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EXISTENCE OF POSITIVE SOLUTIONS FOR SYSTEMS OF SECOND ORDER SINGULAR IMPULSIVE STURM-LIOUVILLE BOUNDARY VALUE PROBLEM

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Abstract. By constructing a cone $K_1 \times K_2$, which is the Cartesian product of two cones in the space $C[0,1]$, using the fixed point index theorem in the $K_1 \times K_2$ and the first eigenvalue, we establish the existence of one or two positive solutions for systems of the impulsive Singular Sturm-Liouville boundary value problem. In particular, we give a number of corollaries and an example to demonstrate the applications of the developed theory.

1. Introduction

Impulsive Sturm-Liouville boundary value problems play a very important role in both theory and application, which have been widely studied by many authors (see [2], [5], [6], [8], [11], [13], [14], [16] and references therein). For example, Zhang and Liu [17] have established unique solution of initial value problems of nonlinear second order impulsive integral differential in Banach spaces. Sun and Zhang [11], have applied the fixed point index theorem and the first eigenvalue to establish the existence of positive solutions.

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Recently, in [9], Lin and Jiang studied the following second-order impulsive differential equation with no singularity

$$\begin{cases} u'' + f(t,u) = 0, & 0 < t < 1, \\ -\Delta u'_{t=t_k} = I_k(u(t_k)), & k = 1, 2, \cdots m, \\ u(0) = u(1) = 0, \end{cases}$$

and obtained two positive solutions by using the fixed point index theorems in cone.

Motivated by the work mentioned above, we study the positive solutions for systems of nonlinear singular boundary value problems for impulsive Sturm-Liouville differential equation:

$$\begin{cases} (p(t)u'(t))' + a_1(t)f_1(t,u,v) = 0, & t \in J', \\ (p(t)v'(t))' + a_2(t)f_2(t,v,u) = 0, & k = 1, 2, \cdots, m, \\ -\Delta u'|_{t=t_k} = I_{1,k}(u(t_k)), & -\Delta v'|_{t=t_k} = I_{2,k}(v(t_k)), \\ \Delta u|_{t=t_k} = \bar{I}_{1,k}(u(t_k)), & \Delta v|_{t=t_k} = \bar{I}_{2,k}(v(t_k)), \\ \alpha_1 u(0) - \beta_1 \lim_{t\to 0+} p(t)u'(t) = 0, & \alpha_1 v(0) - \beta_1 \lim_{t\to 0+} p(t)v'(t) = 0, \\ \alpha_2 u(1) + \beta_2 \lim_{t\to 1-} p(t)u'(t) = 0, & \alpha_2 v(1) + \beta_2 \lim_{t\to 1-} p(t)v'(t) = 0, \end{cases} \tag{1.1}$$

where $J = (0,1), 0 < t_1 < t_2 < \cdots < t_m < 1, J' = J \setminus \{t_1, t_2, \cdots, t_m\}, \bar{J} = [0,1], J_0 = (0, t_1], J_1 = (t_1, t_2] \cdots, J_M = (t_m, 1). I_k, \bar{I}_k \in C(\mathbb{R}^+, \mathbb{R}^+), \Delta u'|_{t=t_k} = u'(t_k^+) - u'(t_k^-), \Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-), u'(t_k^+), u(t_k^+), (u'(t_k^-), u(t_k^-))$ denote the right (left) limit of $u'(t)$ and $u(t)$ at $t = t_k$ respectively, $\Delta v'|_{t=t_k} = v'(t_k^+) - v'(t_k^-), \Delta v|_{t=t_k} = v(t_k^+) - v(t_k^-), v'(t_k^+), v(t_k^+), (v'(t_k^-), v(t_k^-))$ denote the right (left) limit of $v'(t)$ and $v(t)$ at $t = t_k$ respectively. $\alpha_i \geq 0, \beta_i \geq 0$, $f_i \in C(\overline{J} \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+), a_i(t) \in C(J, \mathbb{R}^+)(i = 1, 2)$, among $a_i(t)(i = 1, 2)$ is allowed to be singular at $t = 0$ or $t = 1$, $\mathbb{R}^+ = [0, +\infty), p(t) \in C([0,1], \mathbb{R}^+) \bigcap C^1(J, \mathbb{R}^+)$ and $\int_0^1 \frac{ds}{p(s)} < +\infty, \rho = \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 \int_0^1 \frac{ds}{p(s)} > 0$.

In recent years, many authors studied the existence of positive radial solutions for elliptic systems with no impulse and positive solutions for ordinary differential equations (see [1], [2], [6], [10], [12], [16], [18] and the references therein). Most of the methods used in the studies apply a fixed-point theorem of cone expansion and compression or the fixed-point index theory in cones. Recently, Cheng and Zhang [1] studied the following two-point boundary value problem for a system of nonlinear second-order ordinary differential equations with no singularity and impulse:

$$\begin{cases} -u''(t) = f_1(t, u(t)) + h_1(u(t), v(t)), & 0 < t < 1, \\ -v''(t) = f_2(t, v(t)) + h_2(u(t), v(t)), & 0 < t < 1, \\ u(0) = u(1) = v(0) = v(1), \end{cases} \tag{1.2}$$

where f_i and $h_i (i = 1, 2)$ are superlinear,$f_i \in C(\bar{I} \times \mathbb{R}^+, \mathbb{R}^+), h_i \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)(i = 1, 2), I = [0, 1]$. The authors obtained the existence of at least one positive solution by constructing a cone $K_1 \times K_2$, which is the Cartesian product of two cones in space $C[0, 1]$, and computing the fixed-point index in $K_1 \times K_2$. It is easy to see that system (1.2) contains neither a singularity nor an impulse.

In this paper, by constructing a cone $K \times K$, which is the Cartesian product of two cones in space $C[0, 1]$, and computing the fixed-point index in $K \times K$, under some conditions on $a_i(t)$ concerning the first eigenvalue corresponding to the relevant linear operator, we establish the existence of at least one or at least two positive solutions for the singular and impulsive differential system (1.1). It will be shown that our main results are the extension of those in ([1]-[18]).

The rest of this paper is organized as follows. In Section 2, we provide some preliminaries and establish several lemmas. In Section 3, the main results are formulated and proved. In Section 4, we give a number of corollaries and an example to demonstrate the application of the developed theory.

2. Preliminaries

Now we denote the Green's functions for the following boundary value problems

$$\begin{cases} (p(t)u'(t))' = 0, \quad 0 \le t \le 1, \\ \alpha_1 u(0) - \beta_1 \lim_{t \to 0+} p(t)u'(t) = 0, \\ \alpha_2 u(1) + \beta_2 \lim_{t \to 1-} p(t)u'(t) = 0, \end{cases}$$

by $G(t, s)$. It is well known that $G(t, s)$ can be written by

$$G(t, s) = \frac{1}{\rho} \begin{cases} (\beta_1 + \alpha_1 B(0, s)) (\beta_2 + \alpha_2 B(t, 1)), & 0 \le s \le t \le 1, \\ (\beta_1 + \alpha_1 B(0, t)) (\beta_2 + \alpha_2 B(s, 1)), & 0 \le t \le s \le 1, \end{cases} \tag{2.1}$$

where $B(t, s) = \int_t^s \frac{d\tau}{p(\tau)}, \quad \rho = \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 B(0, 1)$. It is easy to verify the following properties of $G(t, s)$:

(I) $G(t, s) \le G(s, s) \le \frac{1}{\rho} (\beta_1 + \alpha_1 B(0, 1)) (\beta_2 + \alpha_2 B(0, 1)) < +\infty$,

(II) $G(t, s) \ge \sigma G(s, s) > 0$, for any $t \in [a, b], s \in [0, 1]$, where $a \in (0, t_1], b \in [t_m, 1)$ and

$$0 < \sigma = \min \left\{ \frac{\beta_2 + \alpha_2 B(b, 1)}{\beta_2 + \alpha_2 B(0, 1)}, \frac{\beta_1 + \alpha_1 B(0, a)}{\beta_1 + \alpha_1 B(0, 1)} \right\} < 1. \tag{2.2}$$

We denote the first eigenvalue of

$$\begin{cases} -(p(t)\phi'(t))' = \lambda\phi(t)a_i(t), \\ \alpha_1\phi(0) - \beta_1 \lim_{t\to 0+} p(t)\phi'(t) = 0, \\ \alpha_2\phi(1) + \beta_2 \lim_{t\to 1-} p(t)\phi'(t) = 0, \end{cases}$$

by $\lambda_{i,1}$ and the corresponding eigenfunction by $\phi_i(t)$. It is well known that $\lambda_{i,1} > 0$ and $\phi_i(t)$ does not change sign in (0,1) and therefore, without loss of generality, we assume that $\phi_i(t) > 0$ for $0 < t < 1$ and $\|\phi_i\| = \max\limits_{0\le t\le 1} |\phi_i(t)| = 1$ $(i = 1, 2)$.

