) leads to

$$\Phi_m^{(k)} \to \Phi_m \quad \text{strongly in} \quad L^2(Q_T). \tag{2.52}$$

Passing to limit in (2.6), (2.7), we have u_m satisfying (2.3), (2.4) in $L^2(0,T)$. On the other hand, it follows from $(2.3)_1$ and $(2.46)_4$ that

$$u_m'' = \frac{\partial}{\partial x}\left(\mu(x,t)\frac{\partial u_m}{\partial x}\right) - \lambda u_m' + \Phi_m \in L^\infty(0,T;L^2). \tag{2.53}$$

Hence $u_m \in W_1(M,T)$ and Theorem 2.1 is proved. □

Next, in order to obtain the main result in the following theorem, we put

$$W_1(T) = \{v \in L^\infty(0,T;H_0^1) : v' \in L^\infty(0,T;L^2)\},$$

then $W_1(T)$ is a Banach space with respect to the norm

$$\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;H_0^1)} + \|v'\|_{L^\infty(0,T;L^2)}\,. \tag{2.54}$$

Theorem 2.3. *Let (H_1)-(H_3) hold. Then, there exist constants $M > 0$ and $T > 0$ such that*

(i) *Problem (1.1)-(1.3) has a unique weak solution $u \in W_1(M,T)$.*

(ii) *The recurrent sequence $\{u_m\}$, defined by (2.3) and (2.4), converges at a rate of order N to the solution u strongly in the space $W_1(T)$ in the sense*

$$\|u_m - u\|_{W_1(T)} \le C\,\|u_{m-1} - u\|_{W_1(T)}^N\,, \tag{2.55}$$

for all $m \ge 1$, where C is a suitable constant. On the other hand, the estimate is fulfilled

$$\|u_m - u\|_{W_1(T)} \le C_T \beta^{N^m}, \quad \text{for all } m \in \mathbb{N}, \tag{2.56}$$

where C_T and $0 < \beta < 1$ are the constants depending only on T.

Proof. Existence. We can prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Indeed, let $w_m = u_{m+1} - u_m$. Then w_m satisfies the variational problem

$$\begin{cases} \langle w_m''(t), w\rangle + \langle \mu(t) w_{mx}(t), w_x\rangle + \lambda\langle w_m'(t), w\rangle \\ = \langle \Phi_{m+1}(t) - \Phi_m(t), w\rangle\,,\ \forall w \in H_0^1, \\ w_m(0) = w_m'(0) = 0. \end{cases} \tag{2.57}$$

Taking $w = w_m'$ in (2.57), after integrating in t, we get

$$\begin{aligned} Z_m(t) \le 2\,|\lambda| \int_0^t \|w_m'(s)\|^2\,ds + \int_0^t ds \int_0^1 |\mu'(x,s)|\,w_{mx}^2(x,s)dx \\ +2\int_0^t \|\Phi_{m+1}(s) - \Phi_m(s)\|\,\|w_m'(s)\|\,ds, \end{aligned} \tag{2.58}$$

where

$$Z_m(t) = \|w_m'(t)\|^2 + \left\|\sqrt{\mu(t)}w_{mx}(t)\right\|^2. \tag{2.59}$$

It follows from (2.59) that

$$\begin{aligned}\int_0^t ds \int_0^1 |\mu'(x,s)|\, w_{mx}^2(x,s)dx &\le \tilde{K}(\mu) \int_0^t \|w_{mx}(s)\|^2 ds \\ &\le \tfrac{1}{\mu_0}\tilde{K}(\mu)\int_0^t Z_m(s)ds.\end{aligned} \tag{2.60}$$

Using Taylor's expansion of the function $f(x,t,u_m) = f(x,t,u_{m-1}+v_{m-1})$ around the point u_{m-1} up to order N, we obtain

$$\begin{aligned}&f(x,t,u_m) - f(x,t,u_{m-1}) \\ &= \sum_{i=1}^{N-1} \tfrac{1}{i!} D_3^i f(x,t,u_{m-1}) w_{m-1}^i + \tfrac{1}{N!} D_3^N f(x,t,\tilde{\lambda}_m) w_{m-1}^N,\end{aligned} \tag{2.61}$$

where $\tilde{\lambda}_m = \tilde{\lambda}_m(x,t) = u_{m-1} + \theta_1 (u_m - u_{m-1})$, $0<\theta_1<1$. Hence, it follows from (2.4) and (2.61) that

$$\begin{aligned}&\Phi_{m+1}(x,t) - \Phi_m(x,t) \\ &= \sum_{i=1}^{N-1} \tfrac{1}{i!} D_3^i f(x,t,u_m) w_m^i + \tfrac{1}{N!} D_3^N f(x,t,\tilde{\lambda}_m) w_{m-1}^N.\end{aligned} \tag{2.62}$$

So, we have

$$\begin{aligned}&\|\Phi_{m+1}(t) - \Phi_m(t)\| \\ &\le K_M(f) \sum_{i=1}^{N} \tfrac{1}{i!} \|w_{mx}(t)\|^i + \tfrac{1}{N!} K_M(f) \|w_{m-1\,x}(t)\|^N \\ &\le \zeta_T^{(1)} \sqrt{Z_m(t)} + \zeta_T^{(2)} ||w_{m-1}||_{W_1(T)}^N,\end{aligned} \tag{2.63}$$

where

$$\zeta_T^{(1)} = \tfrac{1}{\sqrt{\mu_0}} K_M(f) \sum_{i=1}^{N} \tfrac{1}{i!} M^{i-1}, \;\; \zeta_T^{(2)} = \tfrac{1}{N!} K_M(f). \tag{2.64}$$

