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# SOME COMMON FIXED POINT RESULTS IN PARTIAL METRIC SPACES FOR GENERALIZED RATIONAL TYPE CONTRACTION MAPPINGS

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**Abstract.** The purpose of this paper is to present a fixed point theorem using a contractive condition of rational type in the context of ordered partial metric spaces. We illustrate our results with the help of an example.

## 1. Introduction

One of the most important problems in mathematical analysis is to establish existence and uniqueness theorems for some integral and differential equations. Fixed point theory, in ordered metric spaces, plays a major role in solving such kind of problems. The first result in this direction was obtained by Ran and Reurings [17]. This one was extended for nondecreasing mappings by Nieto and Lopez [10, 11]. Meanwhile, Agarwal et al. [16] and O'Regan and Petrusel [4] studied some results for generalized contractions in ordered metric spaces. Then, many authors obtained fixed point results in ordered metric spaces. For some works in ordered metric spaces, we refer the reader to [1, 29]. Berinde [26, 27] initiated the concept of almost contraction and studied existence fixed point results for almost contraction in complete metric spaces. Later, many authors studied different types of almost contractions and studied fixed point results; for example, see [6, 28].

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In 1994, Matthews [19] introduced the concept of partial metric spaces and proved the Banach contraction principle in these spaces. Then, many authors obtained interesting results in partial metric spaces [14, 18, 30]. Very recently, Haghi et al. [18] proved that some fixed point theorems in partial metric spaces can be obtained from metric spaces.

Now we give preliminaries and basic definitions which are used throughout the paper.

## 2. Preliminaries

First, we start with some preliminaries on partial metric spaces. For more details, we refer the reader to [8, 9, 13, 15], [19]-[25].

**Definition 2.1.** Let $X$ be a nonempty set. A partial metric on $X$ is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(P1) $x = y \iff p(x,x) = p(x,y) = p(y,y)$,
(P2) $p(x,x) \le p(x,y)$,
(P3) $p(x,y) = p(y,x)$,
(P4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair $(X,p)$ such that $X$ is a nonempty set and $p$ is a partial metric on $X$. Each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ which has as a base the family of open p-balls $\{B_p(x,\varepsilon), x \in X, \ \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$for all $x \in X$ and $\varepsilon > 0$.

**Remark 2.2.** It is clear that, if $p(x,y) = 0$, then from (P1), (P2) and (P3), $x = y$. But if $x = y$; $p(x,y)$ may not be 0.
If $p$ is a partial metric on $X$, then the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y), \tag{2.1}$$

is a metric on $X$.

Now, we give an example of partial metric spaces as follows. Consider $X = \mathbb{R}^+$ with $p(x,y) = \max\{x,y\}$. Then $(\mathbb{R}^+, p)$ is a partial metric space. It is clear that $p$ is not a (usual) metric. Note that in this case $p^s(x,y) = |x - y|$.

**Definition 2.3.** Let $(X,p)$ be a partial metric space. Then

(i) a sequence $(x_n)$ in a partial metric space $(X,p)$ converges to a point $x \in X$ if and only if $p(x,x) = \lim\limits_{n\to+\infty} p(x,x_n)$;
(ii) a sequence $(x_n)$ in a partial metric space $(X,p)$ converges properly to a point $x \in X$ if and only if $p(x,x) = \lim\limits_{n\to+\infty} p(x_n,x_n) = \lim\limits_{n\to+\infty} p(x,x_n)$, if and only if $\lim\limits_{n\to+\infty} p^s(x,x_n) = 0$;

(iii) A sequence $(x_n)$ in a partial metric space $(X, p)$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n \to +\infty} p(x_n, x_m)$;

(iv) A partial metric space $(X, p)$ is said to be complete if every Cauchy sequence $(x_n)$ in $X$ converges to a point $x \in X$, that is $p(x, x) = \lim_{n,m \to +\infty} p(x_n, x_m)$.