For convenience and simplicity in the following discussion, for any $y \in R^+$ and $i = 1, 2$, we denote:

$$f_{i,0}(y) = \liminf_{x\to 0^+} \min_{t\in[a,b]} \frac{f_i(t,x,y)}{x}, \qquad I_{i,0}(k) = \liminf_{x\to 0^+} \frac{I_{i,k}(x)}{x},$$

$$f_{i,\infty}(y) = \liminf_{x\to \infty} \min_{t\in[a,b]} \frac{f_i(t,x,y)}{x}, \qquad I_{i,\infty}(k) = \liminf_{x\to \infty} \frac{I_{i,k}(x)}{x},$$

$$f_i^{\infty}(y) = \limsup_{x\to \infty} \max_{t\in[a,b]} \frac{f_i(t,x,y)}{x}, \qquad I_i^{\infty}(k) = \limsup_{x\to \infty} \frac{I_{i,k}(x)}{x},$$

$$f_i^{0}(y) = \limsup_{x\to 0^+} \max_{t\in[a,b]} \frac{f_i(t,x,y)}{x}, \qquad I_i^{0}(k) = \limsup_{x\to 0^+} \frac{I_{i,k}(x)}{x},$$

$$\overline{I}_{i,0}(k) = \liminf_{x\to 0^+} \frac{\overline{I}_{i,k}(x)}{x}, \qquad \overline{I}_{i,\infty}(k) = \liminf_{x\to \infty} \frac{\overline{I}_{i,k}(x)}{x},$$

$$\overline{I}_i^{\infty}(k) = \limsup_{x\to \infty} \frac{\overline{I}_{i,k}(x)}{x}, \qquad \overline{I}_i^{0}(k) = \limsup_{x\to 0^+} \frac{\overline{I}_{i,k}(x)}{x}.$$

(H_1)

$$\inf_{y\in R^+} f_{i,0}(y) + \frac{\sigma \sum\limits_{k=1}^{m} (I_{i,0}(k)\phi_i(t_k) + \overline{I}_{i,0}(k)\phi_i'(t_k))p(t_k)}{\int_0^1 \phi_i(t)a_i(t)dt} > \lambda_{i,1},$$

$$\inf_{y\in R^+} f_{i,\infty}(y) + \frac{\sigma \sum\limits_{k=1}^{m} (I_{i,\infty}(k)\phi_i(t_k) + \overline{I}_{i,\infty}(k)\phi_i'(t_k))p(t_k)}{\int_0^1 \phi_i(t)a_i(t)dt} > \lambda_{i,1}.$$

(H_2)

$$\sup_{y\in R^+} f_i^{0}(y) + \frac{\sum\limits_{k=1}^{m} (I_i^{0}(k)\phi_i(t_k) + \overline{I}_i^{0}(k)\phi_i'(t_k))p(t_k)}{\sigma \int_0^1 \phi_i(t)a_i(t)dt} < \lambda_{i,1},$$

$$\sup_{y\in R^+} f_i^{\infty}(y) + \frac{\sum\limits_{k=1}^{m}(I_i^{\infty}(k)\phi_i(t_k) + \overline{I}_i^{\infty}(k)\phi_i'(t_k))p(t_k)}{\sigma\int_0^1 \phi_i(t)a_i(t)dt} < \lambda_{i,1}.$$

(H_3) There exist $p_i > 0, \eta_i, \eta_{i,k}, \overline{\eta}_{i,k} \geq 0$ such that for all $0 < x \leq p_i, y \in R^+$ and $0 \leq t \leq 1$, $f_i(t,x,y) \leq \eta_i p_i, I_{1,k}(x) \leq \eta_{i,k}p_i, \overline{I}_{i,k}(x) \leq \overline{\eta}_{i,k}p_i$, and

$$\eta_i + \sum_{k=1}^{m}(\eta_{i,k} + \overline{\eta}_{i,k}) > 0,$$

$$\eta_i\int_0^1 G(s,s)a_i(s)ds + \sum_{k=1}^{m} G(t_k,t_k)(\eta_{i,k} + \overline{\eta}_{i,k}) < 1,$$

and there exist $\lambda_i, \lambda_{i,k}, \overline{\lambda}_{i,k} \geq 0$ such that for all $\sigma_i p_i \leq x \leq p_i, y \in R^+$ and $0 \leq t \leq 1$, $f_i(t,x,y) \geq \lambda_i p_i, I_{i,k}(x) \geq \lambda_{i,k}p_i, \overline{I}_{i,k}(x) \geq \overline{\lambda}_{i,k}p_i$, and

$$\lambda_i + \sum_{0<t_k<\frac{1}{2}}(\lambda_{i,k} + \overline{\lambda}_{i,k}) > 0,$$

$$\lambda_i\int_a^b G(\frac{1}{2},s)a_i(s)ds + \sum_{0<t_k<\frac{1}{2}} G(\frac{1}{2},t_k)(\lambda_{i,k} + \overline{\lambda}_{i,k}) > 1.$$

(H_4) $0 < \int_0^1 G(s,s)a_i(s)ds < \infty$.

Let $X = C[\bar{J},\mathbb{R}^+]$ denote the Banach space of all continuous mapping $x : \bar{J} \to \mathbb{R}^+$ with norm $\|x\| = sup_{t\in\bar{J}}|x(t)|$, $PC[\bar{J},\mathbb{R}^+] = \{x : x$ is a map from $\bar{J}$ into $\mathbb{R}^+$ such that $x(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and its right limit at $t = t_k$ $x(t_k^+)$ exists for $k = 1, 2, \cdots m\}$ is a Banach space with the norm $\|x\|_{PC} = sup_{t\in\bar{J}}|x(t)|$, and $PC^1[\bar{J},\mathbb{R}^+] = \{x : x$ is a map from $\bar{J}$ into $\mathbb{R}^+$ such that $x^{'}(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$ and its right limit at $t = t_k$ $x^{'}(t_k^+)$ exists for $k = 1, 2, \cdots m\}$ is a Banach space with the norm $\|x\|_{PC'} = max\{\|x\|_{PC}, \|x^{'}\|_{PC}\}$. $(\bar{J},\mathbb{R}^+) \times (\bar{J},\mathbb{R}^+)$ is also a Banach space with norm

$$\|(u,v)\|_{PC'} = max\{\|u\|, \|v\|\}$$

for any $(u,v) \in (\bar{J},\mathbb{R}^+) \times (\bar{J},\mathbb{R}^+)$.

Let K is a cone in $X = PC[\bar{J},\mathbb{R}^+]$ defined by

$$K = \{x \in PC[\bar{J},\mathbb{R}^+] : x(t) \geq 0,\ t \in [0,1],\ \text{and}\ \ x(t) \geq \sigma\|x\|_{PC},\ t \in [a,b]\}.$$

Definition 2.1. A couple function $(x,y) \in PC^1[\bar{J},\mathbb{R}^+] \cap C^2(J^{'},\mathbb{R}) \times PC^1[\bar{J},\mathbb{R}^+] \cap C^2(J^{'},\mathbb{R}), p(t)x'(t) \in C^1([0,1],\mathbb{R}), p(t)y'(t) \in C^1([0,1],\mathbb{R})$, is called a solution of system (1.1) if it satisfies system (1.1).

Next, for any $(u,v) \in K \times K$, let us define an operator $A_v : K \to C(\overline{J}, R^+)$, $B_u : K \to C(\overline{J}, R^+)$, and $\Phi : K \times K \to C(\overline{J}, R^+) \times C(\overline{J}, R^+)$ as follows

$$\begin{aligned}
A_v(u)(t) &= \int_0^1 G(t,s)a_1(s)f_1(s,u(s),v(s))ds \\
&\quad + \sum_{0<t_k<t} G(t,t_k)(I_{1,k}(u(t_k)) + \bar{I}_{1,k}(u(t_k))), \\
B_u(v)(t) &= \int_0^1 G(t,s)a_2(s)f_2(s,v(s),u(s))ds \\
&\quad + \sum_{0<t_k<t} G(t,t_k)(I_{2,k}(v(t_k)) + \bar{I}_{2,k}(v(t_k))), \\
\Phi(u,v)(t) &= (A_v(u)(t), B_u(v)(t)), \quad t \in [0,1].
\end{aligned} \tag{2.3}$$

Clearly, by (H_1) and (H_4), we know that the operator A_v and B_u are well defined, and so Φ is well defined.

We need the following lemmas in this paper.

Lemma 2.1. *The vector* $(x,y) \in PC^1[\bar{J}, \mathbb{R}^+] \cap C^2(J', \mathbb{R}) \times PC^1[\bar{J}, \mathbb{R}^+] \cap C^2(J', \mathbb{R})$, $p(t)x'(t) \in C^1([0,1], \mathbb{R})$, $p(t)y'(t) \in C^1([0,1], \mathbb{R})$ *is a solution of differential system* (1.1) *if and only if* $(x,y) \in PC^1[\bar{J}, \mathbb{R}^+] \times PC^1[\bar{J}, \mathbb{R}^+]$ *is a solution of the following integral system*

$$\begin{cases}
x(t) = \int_0^1 G(t,s)a_1(s)f_1(s,x(s),y(s))ds \\
\qquad + \sum_{0<t_k<t} G(t,t_k)(I_{1,k}(x(t_k)) + \bar{I}_{1,k}(x(t_k))), \\
y(t) = \int_0^1 G(t,s)a_2(s)f_2(s,y(s),x(s))ds \\
\qquad + \sum_{0<t_k<t} G(t,t_k)(I_{2,k}(y(t_k)) + \bar{I}_{2,k}(y(t_k))).
\end{cases}$$

Lemma 2.2. $\Phi(K \times K) \subset K \times K$.