Then we deduce from (2.58), (2.60) and (2.63) that

$$\begin{aligned}&Z_m(t) \\ &\le \left(2|\lambda| + \tfrac{1}{\mu_0}\tilde{K}(\mu)\right) \int_0^t Z_m(s)ds \\ &\quad +2\int_0^t \|\Phi_{m+1}(s) - \Phi_m(s)\|\, ||w_m'(s)||ds \\ &\le \left(2|\lambda| + \tfrac{1}{\mu_0}\tilde{K}(\mu)\right) \int_0^t Z_m(s)ds \\ &\quad +2\int_0^t \left[\zeta_T^{(1)}\sqrt{Z_m(s)} + \zeta_T^{(2)} ||w_{m-1}||_{W_1(T)}^N\right] \sqrt{Z_m(s)}ds \\ &\le \left(2|\lambda| + \tfrac{1}{\mu_0}\tilde{K}(\mu) + 2\zeta_T^{(1)}\right) \int_0^t Z_m(s)ds \\ &\quad +2\zeta_T^{(2)} \int_0^t ||w_{m-1}||_{W_1(T)}^N \sqrt{Z_m(s)}ds \\ &\le T\zeta_T^{(2)} ||w_{m-1}||_{W_1(T)}^{2N} + \left(2|\lambda| + \tfrac{1}{\mu_0}\tilde{K}(\mu) + 2\zeta_T^{(1)} + \zeta_T^{(2)}\right)\int_0^t Z_m(s)ds.\end{aligned} \tag{2.65}$$

By using Gronwall's lemma, (2.65) leads to

$$||w_m||_{W_1(T)} \le \mu_T ||w_{m-1}||_{W_1(T)}^N, \tag{2.66}$$

where $\mu_T = 2\sqrt{\zeta_T^{(2)} T \exp\left(T\left(2\left|\lambda\right| + \frac{1}{\mu_0}\tilde{K}(\mu) + 2\zeta_T^{(1)} + \zeta_T^{(2)}\right)\right)}$. It follows from (2.66) that, for all m and p,

$$||u_m - u_{m+p}||_{W_1(T)} \leq (1-\beta)^{-1} \left(\mu_T\right)^{\frac{-1}{N-1}} \beta^{N^m}. \tag{2.67}$$

Choosing T small enough such that $\beta = M\mu_T^{\frac{1}{N-1}} < 1$. It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then there exists $u \in W_1(T)$ such that

$$u_m \longrightarrow u \text{ strongly in } W_1(T). \tag{2.68}$$

Note that $u_m \in W_1(M,T)$, then there exists a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{cases} u_{m_j} \to u & \text{in } \; L^\infty(0,T;H_0^1 \cap H^2) \text{ weakly*}, \\ u'_{m_j} \to u' & \text{in } \; L^\infty(0,T;H_0^1) \text{ weakly*}, \\ u''_{m_j} \to u'' & \text{in } \; L^\infty(0,T;L^2) \text{ weakly}, \\ u \in W(M,T). \end{cases} \tag{2.69}$$

We have

$$\begin{aligned} &\|\Phi_m(\cdot,t) - f(\cdot,t,u(t))\| \\ &\leq \|f(\cdot,t,u_{m-1}) - f(\cdot,t,u(t))\| \\ &\quad + \left\|\sum_{i=1}^{N-1} \frac{1}{i!}\frac{\partial^i f}{\partial u^i}(x,t,u_{m-1})(u_m - u_{m-1})^i\right\| \\ &\leq K_M(f)\left\|u_{m-1} - u\right\|_{W_1(T)} + K_M(f)\sum_{i=1}^{N-1} \frac{1}{i!}\left\|u_m - u_{m-1}\right\|_{W_1(T)}^i . \end{aligned} \tag{2.70}$$

Hence, it implies from (2.68) and (2.70) that

$$\Phi_m(t) \to f(\cdot,t,u(t)) \text{ strongly in } L^\infty(0,T;L^2). \tag{2.71}$$

Finally, passing to limit in (2.3), (2.4) as $m = m_j \to \infty$, there exists $u \in W(M,T)$ satisfying the equation

$$\langle u''(t), w\rangle + \langle \mu(t)u_x(t), w_x\rangle + \lambda\langle u'(t), w\rangle = \langle f(\cdot,t,u(t)), w\rangle\,, \tag{2.72}$$

for all $w \in H_0^1$ and the initial conditions

$$u(0) = \tilde{u}_0,\; u'(0) = \tilde{u}_1. \tag{2.73}$$

Uniqueness. Applying a similar argument used in the proof of Theorem 2.1, $u \in W_1(M,T)$ is a unique local weak solution of Prob.(1.1)-(1.3).

Passing to the limit in (2.67) as $p \to +\infty$ for fixed m, we get (2.56). Also with a similar argument, (2.55) follows. Theorem 2.3 is proved completely. □

Remark 2.4. In order to construct a $N-$order iterative scheme, we need the condition $f \in C^N([0,1] \times \mathbb{R}_+ \times \mathbb{R})$. Then, we get a convergent sequence at a rate of order N to a local unique weak solution of problem and the existence follows. This condition of f can be relaxed if we only consider the existence of solutions, see [4], [5], [7].

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