**Lemma 2.4.** *Let $(X, p)$ be a partial metric space.*

(a) *$(x_n)$ is a Cauchy sequence in $(X, p)$ if and only if it is a Cauchy sequence in the metric space $(X, p^s)$;*

(b) *a partial metric space $(X, p)$ is complete if and only if the metric space $(X, p^s)$ is complete. Furthermore, $\lim_{n \to +\infty} p^s(x_n, x) = 0$ if and only if*

$$p(x, x) = \lim_{n \to +\infty} p(x_n, x) = \lim_{n,m \to +\infty} p(x_n, x_m).$$

**Definition 2.5.** Suppose that $(X_1, p_1)$ and $(X_2, p_2)$ are partial metrics. Denote $\tau(p_1)$ and $\tau(p_2)$ their respective topologies. We say $T : (X_1, p_1) \to (X_2, p_2)$ is continuous if both

$$T : (X_1, \tau(p_1)) \to (X_2, \tau(p_2)) \text{ and } T : (X_1, \tau(p_1^s)) \to (X_2, \tau(p_2^s))$$

are continuous.

**Proposition 2.6.** *Let $(X, p)$ be a partial metric space, partially ordered and $T : X \to X$ be a given mapping. We say that $T$ is continuous in $x_0 \in X$ if for every sequence $(x_n)$ in $X$, we have*

(a) *$x_n$ converges to $x_0$ in $(X, p)$ implies $Tx_n$ converges to $Tx_0$ in $(X, p)$.*

(b) *$x_n$ converges properly to $x_0$ in $(X, p)$ implies $Tx_n$ converges properly to $Tx_0$ in $(X, p)$.*

*If $T$ is continuous on each point $x_0 \in X$, then we say that $T$ is continuous on $X$.*

**Example 2.7.** ([3]) Consider $X = [0, \infty)$ endowed with the partial metric $p : X \times X \to [0, \infty)$ defined by $p(x, y) = \max\{x, y\}$ for all $x, y \geq 0$. Let $T : X \to X$ be a non-decreasing function. If $T$ is continuous with respect to the standard metric $d(x, y) = |x - y|$ for all $x, y \geq 0$, then $T$ is continuous with respect to the partial metric $p$.

**Definition 2.8.** ([12]) If $(X, \preceq)$ is a partially ordered set and $f, T : X \to X$, we say that $f$ is T-monotone nondecreasing if $x, y \in X, Tx \preceq Ty$ implies $fx \preceq fy$.

**Definition 2.9.** ([5], Weakly Compatible Mappings) Two mappings $f, T : X \to X$ are weakly compatible if they commute at their coincidence points, that is, if $fx = Tx$ for some $x \in X$ implies that $fTx = Tfx$.

**Definition 2.10.** ([3], Compatible Mappings) Let $(X, p)$ be a partial metric space and $f, T : X \to X$ are mappings of $X$ into itself. We say that the pair $(f, T)$ is partial compatible if the following conditions hold:

(b1) $p(x, x) = 0 \Longrightarrow p(Tx, Tx) = 0$,

(b2) $\lim_{n\to+\infty} p(fTx_n, Tfx_n) = 0$, whenever $(x_n)$ is a sequence in $X$ such that $fx_n \to t$ and $Tx_n \to t$ for some $t \in X$.

**Lemma 2.11.** ([3]) *Let $(X, p)$ be a partial metric space. The function $T : X \to X$ is continuous if given a sequence $(x_n)$ and $x \in X$ such that $p(x, x) = \lim_{n\to+\infty} p(x, x_n)$, then $p(Tx, Tx) = \lim_{n\to+\infty} p(Tx, Tx_n)$.*

**Definition 2.12.** ([2], $f$-Non Decreasing Mapping) Suppose $(X, \preceq)$ is a partially ordered set and $f, T : X \to X$ are mappings of $X$ to itself. $T$ is said to be f-non-decreasing if for $x, y \in X$, $fx \preceq fy$ implies $Tx \preceq Ty$.