Proof. We show that for any $(u,v) \in K \times K$, we prove $\Phi(u,v) \in K \times K$, i.e. $A_v(u) \in K$ and $B_u(v) \in K$. From the property (**I**) of $G(t,s)$, we know

$$\begin{aligned}
\|A_v(u)\|_{PC} &\le \int_0^1 G(s,s)a_1(s)f_1(s,u(s),v(s))ds \\
&\quad + \sum_{0<t_k<t} G(t_k,t_k)(I_{1,k}(u(t_k)) + \bar{I}_{1,k}(u(t_k))) < +\infty. \\
\|B_u(v)\|_{PC} &\le \int_0^1 G(s,s)a_2(s)f_2(s,v(s),u(s))ds \\
&\quad + \sum_{0<t_k<t} G(t_k,t_k)(I_{2,k}(v(t_k)) + \bar{I}_{2,k}(v(t_k))) < +\infty.
\end{aligned}$$

On the other hand, by the property (**II**) of $G(t,s)$, for any $t \in [a,b]$, we have

$$\begin{aligned} A_v u(t) &= \int_0^1 G(t,s)a_1(s)f_1(s,u(s),v(s))ds \\ &\quad + \sum_{0<t_k<t} G(t,t_k)(I_{1,k}(u(t_k)) + \bar{I}_{1,k}(u(t_k))) \\ &\geq \sigma \int_a^b G(s,s)a_1(s)f_1(s,u(s),v(s))ds \\ &\quad + \sigma \sum_{0<t_k<t} G(t_k,t_k)(I_{1,k}(u(t_k)) + \bar{I}_{1,k}(u(t_k))) \\ &\geq \sigma \|A_v(u)\|_{PC}. \end{aligned}$$

Similarly, $B_u(v)(t) \geq \sigma \|B_u(v)\|_{PC}$. Thus, $A_v(u) \in K$ and $B_u(v) \in K$. Therefore, $\Phi(u,v) \in K \times K$. □

Lemma 2.3. $\Phi(u,v) : K \times K \to K \times K$ *is a completely continuous operator.*

Proof. For any $n \geq 2$, we defined a continuous function $a_{(1,n)}$ by

$$a_{(1,n)}(t) = \begin{cases} \inf\left\{a_1(t), a_1(\frac{1}{n})\right\}, & 0 < t \leq \frac{1}{n}, \\ a_1(t), & \frac{1}{n} \leq t \leq 1 - \frac{1}{n}, \\ \inf\left\{a_1(t), a_1(1-\frac{1}{n})\right\}, & 1 - \frac{1}{n} \leq t \leq 1. \end{cases}$$

Next, for $n \geq 2$, we define an operator $(A_v)_n : K \to K$ by

$$\begin{aligned} (A_v)_n(t) &= \int_0^1 G(t,s)a_{(1,n)}(s)f_1(s,u(s),v(s))ds \\ &\quad + \sum_{0<t_k<t} G(t,t_k)((I_{1,k}(u(t_k)) + \bar{I}_{1,k}(u(t_k))), \quad t \in [0,1]. \end{aligned}$$

Obviously, for any $n \geq 2$, $(A_v)_n$ is completely continuous on K by an application of the Ascoli-Arzela theorem (see [3]). Then $\|(A_v)_n - A_v\|_{PC} \to 0$, as $n \to +\infty$. In fact, Suppose $D_1, D_2 \subset K$ are any bounded sets, then for any $(u,v) \in D_1 \times D_2$, there exists a constant $M_0 > 0$ such that $0 \leq u(t) \leq M_0$ and $0 \leq v(t) \leq M_0$ for any $t \in [0,1]$. Thus by $f_1 \in C([0,1] \times [0,M_0] \times [0,M_0], R^+)$, there exist $M = \max\limits_{0\leq t\leq 1 \quad u,v\in[0,M_0]} f_1(t,u(t),v(t))$. for any $(u,v) \in D_1 \times D_2$, from (H_1), (H_4) and the property (**I**) of $G(t,s)$, we obtain

$$\begin{aligned}\|(A_v)_n u - A_v u\|_{PC} &= \max_{t\in[0,1]} \left| \int_0^1 G(t,s)[a_1(s) - a_{(1,n)}(s)] f_1(s,u(s),v(s))ds \right| \\ &\le \int_0^{\frac{1}{n}} G(s,s)|a_1(s) - a_{(1,n)}(s)| f_1(s,u(s),v(s))ds \\ &\quad + \int_{1-\frac{1}{n}}^1 G(s,s)|a_1(s) - a_{(1,n)}(s)| f_1(s,u(s),v(s))ds \\ &\le M \int_0^{\frac{1}{n}} G(s,s)|a_1(s) - a_{(1,n)}(s)|ds \\ &\quad + M \int_{1-\frac{1}{n}}^1 G(s,s)|a_1(s) - a_{(1,n)}(s)|ds \\ &\to 0,\ n \to +\infty.\end{aligned}$$

Hence $\|(A_v)_n - A_v\|_{PC} \to 0$, as $n \to +\infty$. Therefore, A_v is completely continuous. Similarly, B_u is completely continuous to. To sum up,$\Phi(u,v)$ is completely continuous operator. □

For $r > 0$, let $K_r = \{x \in K : \|x\| < r\}$ and $\partial K_r = \{x \in K : \|x\| = r\}$. The following Lemma is needed in this paper.

Lemma 2.4. ([3]) *Let $\Phi : K \to K$ be a completely continuous operator, assume $\Phi x \neq x$ for every $x \in \partial K_r$. Then the following conclusions hold.*

(i) *If $\|x\| \le \|\Phi x\|$ for $x \in \partial K_r$, then $i(\Phi, K_r, K) = 0$.*
(ii) *If $\|x\| \ge \|\Phi x\|$ for $x \in \partial K_r$, then $i(\Phi, K_r, K) = 1$.*

Lemma 2.5. ([3], [5]) *Let $\Phi : K \to K$ be a completely continuous mapping and $\mu\Phi x \neq x$ for $x \in \partial K_r$ and $0 < \mu \le 1$. Then $i(\Phi, K_r, K) = 1$.*

Lemma 2.6. ([3], [5]) *Let $\Phi : K \to K$ be a completely continuous mapping. Suppose the following two conditions are satisfied:*

(i) $\inf_{x\in\partial K_r} \|\Phi x\| > 0$;
(ii) *$\mu\Phi x \neq x$ for every $x \in \partial K_r$ and $\mu \ge 1$.*

Then $i(\Phi, K_r, K) = 0$.

Lemma 2.7. ([1]) *Let E be a Banach space and $K_i \subset K (i = 1, 2)$ be a closed set in E. For $r_i > 0 (i = 1, 2)$, denote $K_{r_i} = \{x \in K_i : \|x\| < r_i\}, \partial K_{r_i} = \{x \in K_i : \|x\| = r_i\}$. Suppose $A_i : K_i \to K_i$ is completely continuous. If $x_i \neq A_i x_i$ for any $x_i \in \partial K_{r_i}$, then $i(A, K_{r_1} \times K_{r_2}, K_1 \times K_2) = i(A_1, K_{r_1}, K_1) \times i(A_2, K_{r_2}, K_2)$, where $A(u,v) =: (A_1 u, A_2 v)$ for any $(u,v) \in K_1 \times K_2$.*

3. Main Results

Theorem 3.1. *Suppose that* (H_1)*-*(H_4) *hold. Then system* $BVP(1.1)$ *has at least two positive solutions* (u_1, v_1) *and* (u_2, v_2)*, satisfying* $0 \le \|(u_1, v_1)\|_{pc} \le p \le \|(u_2, v_2)\|_{pc}$.

Proof. The first step, suppose that (H_3) holds, then $i(A_v, K_p, K) = 1$. In fact let $u \in K$ with $\|u\|_{pc} = p$. From (H_3) and (2.3) we have

$$\begin{aligned}
&\|A_v u\|_{pc} \\
&\le \int_0^1 G(s,s)a_1(s)f_1(s,u(s),v(s))ds \\
&\quad + \sum_{k=1}^m G(t_k,t_k)(I_{1,k}(u(t_k)) + \bar{I}_{1,k}(u(t_k))) \\
&\le p\left(\eta\int_0^1 G(s,s)a_1(s)ds + \sum_{k=1}^m G(t_k,t_k)(\eta_{1,k} + \bar{\eta}_{1,k})\right) \\
&< p = \|u\|_{pc}.
\end{aligned}$$

That is $\|u\|_{pc} \ge \|A_v u\|_{pc}$ for $u \in \partial K_p$. Therefore, by Lemma 2.3, we obtain

$$i(A_v, K_p, K) = 1. \tag{3.1}$$

The second step, we prove that there exists $0 < r < p$ such that $i(A_v, K_r, K) = 0$. We first prove $\inf_{u\in\partial K_r} \|A_v u\| > 0$.

By (H_1), there exists $0 < \varepsilon_0 < 1$ such that

$$\begin{aligned}
&(1-\varepsilon_0)\left(\inf_{y\in R^+} f_{1,0}(y) + \frac{\sigma\sum_{k=1}^m (I_{1,0}(k)\phi_1(t_k)+\overline{I}_{1,0}(k)\phi_1'(t_k))p(t_k)}{\int_0^1 \phi_1(t)a_1(t)dt}\right) > \lambda_{1,1}, \\
&(1-\varepsilon_0)\left(\inf_{y\in R^+} f_{1,\infty}(y) + \frac{\sigma\sum_{k=1}^m (I_{1,\infty}(k)\phi_1(t_k)+\overline{I}_{1,\infty}(k)\phi_1'(t_k))p(t_k)}{\int_0^1 \phi_1(t)a_1(t)dt}\right) > \lambda_{1,1}.
\end{aligned} \tag{3.2}$$

By the definitions of $f_{1,0}, I_{1,0}$ and $\overline{I}_{1,0}$, there exists $0 < r_0 < p$ such that for any $t \in [a,b], 0 \le x \le r_0$,

$$\begin{aligned}
f_1(t,x,y) &\ge f_{1,0}(y)(1-\varepsilon_0)x, \\
I_{1,k}(x) &\ge I_{1,0}(k)(1-\varepsilon_0)x, \\
\overline{I}_{1,k}(x) &\ge \overline{I}_{1,0}(k)(1-\varepsilon_0)x.
\end{aligned} \tag{3.3}$$

Let $r \in (0, r_0)$ then for $u \in \partial K_r$, we have

$$r_0 > \|u\|_{pc} \ge u(t) \ge \sigma\|u\|_{pc} = \sigma r > 0, \quad t \in [a,b]. \tag{3.4}$$

So, by (3.3)and (3.4) we get

$$
\begin{aligned}
\|A_v u\|_{pc} &\geq A_v u(\frac{1}{2}) \\
&= \int_0^1 G(\frac{1}{2}, s) a_1(s) f_1 s, u(s), v(s)) ds \\
&\quad + \sum_{0<t_k<t} G(\frac{1}{2}, t_k)(I_{1,k}(u(t_k)) + \bar{I}_{1,k}(u(t_k))) \\
&\geq \int_a^b G(\frac{1}{2}, s) a_1(s) f_1(s, u(s), v(s)) ds \\
&\quad + \sum_{0<t_k<t} G(\frac{1}{2}, t_k)(I_{1,k}(u(t_k)) + \bar{I}_{1,k}(u(t_k))) \\
&\geq (1-\varepsilon_0) \int_a^b G(\frac{1}{2}, s) a_1(s) f_{1,0}(v(s)) u(s) ds \\
&\quad + (1-\varepsilon_0) \sum_{0<t_k<\frac{1}{2}} G(\frac{1}{2}, t_k) \left(I_{1,0}(k)(u(t_k)) + \bar{I}_{1,0}(k)(u(t_k))\right) \\
&\geq (1-\varepsilon_0)\sigma r \left(\inf_{y \in R^+} f_{1,0}(y) \int_a^b G(\frac{1}{2}, s) a_1(s) ds \right. \\
&\quad \left. + \sum_{0<t_k<\frac{1}{2}} G(\frac{1}{2}, t_k)(I_{1,0}(k) + \bar{I}_{1,0}(k)) \right) > 0,
\end{aligned}
\tag{3.5}
$$

this implies that $\inf_{u \in \partial K_r} \|A_v u\| > 0$.

Next we show $\mu A_v u \neq u$ for every $u \in \partial K_r$ and $\mu \geq 1$. If it is not true, then there exist $u_0 \in \partial K_r$ and $\mu_0 \geq 1$ such that $\mu_0 A_v u_0 = u_0$. It is easy to see that $u_0(t)$ satisfies

$$
\begin{cases}
(p(t)u_0'(t))' + \mu_0 a_1(t) f_1(t, u_0(t), v(t)) = 0, \qquad t \in J', \\
-\Delta u_0'|_{t=t_k} = I_{1,k}(u_0(t_k)), \\
\Delta u_0|_{t=t_k} = \bar{I}_{1,k}(u_0(t_k)), \qquad k = 1, 2, \cdots, m, \\
\alpha_1 u_0(0) - \beta_1 \lim_{t \to 0+} p(t) u_0'(t) = 0, \\
\alpha_2 u_0(1) + \beta_2 \lim_{t \to 1-} p(t) u_0'(t) = 0.
\end{cases}
\tag{3.6}
$$

Multiplying $(p(t)u_0'(t))'$ by $\phi_1(t)$ and then integrating the product from a to b with respect to t, we get

$$
\begin{aligned}
&\textstyle\int_a^b (p(t)u_0'(t))'\phi_1(t)dt \\
&= \textstyle\int_a^{t_1} \phi_1(t)d(p(t)u_0'(t)) + \sum_{k=1}^{m-1} \int_{t_k}^{t_{k+1}} \phi_1(t)d(p(t)u_0'(t)) \\
&\quad + \textstyle\int_{t_m}^b \phi_1(t)d(p(t)u_0'(t)) \\
&= \phi_1(t_1)p(t_1)u_0'(t_1-0) - \textstyle\int_a^{t_1} \phi_1'(t)p(t)u_0'(t)dt \\
&\quad + \sum_{k=1}^{m-1} (\phi_1(t_{k+1})p(t_{k+1})u_0'(t_{k+1}-0) - \phi_1(t_k)p(t_k)u_0'(t_k+0)) \\
&\quad - \sum_{k=1}^{m-1} \textstyle\int_{t_k}^{t_{k+1}} \phi_1'(t)p(t)u_0'(t)dt - \phi_1(t_m)p(t_m)u_0'(t_m+0) \\
&\quad - \textstyle\int_{t_m}^b \phi_1'(t)p(t)u_0'(t)dt \\
&= -\sum_{k=1}^{m} \Delta u_0'(t_k)\phi_1(t_k)p(t_k) - \textstyle\int_a^b p(t)u_0'(t)\phi_1'(t)dt.
\end{aligned}
\tag{3.7}
$$

Similarly, we have

$$
\begin{aligned}
&\textstyle\int_a^b p(t)u_0'(t)\phi_1'(t)dt \\
&= \textstyle\int_a^{t_1} \phi_1'(t)p(t)du_0(t) + \sum_{k=1}^{m-1} \int_{t_k}^{t_{k+1}} \phi_1'(t)p(t)du_0(t) \\
&\quad + \textstyle\int_{t_m}^b \phi_1'(t)p(t)du_0(t) \\
&= -\sum_{k=1}^{m} \Delta u_0(t_k)\phi_1'(t_k)p(t_k) - \textstyle\int_a^b (p(t)\phi_1'(t))'u_0(t)dt \\
&= -\sum_{k=1}^{m} \Delta u_0(t_k)\phi_1'(t_k)p(t_k) + \lambda_{1,1}\textstyle\int_a^b a_1(t)\phi_1(t)u_0(t)dt.
\end{aligned}
\tag{3.8}
$$

Then, from (3.7)and (3.8), we get

$$
\begin{aligned}
&\textstyle\int_a^b (p(t)u_0'(t))'\phi_1(t)dt \\
&= -\sum_{k=1}^{m} \Delta u_0'(t_k)\phi_1(t_k)p(t_k) + \sum_{k=1}^{m} \Delta u_0(t_k)\phi_1'(t_k)p(t_k) \\
&\quad -\lambda_{1,1}\textstyle\int_a^b a_1(t)\phi_1(t)u_0(t)dt \\
&= \mu_0 \sum_{k=1}^{m} \left(I_{1,k}(u_0(t_k))\phi_1(t_k) + \overline{I}_{1,k}(u_0(t_k))\phi_1'(t_k)\right) p(t_k) \\
&\quad -\lambda_{1,1}\textstyle\int_a^b a_1(t)\phi_1(t)u_0(t)dt.
\end{aligned}
\tag{3.9}
$$

From (3.6), we obtain $(p(t)u_0'(t))' = -\mu_0 a_1(t)f_1(t,u_0(t),v)$, so

$$
\int_a^b (p(t)u_0'(t))'\phi_1(t)dt = -\mu_0 \int_a^b \phi_1(t)a_1(t)f_1(t,u_0(t),v)dt. \tag{3.10}
$$

Then, from (3.9)and (3.10), we get

$$
\begin{aligned}
&\lambda_{1,1} \int_a^b a_1(t)\phi_1(t)u_0(t)dt \\
&= \mu_0 \sum_{k=1}^{m} \left(I_{1,k}(u_0(t_k))\phi_1(t_k) + \overline{I}_{1,k}(u_0(t_k))\phi_1'(t_k)\right) p(t_k) \\
&\quad +\mu_0 \int_a^b \phi_1(t)a_1(t)f_1(t,u_0(t),v)dt \\
&\geq (1-\varepsilon_0)\Bigg(\sum_{k=1}^{m} \left(I_{1,0}(k)\phi_1(t_k) + \overline{I}_{1,0}(k)\phi_1'(t_k)\right) u_0(t_k)p(t_k) \\
&\quad + \inf_{y\in R^+} f_{1,0}(y) \int_a^b \phi_1(t)a_1(t)u_0(t)dt \Bigg).
\end{aligned} \tag{3.11}
$$

Since $u_0(t) \geq \sigma\|u\|_{pc} > 0$ for all $t \in [a,b]$, we have $\int_a^b \phi_1(t)a_1(t)u_0(t)dt > 0$ and $\sum_{k=1}^{m} \left(I_{1,0}(k)\phi_1(t_k) + \overline{I}_{1,0}(k)\phi_1'(t_k)\right) u_0(t_k)p(t_k) > 0$. By (3.11) we know

$$
\lambda_{1,1} > (1-\varepsilon_0) \inf_{y\in R^+} f_{1,0}(y),
$$

and hence we obtain

$$
\begin{aligned}
&(\lambda_{1,1} - (1-\varepsilon_0) \inf_{y\in R^+} f_{1,0}(y)) \int_a^b \phi_1(t)a_1(t)\|u_0(t)\|dt \\
&\geq (\lambda_{1,1} - (1-\varepsilon_0) \inf_{y\in R^+} f_{1,0}(y)) \int_a^b \phi_1(t)a_1(t)u_0(t)dt \\
&\geq (1-\varepsilon_0) \sum_{k=1}^{m} \left(I_{1,0}(k)\phi_1(t_k) + \overline{I}_{1,0}(k)\phi_1'(t_k)\right) u_0(t_k)p(t_k) \\
&\geq (1-\varepsilon_0)\sigma\|u_0(t)\| \sum_{k=1}^{m} \left(I_{1,0}(k)\phi_1(t_k) + \overline{I}_{1,0}(k)\phi_1'(t_k)\right) p(t_k).
\end{aligned} \tag{3.12}
$$