## 3. Main Results

**Theorem 3.1.** *Let $(X, \preceq)$ be a partially ordered set and $p$ be a partial metric on $X$ such that $(X, p)$ is a complete partial metric space. Suppose that $T$ and $f$ are continuous self mappings on $X$, $T(X) \subseteq f(X)$, $T$ is a f-monotone non-decreasing mapping and*

$$\begin{aligned} &p(Tx, Ty) \\ &\leq \alpha\Big(\frac{p(fx, Tx)p(fy, Ty)}{p(fx, fy)}\Big) + \beta p(fx, fy) \\ &\quad + \gamma\big[p(fx, Tx) + p(fy, Ty)\big] + \delta\big[p(fx, Ty) + p(fy, Tx)\big] \end{aligned} \tag{3.1}$$

*for all $x, y \in X$ with $fx \succeq fy$, $fx \neq fy$ and for some $\alpha, \beta, \gamma, \delta \in [0, 1)$ with $\alpha + \beta + 2\gamma + 2\delta < 1$. If there exists $x_0 \in X$ such that $fx_0 \preceq Tx_0$, $T$ and $f$ are compatible, then $T$ and $f$ have a coincidence point.*

*Proof.* Since $T(X) \subseteq f(X)$, we can choose $x_1 \in X$ so that $fx_1 = Tx_0$. Since $Tx_1 \in f(X)$, there exists $x_2 \in X$ such that $fx_2 = Tx_1$. By induction, we can construct a sequence $(x_n)$ in $X$ such that $fx_{n+1} = Tx_n$ for every $n \geq 0$. Since $fx_0 \preceq Tx_0$, $Tx_0 = fx_1$, $fx_0 \preceq fx_1$, $T$ is a $f$-monotone non-decreasing mapping, $Tx_0 \preceq Tx_1$. Similarly $fx_1 \preceq fx_2$, $Tx_1 \preceq Tx_2$, $fx_2 \preceq$

$fx_3$. Continuing this process, we obtain

$$Tx_0 \preceq Tx_1 \preceq Tx_2 \preceq \cdots \preceq Tx_n \preceq Tx_{n+1} \preceq \cdots .$$

We suppose that $Tx_n \neq Tx_{n+1}$ for all $n$. If not, then $Tx_{n+1} = Tx_n$ for some $n$, $Tx_{n+1} = fx_{n+1}$, i.e., $T$ and $f$ have a coincidence point $x_{n+1}$, and so we have the result. Consider

$$\begin{aligned}
&p(Tx_{n+1}, Tx_n) \\
&\leq \alpha\left(\frac{p(fx_{n+1}, Tx_{n+1})p(fx_n, Tx_n)}{p(fx_{n+1}, fx_n)}\right) + \beta p(fx_{n+1}, fx_n) \\
&\quad + \gamma\big[p(fx_{n+1}, Tx_{n+1}) + p(fx_n, Tx_n)\big] \\
&\quad + \delta\big[p(fx_{n+1}, Tx_n) + p(fx_n, Tx_{n+1})\big] \\
&= \alpha\left(\frac{p(fx_{n+1}, fx_{n+2})p(fx_n, fx_{n+1})}{p(fx_{n+1}, fx_n)}\right) + \beta p(fx_{n+1}, fx_n) \\
&\quad + \gamma\big[p(fx_{n+1}, fx_{n+2}) + p(fx_n, fx_{n+1})\big] \\
&\quad + \delta\big[p(fx_{n+1}, fx_{n+1}) + p(fx_n, fx_{n+2})\big] \\
&\leq \alpha p(fx_{n+1}, fx_{n+2}) + \beta p(fx_{n+1}, fx_n) \\
&\quad + \gamma\big[p(fx_{n+1}, fx_{n+2}) + p(fx_n, fx_{n+1})\big] \\
&\quad + \delta\big[p(fx_{n+1}, fx_{n+1}) + p(fx_n, fx_{n+1}) \\
&\quad + p(fx_{n+1}, fx_{n+2}) - p(fx_{n+1}, fx_{n+1})\big] \\
&= \alpha p(fx_{n+1}, fx_{n+2}) + \beta p(fx_{n+1}, fx_n) \\
&\quad + \gamma\big[p(fx_{n+1}, fx_{n+2}) + p(fx_n, fx_{n+1})\big] \\
&\quad + \delta\big[p(fx_n, fx_{n+1}) + p(fx_{n+1}, fx_{n+2})\big] \\
&= (\alpha + \gamma + \delta)p(fx_{n+1}, fx_{n+2}) + (\beta + \gamma + \delta)p(fx_n, fx_{n+1}) \\
&= (\alpha + \gamma + \delta)p(Tx_n, Tx_{n+1}) + (\beta + \gamma + \delta)p(Tx_{n-1}, Tx_n)
\end{aligned} \tag{3.2}$$

which implies that

$$p(Tx_{n+1}, Tx_n) \leq \frac{(\beta + \gamma + \delta)}{1 - (\alpha + \gamma + \delta)} p(Tx_n, Tx_{n-1}).$$

Using mathematical induction we have

$$p(Tx_{n+1}, Tx_n) \leq \left(\frac{(\beta + \gamma + \delta)}{1 - (\alpha + \gamma + \delta)}\right)^n p(Tx_1, Tx_0).$$

Put $k = \frac{(\beta+\gamma+\delta)}{1-(\alpha+\gamma+\delta)} < 1$. Moreover, by (P4), we have, for $m \geq n$,

$$\begin{array}{rcl} p(Tx_m, Tx_n) & \leq & p(Tx_m, Tx_{m-1}) + p(Tx_{m-1}, Tx_n) - p(Tx_{m-1}, Tx_{m-1}) \\ & \leq & p(Tx_m, Tx_{m-1}) + p(Tx_{m-1}, Tx_n) \\ & \leq & p(Tx_m, Tx_{m-1}) + p(Tx_{m-1}, Tx_{m-2}) + p(Tx_{m-2}, Tx_n) \\ & & -p(Tx_{m-2}, Tx_{m-2}) \\ & \leq & p(Tx_m, Tx_{m-1}) + p(Tx_{m-1}, Tx_{m-2}) + p(Tx_{m-2}, Tx_n) \\ & \vdots & \\ & \leq & p(Tx_m, Tx_{m-1}) + p(Tx_{m-1}, Tx_{m-2}) + \cdots \\ & & +p(Tx_{n+2}, Tx_{n+1}) + p(Tx_{n+1}, Tx_n) \\ & \leq & \left(k^{m-1} + k^{m-2} + \cdots + k^{n+1} + k^n\right) \cdot p(Tx_1, Tx_0) \\ & \leq & \left(\frac{k^n}{1-k}\right) \cdot p(Tx_1, Tx_0). \end{array} \tag{3.3}$$

Letting $m, n \to +\infty$ in (3.3), we get

$$\lim_{m,n\to+\infty} p(Tx_m, Tx_n) = 0. \tag{3.4}$$

By (2.1), we have

$$p^s(Tx_m, Tx_n) \leq 2p(Tx_m, Tx_n). \tag{3.5}$$

Taking $m, n \to +\infty$ in (3.5) and using (3.4), we get that

$$\lim_{m,n\to+\infty} p^s(Tx_m, Tx_n) = 0. \tag{3.6}$$

Then $(Tx_n)$ is a Cauchy sequence in the metric space $(X, p^s)$. Since $(X, p)$ is complete, from Lemma 2.4, $(X, p^s)$ is a complete metric space. Then, there exists $u \in X$ such that

$$\lim_{n\to+\infty} p^s(Tx_n, u) = \lim_{n\to+\infty} p^s(fx_{n+1}, u) = 0. \tag{3.7}$$

On the other hand, we have

$$p^s(Tx_n, u) = 2p(Tx_n, u) - p(Tx_n, Tx_n) - p(u, u).$$

Letting $n \to +\infty$ in the above equation, using (3.4) and (3.7), we get

$$\lim_{n\to+\infty} p(Tx_n, u) = \frac{1}{2} p(u, u). \tag{3.8}$$

Furthermore, by (p2) ; we have $p(u, u) \leq p(u, Tx_n)$ for all $n \in \mathbb{N}$. On letting $n \to +\infty$, we get that

$$p(u, u) \leq \lim_{n\to+\infty} p(u, Tx_n). \tag{3.9}$$

Using (3.8), (3.9) and from Lemma 2.4 we have

$$\lim_{n\to+\infty} p(u, Tx_n) = \lim_{n,m\to+\infty} p(Tx_n, Tx_m) = p(u, u) = 0. \tag{3.10}$$

Now, since $T$ is continuous, from (3.10) and using Lemma 3.1, we get

$$\lim_{n\to+\infty} p(T(Tx_n), Tu) = p(Tu, Tu). \tag{3.11}$$

Using the triangular inequality, we obtain

$$p(Tu, fu) \leq p\big(Tu, T(fx_n)\big) + p\big(T(fx_n), f(Tx_n)\big) + p\big(f(Tx_n), fu\big). \tag{3.12}$$