This implies that

$$
\begin{aligned}
&(\lambda_{1,1} - (1-\varepsilon_0) \inf_{y\in R^+} f_{1,0}(y)) \int_a^b \phi_1(t)a_1(t)dt \\
&\geq (1-\varepsilon_0)\sigma \sum_{k=1}^{m} \left(I_{1,0}(k)\phi_1(t_k) + \overline{I}_{1,0}(k)\phi_1'(t_k)\right) p(t_k).
\end{aligned} \tag{3.13}
$$

So,

$$
\lambda_{1,1} \geq (1-\varepsilon_0) \left(\inf_{y\in R^+} f_{1,0}(y) + \frac{\sigma \sum_{k=1}^{m} \left(I_{1,0}(k)\phi_1(t_k) + \overline{I}_{1,0}(k)\phi_1'(t_k)\right) p(t_k)}{\int_a^b \phi_1(t)a_1(t)dt} \right)
$$

which is in contradiction with (3.2). So we obtain $\mu A_v u \neq u$ for every $u \in \partial K_r$ and $\mu \geq 1$. Hence, by Lemma 2.5, we get

$$
i(A_v, K_r, K) = 0. \tag{3.14}
$$

The third step, we prove that there exists large enough R such that

$$
i(A_v, K_R, K) = 0.
$$

Firstly, we show $\inf_{u\in\partial K_R}\|A_v u\|>0$. From the definitions of $f_{1,\infty}, I_{1,\infty}$ and $\overline{I}_{1,\infty}$, there exists $H>p>0$ such that for any $t\in[a,b]$ and $x\geq H$,

$$\begin{aligned}&f_1(t,x,y)\geq f_{1,\infty}(y)(1-\varepsilon_0)x,\\&I_{1,k}(x)\geq I_{1,\infty}(k)(1-\varepsilon_0)x,\\&\overline{I}_{1,k}(x)\geq \overline{I}_{1,\infty}(k)(1-\varepsilon_0)x.\end{aligned}\tag{3.15}$$

Let

$$\begin{aligned}c\ &=\max_{0\leq x\leq H}\max_{a\leq t\leq b}|f_1(t,x,y)-f_{1,\infty}(y)(1-\varepsilon_0)x|\\&\quad+\sum_{k=1}^{m}\max_{0\leq x\leq H}|I_{1,k}(x)-I_{1,\infty}(k)(1-\varepsilon_0)x|\\&\quad+\sum_{k=1}^{m}\max_{0\leq x\leq H}|\overline{I}_{1,k}(x)-\overline{I}_{1,\infty}(k)(1-\varepsilon_0)x|.\end{aligned}\tag{3.16}$$

Then, from (3.15) and (3.16), we have

$$\begin{aligned}&f_1(t,x,y)\geq f_{1,\infty}(y)(1-\varepsilon_0)x-c,\\&I_{1,k}(x)\geq I_{1,\infty}(k)(1-\varepsilon_0)x-c,\\&\overline{I}_{1,k}(x)\geq \overline{I}_{1,\infty}(k)(1-\varepsilon_0)x-c,\quad \text{for all}\quad t\in[a,b],x>0.\end{aligned}\tag{3.17}$$

Choose $R>R_0=\max\{\frac{H}{\sigma},p\}$. Let $u\in\partial K_R$, then $u(t)\geq\sigma\|u\|_{pc}=\sigma R>H$ for all $t\in[a,b]$, by (3.15) and (**II**) we have

$$\begin{aligned}&f_1(t,u,v)\geq f_{1,\infty}(v(t))(1-\varepsilon_0)\sigma R,\\&I_{1,k}(u(t_k))\geq I_{1,\infty}(k)(1-\varepsilon_0)\sigma R,\\&\overline{I}_{1,k}(u(t_k))\geq \overline{I}_{1,\infty}(k)(1-\varepsilon_0)\sigma R.\end{aligned}$$

Proceeding as in second step, we can get $\inf_{u\in\partial K_R}\|A_v u\|>0$.

Secondly, we show that if R is large enough, then we have $\mu A_v u\neq u$ for every $u\in\partial K_R$ and $\mu\geq 1$. In fact, if it is not true, then there exist $u_0\in\partial K_R$ and $\mu_0\geq 1$ such that $\mu_0 A_v u_0=u_0$. It is easy to see that $u_0(t)$ satisfies (3.6), and similar to the analysis in second step, by (3.17), we obtain

$$\begin{aligned}&\lambda_{1,1}\int_a^b a_1(t)\phi_1(t)u_0(t)dt\\&=\mu_0\sum_{k=1}^{m}\left(I_{1,k}(u_0(t_k))\phi_1(t_k)+\overline{I}_{1,k}(u_0(t_k))\phi_1'(t_k)\right)p(t_k)\\&\quad+\mu_0\int_a^b\phi_1(t)a_1(t)f_1(t,u_0(t),v)dt\\&\geq(1-\varepsilon_0)\sum_{k=1}^{m}\left(I_{1,\infty}(k)\phi_1(t_k)+\overline{I}_{1,\infty}(k)\phi_1'(t_k)\right)u_0(t_k)p(t_k)\\&\quad+(1-\varepsilon_0)f_{1,\infty}(y)\int_a^b\phi_1(t)a_1(t)u_0(t)dt\\&\quad-c\left(\sum_{k=1}^{m}\left(\phi_1(t_k)+\phi_1'(t_k)\right)p(t_k)+\int_a^b\phi_1(t)a_1(t)dt\right).\end{aligned}$$

(I) If $(1-\varepsilon_0)f_{1,\infty}(y) \leq \lambda_{1,1}$, then

$$\begin{aligned}&(\lambda_{1,1}-(1-\varepsilon_0)f_{1,\infty}(y))\int_a^b \phi_1(t)a_1(t)u_0(t)dt\\ &\quad +c\left(\sum_{k=1}^{m}\left(\phi_1(t_k)+\phi_1'(t_k)\right)p(t_k)+\int_a^b \phi_1(t)a_1(t)dt\right)\\ &\geq (1-\varepsilon_0)\sum_{k=1}^{m}\left(I_{1,\infty}(k)\phi_1(t_k)+\overline{I}_{1,\infty}(k)\phi_1'(t_k)\right)u_0(t_k)p(t_k),\end{aligned}$$

such that

$$\begin{aligned}&\|u_0\|_{pc}(\lambda_{1,1}-(1-\varepsilon_0)f_{1,\infty}(y))\int_a^b \phi_1(t)a_1(t)dt\\ &\quad +c\left(\sum_{k=1}^{m}\left(\phi_1(t_k)+\phi_1'(t_k)\right)p(t_k)+\int_a^b \phi_1(t)a_1(t)dt\right)\\ &\geq (1-\varepsilon_0)\sigma\|u_0\|_{pc}\sum_{k=1}^{m}\left(I_{1,\infty}(k)\phi_1(t_k)+\overline{I}_{1,\infty}(k)\phi_1'(t_k)\right)p(t_k).\end{aligned}$$

This implies

$$\begin{aligned}&\|u_0\|_{pc}\\ &\leq \frac{c\left(\sum_{k=1}^{m}\left(\phi_1(t_k)+\phi_1'(t_k)\right)p(t_k)+\int_a^b \phi_1(t)a_1(t)dt\right)}{(1-\varepsilon_0)\sigma\sum_{k=1}^{m}\left(I_{1,\infty}(k)\phi_1(t_k)+\overline{I}_{1,\infty}(k)\phi_1'(t_k)\right)p(t_k)-(\lambda_{1,1}-(1-\varepsilon_0)f_{1,\infty}(y))\int_a^b \phi_1(t)a_1(t)dt}\\ &=: R_1.\end{aligned}$$

(II) If $(1-\varepsilon_0)f_{1,\infty}(y) > \lambda_{1,1}$, then

$$\begin{aligned}&c\left(\sum_{k=1}^{m}\left(\phi_1(t_k)+\phi_1'(t_k)\right)p(t_k)+\int_a^b \phi_1(t)a_1(t)dt\right)\\ &\geq ((1-\varepsilon_0)f_{1,\infty}(y)-\lambda_{1,1})\int_a^b \phi_1(t)a_1(t)u_0(t)dt\\ &\geq ((1-\varepsilon_0)f_{1,\infty}(y)-\lambda_{1,1})\sigma\|u_0\|_{pc}\int_a^b \phi_1(t)a_1(t)dt,\end{aligned}$$

thus

$$\|u_0\|_{pc} \leq \frac{c\left(\sum_{k=1}^{m}\left(\phi_1(t_k)+\phi_1'(t_k)\right)p(t_k)+\int_a^b \phi_1(t)a_1(t)dt\right)}{((1-\varepsilon_0)f_{1,\infty}(y)-\lambda_1)\sigma\int_a^b \phi_1(t)a_1(t)dt} =: R_2.$$