Letting $n \to +\infty$ in the above inequality, we get that $p(fu, Tu) = 0$. By (P1) and (P2), we have $fu = Tu$. This completes the proof. □

In what follows, we prove that Theorem 3.1 is still valid for $T$ not necessarily continuous, assuming the following hypothesis in $X$ :

$$\begin{array}{c} if\ (x_n)\ \text{ is a nondecreasing sequence in } X \text{such that} \\ x_n \to x, \text{then } x = \sup\{x_n\} \ \text{ for all}\, n \in \mathbb{N}. \end{array} \tag{3.13}$$

**Theorem 3.2.** *Let $(X, \preceq)$ be a partially ordered set and $p$ be a partial metric on $X$ such that $(X, p)$ is a complete partial metric space. Assume that $X$ satisfies* (3.13). *Let $T : X \to X$ be a nondecreasing mapping such that*

$$\begin{aligned} &p(Tx, Ty) \\ &\leq \alpha\left(\frac{p(fx, Tx)p(fy, Ty)}{p(fx, fy)}\right) + \beta p(fx, fy) \\ &\quad + \gamma\big[p(fx, Tx) + p(fy, Ty)\big] + \delta\big[p(fx, Ty) + p(fy, Tx)\big] \end{aligned} \tag{3.14}$$

*for all $x, y \in X$ with $x \succeq y$, $x \neq y$ and for some $\alpha, \beta, \gamma, \delta \in [0, 1)$ with $\alpha + \beta + 2\gamma + 2\delta < 1$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has a fixed point.*

*Proof.* Following the proof of Theorem 3.1 we have $(Tx_n)$ is a Cauchy sequence and so is $(fx_n)$. Since $f(X)$ is closed and $X$ is complete, $\lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} fx_n = fu$ for some $u \in X$. Notice that the sequences $(Tx_n)$ and $(fx_n)$ are non-decreasing. Then from our assumptions we have $Tx_n \preceq fu$ and $fx_n \preceq fu$ for all $n$. Keeping in mind that $T$ is f - monotone non-decreasing we get $Tx_n \preceq Tu$ for all $n$. Letting $n$ to $+\infty$ we obtain $fu \preceq Tu$. Suppose $fu \prec Tu$ (otherwise we are done). Construct a sequence $(u_n)$ as $u_0 = u$ and $fu_{n+1} = Tu_n$ for all $n$. A similar argument as in the proof of Theorem 3.1 yields $(fu_n)$ is a non-decreasing sequence and $\lim_{n\to+\infty} fu_n = \lim_{n\to+\infty} Tu_n = fv$ for some $v \in X$. From our assumptions it follows that $\sup_n fu_n \preceq fv$ and $\sup_n Tu_n \preceq fv$. Notice that

$$fx_n \preceq fu \prec fu_1 \preceq \ldots \preceq fu_n \preceq \ldots \preceq fv.$$

We distinguish two cases:
**Case 1.** Suppose there is $n_0 \geq 1$ with $fx_{n_0} = fu_{n_0}$. Then

$$fx_{n_0} = fu = fu_{n_0} = fu_1 = Tu.$$

We are done.
**Case 2.** Suppose $fu_n \neq fx_n$ for all $n \geq 1$. Then from the contraction assumption we obtain

$$\begin{aligned} p(fx_{n+1}, fu_{n+1}) &= p(Tx_n, Tu_n) \\ &\leq \alpha\left(\frac{p(fx_n, Tx_n)p(fu_n, Tu_n)}{p(fx_n, fu_n)}\right) + \beta p(fx_n, fu_n) \\ &\quad + \gamma\big[p(fx_n, Tx_n) + p(fu_n, Tu_n)\big] \\ &\quad + \delta\big(p(fx_n, Tu_n) + p(fu_n, Tx_n)\big). \end{aligned}$$

Letting $n$ to $+\infty$ we get $p(fu, fv) \leq (\beta + 2\delta)p(fu, fv)$, which implies that $fu = fv$ since $\beta + 2\delta < 1$. Hence $fu = fv = fu_1 = Tu$, the proof is complete. □

If $\beta = \gamma = \delta = 0$, in Theorem 3.1 (or Theorem 3.2), we obtain the following fixed point theorem in ordered partial complete metric spaces.