Let $R > \max\{R_0, R_1, R_2\}$, then for all $u \in \partial K_R$ and $\mu \geq 1$, $\mu A_v u \neq u$. Hence, by Lemma 2.5, we have

$$i(A_v, K_R, K) = 0. \tag{3.18}$$

By (3.1), (3.14), (3.22) and the property of the fixed points index, we obtain

$$i(A_v, K_R\backslash\overline{K}_P, K) = -1, \quad i(A_v, K_p\backslash\overline{K}_r, K) = 1. \tag{3.19}$$

The fourth step, establishes by (H_3), then $i(B_u, K_p, K) = 0$. Let $v \in K$ with $\|v\|_{pc} = p$. From (H_3) and (2.3) we have

$$\begin{aligned}
\|B_u v\|_{pc} &\geq B_u v(\frac{1}{2}) \\
&= \int_0^1 G(\frac{1}{2}, s) a_2(s) f_2(s, v(s), u(s)) ds \\
&\quad + \sum_{0<t_k<\frac{1}{2}} G(\frac{1}{2}, t_k)(I_{2,k}(v(t_k)) + \bar{I}_{2,k}(v(t_k))) \\
&\geq \int_a^b G(\frac{1}{2}, s) a_2(s) f_2(s, v(s), u(s)) ds \\
&\quad + \sum_{0<t_k<\frac{1}{2}} G(\frac{1}{2}, t_k)(I_{2,k}(v(t_k)) + \bar{I}_{2,k}(v(t_k))) \\
&\geq P\bigg(\lambda_2 \int_a^b G(\frac{1}{2}, s) a_2(s) ds + \sum_{0<t_k<\frac{1}{2}} G(\frac{1}{2}, t_k) \left(\lambda_{2,k} + \overline{\lambda}_{2,k}\right)\bigg) \\
&> p = \|v\|_{pc}.
\end{aligned}$$

This implies that for any $v \in \partial K_p$, we have $\|v\|_{pc} \leq \|B_u v\|_{pc}$. Therefore, by Lemma 2.3, we obtain

$$i(B_u, K_p, K) = 0. \tag{3.20}$$

The fifth step, suppose that $0 < r' < p$ holds, then $i(B_u, K_{r'}, K) = 1$. In fact, by (H_2), there exists

$$0 < \varepsilon_1 < \min\left\{\lambda_{2,1} - \sup_{y\in R^+} f_2^0(y), \lambda_{2,1} - \sup_{y\in R^+} f_2^\infty(y)\right\}$$

such that

$$\begin{aligned}
&(\lambda_{2,1} - \varepsilon_1 - \sup_{y\in R^+} f_2^0(y))\sigma \textstyle\int_a^b \phi_2(t) a_2(t) dt \\
&> \sum_{k=1}^m \left(\left(I_2^0(k) + \varepsilon_1\right) \phi_2(t_k) + \left(\overline{I}_2^0(k) + \varepsilon_1\right) \phi_2'(t_k)\right) p(t_k), \\
&(\lambda_{2,1} - \varepsilon_1 - \sup_{y\in R^+} f_2^\infty(y))\sigma \textstyle\int_a^b \phi_2(t) a_2(t) dt \\
&> \sum_{k=1}^m \left(\left(I_2^\infty(k) + \varepsilon_1\right) \phi_2(t_k) + \left(\overline{I}_2^\infty(k) + \varepsilon_1\right) \phi_2'(t_k)\right) p(t_k).
\end{aligned} \tag{3.21}$$

By the definitions of $f_2^0(y), I_2^0$ and $\overline{I}_2^0$, there exists $0 < r_0' < p$ such that for any $t \in [a, b], 0 \leq x \leq r_0'$, we have

$$\begin{aligned}
&f_2(t, x, y) \leq (f_2^0(y) + \varepsilon_1)x, \\
&I_{2,k}(x) \leq (I_2^0(k) + \varepsilon_1)x, \\
&\overline{I}_{2,k}(x) \leq (\overline{I}_2^0(k) + \varepsilon_1)x.
\end{aligned} \tag{3.22}$$

Let $r' \in (0, r'_0)$. We now show that $\mu B_u v \neq v$ for $v \in \partial K_{r'}$ and $0 < \mu \leq 1$. If this is not true, then there exist $v_0 \in \partial K_{r'}$ and $0 < \mu_0 \leq 1$ such that $\mu_0 B_u v_0 = v_0$. Then $v_0(t)$ satisfies $BVP(3.6)$. From (3.22), multiplying $(p(t)v_0'(t))'$ by $\phi_2(t)$ and then integrating the product from a to b with respect to t, and then proceeding as in the second step of proof of Theorem 3.1, we have

$$\begin{aligned}
&\lambda_{2,1} \int_a^b a_2(t)\phi_2(t)v_0(t)dt \\
&= \mu_0 \sum_{k=1}^m \left(I_{2,k}(v_0(t_k))\phi_2(t_k) + \overline{I}_{2,k}(u_0(t_k))\phi_2'(t_k)\right) p(t_k) \\
&\quad +\mu_0 \int_a^b \phi_2(t)a_2(t)f_2(t, v_0(t), u)dt \\
&\leq \sum_{k=1}^m \left((I_2^0(k) + \varepsilon_1)\phi_2(t_k) + (\overline{I}_2^0(k) + \varepsilon_1)\phi_2'(t_k)\right) u_0(t_k)p(t_k) \\
&\quad +(f_2^0(y) + \varepsilon_1) \int_a^b \phi_2(t)a_2(t)v_0(t)dt.
\end{aligned}$$

Since $v_0(t) \geq \sigma\|v_0\|_{pc} = \sigma r'$ for $t \in [a, b]$, we have

$$\begin{aligned}
&r(\lambda_{2,1} - \sup_{y\in R^+} f_2^0(y) - \varepsilon_1) \int_a^b \sigma a_2(t)\phi_2(t)dt \\
&\leq (\lambda_{2,1} - \sup_{y\in R^+} f_2^0(y) - \varepsilon_1) \int_a^b \phi_2(t)a_2(t)v_0(t)dt \\
&\leq \sum_{k=1}^m \left((I_2^0(k) + \varepsilon_1)\phi_2(t_k) + (\overline{I}_2^0(k) + \varepsilon_1)\phi_2'(t_k)\right) v_0(t_k)p(t_k) \\
&\leq r \sum_{k=1}^m \left((I_2^0(k) + \varepsilon_1)\phi_2(t_k) + (\overline{I}_2^0(k) + \varepsilon_1)\phi_2'(t_k)\right) p(t_k).
\end{aligned}$$

This is a contradiction with (3.21). Hence, by Lemma 2.4, we have

$$i(B_u, K_{r'}, K) = 1. \tag{3.23}$$

The sixth step, we prove $i(B_u, K_{R'}, K) = 1$. From the definitions of $f_2^\infty(y), I_2^\infty$ and $\overline{I}_2^\infty$, there exists $H > p > 0$ such that for any $t \in [a, b]$ and $x \geq H$,

$$\begin{aligned}
&f_2(t, x, y) \leq (f_2^\infty(y) + \varepsilon_1)x, \\
&I_{2,k}(x) \leq (I_2^\infty(k) + \varepsilon_1)x, \\
&\overline{I}_{2,k}(x) \leq (\overline{I}_2^\infty(k) + \varepsilon_1)x.
\end{aligned} \tag{3.24}$$

Proceeding as in the third step of proof of theorem 3.1, for any $t \in [a, b]$ and $x \geq H$, let

$$\begin{aligned}
l \;=\; &\max_{0\leq x\leq H, y\in R^+} \max_{a\leq t\leq b} |f_2(t, x, y) - (f_2^\infty + \varepsilon_1)x| \\
&+ \sum_{k=1}^m \max_{0\leq x\leq H} |I_{2,k}(x) - (I_2^\infty(k) + \varepsilon_1)x| \\
&+ \sum_{k=1}^m \max_{0\leq x\leq H} |\overline{I}_{2,k}(x) - (\overline{I}_2^\infty(k) + \varepsilon_1)x|.
\end{aligned} \tag{3.25}$$

Then, from (3.24), we have

$$\begin{aligned} &f_2(t,x,y) \le (f_2^\infty + \varepsilon_1)x + l, \quad I_{2,k}(x) \le (I_2^\infty(k) + \varepsilon_1)x + l, \\ &\overline{I}_{2,k}(x) \le (\overline{I}_2^\infty(k) + \varepsilon_1)x + l, \quad \text{for all } \ t \in [a,b], x > 0. \end{aligned} \tag{3.26}$$

Then we show that if R' is large enough, we have $\mu B_u v \neq v$ for every $v \in \partial K_{R'}$ and $0 < \mu \le 1$. In fact, if it is not true, then there exist $v_0 \in \partial K_{R'}$ and $\mu_0 \ge 1$ such that $\mu_0 B_u v_0 = v_0$. It is easy to see that $v_0(t)$ satisfies (3.6), and similar to the proof of the third step of Theorem 3.1, we obtain

$$\begin{aligned} &\|v_0\|_{pc}(\lambda_{2,1} - \sup_{y\in R^+} f_2^\infty(y) - \varepsilon_1) \int_a^b \sigma a_2(t)\phi_2(t)dt \\ &\le (\lambda_{2,1} - \sup_{y\in R^+} f_2^\infty(y) - \varepsilon_1) \int_a^b \phi_2(t)a_2(t)v_0(t)dt \\ &\le \sum_{k=1}^m \left((I_2^\infty(k) + \varepsilon_1)\phi_2(t_k) + (\overline{I}_2^\infty(k) + \varepsilon_1)\phi_2'(t_k)\right) v_0(t_k)p(t_k) \\ &\quad + l\left(\sum_{k=1}^m (\phi_2(t_k) + \phi_2'(t_k))\, p(t_k) + \int_a^b \phi_2(t)a_2(t)dt\right) \\ &\le \|v_0\|_{pc} \sum_{k=1}^m \left((I_2^\infty(k) + \varepsilon_1)\phi_2(t_k) + (\overline{I}_2^\infty(k) + \varepsilon_1)\phi_2'(t_k)\right) p(t_k) \\ &\quad + l\left(\sum_{k=1}^m (\phi_2(t_k) + \phi_2'(t_k))\, p(t_k) + \int_a^b \phi_2(t)a_2(t)dt\right). \end{aligned}$$

So,

$$\begin{aligned} &\|v_0\|_{pc} \\ &\le \frac{l\left(\sum_{k=1}^m \left(\phi_2(t_k)+\phi_2'(t_k)\right)p(t_k)+\int_a^b \phi_2(t)a_2(t)dt\right)}{(\lambda_{2,1}- \sup_{y\in R^+} f_2^\infty(y)-\varepsilon_1) \int_a^b \sigma a_2(t)\phi_2(t)dt - \sum_{k=1}^m \left((I_2^\infty(k)+\varepsilon_1)\phi_2(t_k)+(\overline{I}_2^\infty(k)+\varepsilon_1)\phi_2'(t_k)\right)p(t_k)} \\ &=: R_1. \end{aligned}$$

Let $R' > \max\{H, R_1\}$, then for all $v \in \partial K_{R'}$ and $0 < \mu \le 1$, $\mu B_u v \neq v$. Hence, by Lemma 2.5, we have

$$i(B_u, K_{R'}, K) = 1. \tag{3.27}$$

By (3.20), (3.23), (3.27) and the property of the fixed points index, we obtain

$$i(B_u, K_{R'}\backslash\overline{K}_P, K) = 1, \quad i(B_u, K_p\backslash\overline{K}_{r'}, K) = -1. \tag{3.28}$$

By (3.19), (3.28), and Lemma 2.7, we have

$$\begin{aligned} i(\Phi, K_P\backslash\overline{K}_r \times K_P\backslash\overline{K}_{r'}, K\times K) &= i(A_v, K_P\backslash\overline{K}_r, K) \times i(B_v, K_P\backslash\overline{K}_{r'}, K) \\ &= -1, \end{aligned}$$

$$\begin{aligned} i(\Phi, K_R\backslash\overline{K}_P \times K_{R'}\backslash\overline{K}_P, K\times K) &= i(A_v, K_R\backslash\overline{K}_P, K) \times i(B_v, K_{R'}\backslash\overline{K}_P, K) \\ &= -1. \end{aligned}$$

By Lemma 2.3, we know that A_v, B_u and $\Phi(u,v)$ are completely continuous operator. Thus, $BVP(1.1)$ has at least two positive solutions (u_1, v_1) and

(u_2, v_2) satisfying $0 \le \|(u_1, v_1)\|_{pc} \le p \le \|(u_2, v_2)\|_{pc}$. The proof is completed. □

(H_1^*)

$$\inf_{y\in R^+} f_{i,0}(y) + \frac{\sigma \sum\limits_{k=1}^{m} (I_{i,0}(k)\phi_i(t_k) + \overline{I}_{i,0}(k)\phi_i'(t_k))p(t_k)}{\int_0^1 \phi_i(t)a_i(t)dt} < \lambda_{i,1},$$

$$\inf_{y\in R^+} f_{i,\infty}(y) + \frac{\sigma \sum\limits_{k=1}^{m} (I_{i,\infty}(k)\phi_i(t_k) + \overline{I}_{i,\infty}(k)\phi_i'(t_k))p(t_k)}{\int_0^1 \phi_i(t)a_i(t)dt} < \lambda_{i,1}.$$

(H_3^*) There exist $p > 0, \eta_i, \eta_{i,k}, \overline{\eta}_{i,k} \ge 0$ such that for all $\sigma p \le x \le p$, $y \in R^+$ and $0 \le t \le 1$, $f_i(t, x, y) \ge \eta_i p$, $I_{i,k}(x) \ge \eta_{i,k} p, \overline{I}_{i,k}(x) \ge \overline{\eta}_{i,k} p$, and

$$\eta_i + \sum_{0<t_k<\frac{1}{2}} (\eta_{i,k} + \overline{\eta}_{i,k}) > 0,$$

$$\eta_i \int_0^1 G(\frac{1}{2}, s)a_i(s)ds + \sum_{0<t_k<\frac{1}{2}} G(\frac{1}{2}, t_k)(\eta_{i,k} + \overline{\eta}_{i,k}) > 1,$$

for $i = 1, 2$.

Theorem 3.2. *Suppose that* (H_1^*), (H_2), (H_3^*), (H_4) *hold. Then system* $BVP(1.1)$ *has at least two positive solutions* (u_1, v_1) *and* (u_2, v_2), *satisfying*

$$0 \le \|(u_1, v_1)\|_{pc} \le p \le \|(u_2, v_2)\|_{pc}.$$

Proof. Similar to the proof of Theorem 3.1, and the proof is omitted. □

(H_2^*)

$$\sup_{y\in R^+} f_i^0(y) + \frac{\sum\limits_{k=1}^{m} (I_i^0(k)\phi_i(t_k) + \overline{I}_i^0(k)\phi_i'(t_k))p(t_k)}{\sigma \int_0^1 \phi_i(t)a_i(t)dt} > \lambda_{i,1},$$

$$\sup_{y\in R^+} f_i^\infty(y) + \frac{\sum\limits_{k=1}^{m} (I_i^\infty(k)\phi_i(t_k) + \overline{I}_i^\infty(k)\phi_i'(t_k))p(t_k)}{\sigma \int_0^1 \phi_i(t)a_i(t)dt} > \lambda_{i,1}.$$

(H_3^{**}) There exist $p > 0, \eta_i, \eta_{i,k}, \overline{\eta}_{i,k} \ge 0 (i = 1, 2)$ such that for all $0 \le x \le p, y \in R^+$ and $0 \le t \le 1, f_i(t, x, y) \le \eta_i p, I_{i,k}(x) \le \eta_{i,k} p, \overline{I}_{i,k}(x) \le \overline{\eta}_{i,k} p$, and

$$\eta_i + \sum_{k=1}^{m} (\eta_{i,k} + \overline{\eta}_{i,k}) > 0,$$

$$\eta_i \int_0^1 G(s,s)a_i(s)ds + \sum_{k=1}^{m} G(t_k,t_k)(\eta_{i,k} + \overline{\eta}_{i,k}) < 1.$$

Theorem 3.3. *Suppose that* (H_1), (H_2^*), (H_3^{**}), (H_4) *hold. Then system* $BVP(1.1)$ *has at least two positive solutions* (u_1, v_1) *and* (u_2, v_2), *satisfying*

$$0 \le \|(u_1, v_1)\|_{pc} \le p \le \|(u_2, v_2)\|_{pc}.$$

Proof. Similar to the proof of Theorem 3.1, and the proof is omitted too. □

4. Some corollaries and an example

Corollary 4.1. *Suppose that* $(H_1^*), (H_2^*)$ *hold. Then system* $BVP(1.1)$ *has at least one positive solutions.*

Corollary 4.2. *Theorem* 3.1 *is valid if* (H_1) *is replaced by*

$$\begin{aligned}
&\inf_{y\in R^+} f_{1,0}(y) = \infty, \quad or\\
&\sum_{k=1}^{m} I_{1,0}(k)\phi_1(t_k)p(t_k) = \infty, \quad or\\
&\sum_{k=1}^{m} \overline{I}_{1,0}(k)\phi_1'(t_k)p(t_k) = \infty,
\end{aligned}$$

and

$$\begin{aligned}
&\inf_{y\in R^+} f_{1,\infty}(y) = \infty, \quad or\\
&\sum_{k=1}^{m} I_{1,\infty}(k)\phi_1(t_k)p(t_k) = \infty, \quad or\\
&\sum_{k=1}^{m} \overline{I}_{1,\infty}(k)\phi_1'(t_k)p(t_k) = \infty,
\end{aligned}$$

and Theorem 3.1 *is valid if* (H_2) *is replaced by*

$$\begin{aligned}
&\sup_{y\in R^+} f_2^0(y) = 0, \quad I_2^0(k) = 0, \quad \overline{I}_2^0(k) = 0, \quad or\\
&\sup_{y\in R^+} f_2^\infty(y) = 0, \quad I_2^\infty(k) = 0, \quad \overline{I}_2^\infty(k) = 0, \quad k = 1, 2 \ldots, m.
\end{aligned}$$

Then system $BVP(1.1)$ *has at least one positive solutions.*

Corollary 4.3. *Theorem* 3.1 *is valid if* (H_1) *is replaced by*

$$\begin{aligned}
&\sup_{y\in R^+} f_1^\infty(y) = 0, \qquad I_1^\infty(k) = 0, \qquad \overline{I}_1^\infty(k) = 0, \quad and\\
&\inf_{y\in R^+} f_{1,0}(y) = \infty, \quad or \quad \sum_{k=1}^{m} I_{1,0}(k) = \infty, \quad or \quad \sum_{k=1}^{m} \overline{I}_{1,0}(k) = \infty.
\end{aligned}$$

Theorem 3.1 *is valid if* (H_2) *is replaced by*

$$\sup_{y\in R^+} f_2^{\infty}(y)=0, \quad I_2^{\infty}(k)=0, \quad \overline{I}_2^{\infty}(k)=0, \quad and$$
$$\inf_{y\in R^+} f_{2,0}(y)=\infty, \quad or \ \sum_{k=1}^{m} I_{2,0}(k)=\infty, \quad or \ \sum_{k=1}^{m} \overline{I}_{2,0}(k)=\infty.$$

Then system $BVP(1.1)$ *has at least one positive solutions.*

Example 4.1. Now we give an example

$$\begin{cases} ((t-2)^8u'(t))'+(u^{\alpha}+u^{\beta})v=0, t\in J', 0<\alpha<1<\beta, \\ ((t-2)^8v'(t))'+(v^{\alpha}+v^{\beta})u=0, \\ -\Delta u'|_{t=t_k}=c_ku(t_k), c_k\geq 0, k=1,2,\ldots m, \\ -\Delta v'|_{t=t_k}=c'_kv(t_k), c'_k\geq 0, \\ \Delta u|_{t=t_k}=d_ku(t_k), d_k\geq 0, \\ \Delta v|_{t=t_k}=d'_kv(t_k), d'_k\geq 0, \\ u(0)=0, \quad v(0)=0, \\ \frac{12}{7}u(1)+\frac{1}{2}u'(1)=0, \quad \frac{12}{7}v(1)+\frac{1}{2}v'(1)=0. \end{cases} \tag{4.1}$$

Then $BVP(4.1)$ has at least two positive solutions (u_1,v_1) and (u_2,v_2), satisfying $0\leq \|(u_1,v_1)\|_{pc}\leq p\leq \|(u_2,v_2)\|_{pc}$. Where

$$\begin{gathered} \beta_1=0, \ \ \alpha_1=\beta_2=\frac{1}{2}, \ \ \alpha_2=\frac{12}{7}, \\ \sum_{k=1}^{m} G(t_k,t_k)c_k<\frac{1}{5}, \ \ \sum_{k=1}^{m} G(t_k,t_k)d_k<\frac{1}{5}. \end{gathered} \tag{4.2}$$

By (4.2),choose $\eta_1>0$ such that

$$2<\eta_1<5\left(1-\sum_{k=1}^{m} G(t_k,t_k)c_k-\sum_{k=1}^{m} G(t_k,t_k)d_k\right),$$
$$2<\eta_2<5\left(1-\sum_{k=1}^{m} G(t_k,t_k)c'_k-\sum_{k=1}^{m} G(t_k,t_k)d'_k\right).$$

Since

$$f_1(t,u,v)=(u^{\alpha}+u^{\beta})v, \quad f_2(t,v,u)=(v^{\alpha}+v^{\beta})u,$$

thus

$$\sup_{v\in R^+} f_{1,0}(v)=\infty, \quad \sup_{v\in R^+} f_{1,\infty}(v)=\infty,$$
$$\sup_{u\in R^+} f_{1,0}(u)=\infty, \quad \sup_{u\in R^+} f_{1,\infty}(u)=\infty,$$

so (H_1) holds. From $B(t,s)=\int_t^s \frac{d\tau}{p(\tau)}$, and $\rho=\alpha_2\beta_1+\alpha_1\beta_2+\alpha_1\alpha_2 B(0,1)$.

$$\begin{aligned} G(s,s) &= \frac{1}{\rho}\left(\beta_1+\alpha_1 B(0,s)\right)\left(\beta_2+\alpha_2 B(s,1)\right)\\ &\le \frac{\left(\beta_1+\alpha_1 B(0,1)\right)\left(\beta_2+\alpha_2 B(0,1)\right)}{\alpha_2\beta_1+\alpha_1\beta_2+\alpha_1\alpha_2 B(0,1)}\\ &= \frac{1}{6}. \end{aligned}$$

So $\int_0^1 G(s,s)ds \le \int_0^1 \frac{(\beta_1+\alpha_1 B(0,1))(\beta_2+\alpha_2 B(0,1))}{\alpha_2\beta_1+\alpha_1\beta_2+\alpha_1\alpha_2 B(0,1)}ds = \frac{1}{6}$. Let $\eta_{1,k}=c_k, \overline{\eta}_{1,k}=d_k, \eta_{2,k}=c_k', \overline{\eta}_{2,k}=d_k'$ such that $\eta_i, \eta_{i,k}, \overline{\eta}_{i,k} (i=1,2)$ satisfying

$$\eta_1+\sum_{k=1}^{m}(\eta_{1,k}+\overline{\eta}_{1,k})>0,$$

$$\eta_1\int_0^1 G(s,s)a_1(s)ds+\sum_{k=1}^{m}G(t_k,t_k)(\eta_{1,k}+\overline{\eta}_{1,k})<1,$$

$$\eta_2+\sum_{k=1}^{m}(\eta_{2,k}+\overline{\eta}_{2,k})>0,$$

$$\eta_2\int_0^1 G(s,s)a_2(s)ds+\sum_{k=1}^{m}G(t_k,t_k)(\eta_{2,k}+\overline{\eta}_{2,k})<1.$$

Let $p=1$ for every $0<u\le p, 0<v\le p$ we have

$$\begin{aligned} &f_1(t,u,v)=(u^\alpha+u^\beta)v\le p^\alpha+p^\beta=2<\eta_1 p,\\ &f_2(t,v,u)=(v^\alpha+v^\beta)u\le p^\alpha+p^\beta=2<\eta_2 p,\\ &I_{1,k}(u)=c_k u=\eta_{1,k}u\le \eta_{1,k}p, \quad I_{2,k}(u)=c_k' u=\eta_{2,k}u\le\eta_{2,k}p,\\ &\overline{I}_{1,k}(v)=d_k v=\overline{\eta}_{1,k})v\le\overline{\eta}_{1,k}p, \quad \overline{I}_{2,k}(v)=d_k' v=\overline{\eta}_{2,k})v\le\overline{\eta}_{2,k}p. \end{aligned}$$

So (H_3^{**}) hlods. From Theorem 3.3, the conclusion is established.

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References

[1] X.Y. Cheng and C.K. Zhong, *Existence of positive solutions for a second-order ordinary differential system*, J. Math. Anal. Appl., **312** (2005), 14–23.

[2] Weigao Ge and Jingli Ren, *New existence theorem of positive solutions for Sturm-Liouville boundary value problems*, Appl. Math. Comput., **148** (2004), 631–644.

[3] Dajun Guo, Nonlinear Functional Analysis, Shandong Sci. Tech. Jinan., (2004) (in Chinese).

[4] Dajun Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, (1988).

[5] Lili Hu, Lishan Liu and Yonghong Wu, *Positive solutions of nonlinear singular two-point boundary value problems for second-order impulsive differential equations*, Appl. Math. Comput., **196** (2008), 550–562.

[6] Y.H. Lee, *Multiplicity of positive radial solutions for multiparameter semilinear elliptic system on annulus*, J. Diff. Equa., **174** (2001), 420–441.

[7] Eun Kyoung Lee and Yong-Hoon Lee, *Multiple positive solutions of singular two point boundary value problems for second order implusive differential equations*, Appl. Math. Comput., **156** (2004), 745–759.

[8] Hairong Lian and Weigao Ge, *Existence of positive solutions for Sturm-Liouville boundary value problems on the half-line*, J. Math. Anal. Appl., **321** (2006), 781–792.

[9] Xiaoning Lin and Daqing Jiang, *Multiple positive solutions of Dirichlet boundary value problems for second order implusive differential equations*, J. Math. Anal. Appl., **321** (2006), 501–514.

[10] R.Y. Ma, *Existence of positive radial solutions for elliptic systems*, J. Math. Anal. Appl., **201** (1996), 375–386.

[11] Jingxian Sun and Guowei Zhang, *Positive solutions of singular nonlinear Sturm-Liouville problems*, Aata Math. Sinica., **48** (2005), 1196–1104 (in Chinese).

[12] Guotao Wang, Lihong Zhang and Guangxing Song, *Systems of first order impulsive functional differential equations with deviating arguments and nonlinear boundary conditions*, Nonl. Anal., **74** (2011), 974–982.

[13] Junyu Wang, Wenjie Gao and Zhongxin Zhang, *Nonexistence existence and multiplicity results for Sturm-Liouville boundary value problem*, Aata Math. Sinica., **48** (2005), 739–746 (in Chinese).

[14] Xi'an Xu, *Positive solutions for semipositone implusive differential boundary value problems*, Chinese Annals of Mathematics, **25A6** (2004), 717–724 (in Chinese).

[15] K. Yosida, Functional Analysis, fourth ed, Springer-Verlag, Berlin, 1978.

[16] Xinguang Zhang, Lishan Liu and Huichao Zou, *Eigenvalues of fourth-order singular Sturm-Liouville boundary value problems*, Nonl. Anal., **68** (2008), 384–392.

[17] Xinguang Zhang and Lishan Liu, *Initial value problems for nonlinear second order impulsive integro-differential equations of mixed type in Banach spaces*, Nonl. Anal., **64** (2006), 2562–2574.

[18] Fengling Zhu, Lishan Liu and Yonghong Wu, *Positive solutions for systems of a nonlinear fourth-order singular semipositone boundary value problems*, Appl. Math. Comput., **216** (2010), 448–457.