**Corollary 3.3.** ([7]) *Let $(X, \preceq)$ be a partially ordered set and $p$ be a partial metric on $X$ such that $(X, p)$ is a complete partial metric space. Let $T : X \longrightarrow X$ be a nondecreasing mapping such that*

$$p(Tx, Ty) \quad \leq \quad \alpha\big(\tfrac{p(x,x)p(y,y)}{p(x,y)}$$

*for all $x, y \in X$ with $x \succeq y$, $x \neq y$ and for some $\alpha \in [0, 1)$ with $\alpha < 1$. Suppose also that either $T$ is continuous or $X$ satisfies condition* (3.11)*. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has a fixed point.*

If $\gamma = 0 = \delta$, in Theorem 3.1 (or Theorem 3.2), we obtain the following fixed point theorem in ordered partial complete metric spaces.

**Corollary 3.4.** ([7]) *Let $(X, \preceq)$ be a partially ordered set and $p$ be a partial metric on $X$ such that $(X, p)$ is a complete partial metric space. Let $T : X \to X$ be a nondecreasing mapping such that*

$$p(Tx, Ty) \quad \leq \quad \alpha\big(\tfrac{p(x,x)p(y,y)}{p(x,y)}\big) + \beta p(x, y)$$

*for all $x, y \in X$ with $x \succeq y$, $x \neq y$ and for some $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$. Suppose also that either $T$ is continuous or $X$ satisfies condition* (3.11)*. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has a fixed point.*

If $\beta = 0$ in Theorem 3.1 (or Theorem 3.2), we obtain the following fixed point theorem in ordered partial complete metric spaces.

**Corollary 3.5.** *Let $(X, \preceq)$ be a partially ordered set and $p$ be a partial metric on $X$ such that $(X, p)$ is a complete partial metric space. Let $T : X \to X$ be a nondecreasing mapping such that*

$$p(Tx, Ty) \le \alpha \left( \frac{p(x,x)p(y,y)}{p(x,y)} \right) + \gamma\big[p(x,Tx) + p(y,Ty)\big] + \delta\big[p(x,Ty) + p(y,Tx)\big]$$

*for all $x, y \in X$ with $x \succeq y$, $x \neq y$ and for some $\alpha, \gamma, \delta \in [0,1)$ with $\alpha + 2\gamma + 2\delta < 1$. Suppose also that either $T$ is continuous or $X$ satisfies condition (3.11). If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has a fixed point.*

**Example 3.6.** Let $X = [0, +\infty[$ endowed with the usual partial metric $p$ defined by $p : X \times X \to [0, +\infty[$ with $p(x, y) = \max\{x, y\}$. We give the partial order on $X$ by

$$x \preceq y \Longleftrightarrow p(x,x) = p(x,y) \Longleftrightarrow x = \max\{x,y\} \Longleftarrow y \le x.$$

It is clear that $(X, \preceq)$ is ordered. The partial metric space $(X, p)$ is complete because $(X, p^s)$ is complete. Indeed, for any $x, y \in X$,

$$p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y) = 2\max\{x,y\} - (x+y) = |x-y|,$$

Thus, $(X, p^s) = ([0, +\infty[, |.|)$ is the usual metric space, which is complete. Again, we define $T(x) = 0$ for all $x \in X$. Then $p(Tx, Ty) = 0$ Any $x, y \in X$ are comparable, so for example we take $y \preceq x$, then $p(x,x) = x$, $p(x,y) = y$, so $0 \le x < y$.

$$\begin{aligned}&\alpha\big(\tfrac{p(x,Tx).p(y,Ty)}{p(x,y)}\big) + \beta p(x,y) + \gamma\big[p(x,Tx) + p(y,Ty)\big] + \delta\big[p(x,Ty) + p(y,Tx)\big]\\ &= \alpha(x) + \beta(y) + \gamma[x+y] + \delta[x+y] = (\alpha+\gamma+\delta)x + (\beta+\gamma+\delta)y\\ &> 0.\end{aligned}$$

Hence the inequality holds if $\beta + \gamma + \delta \neq 0$. On the other hand, it is obvious that $T$ is a non-decreasing mapping with respect to $\preceq$ and there exists $x_0 = 0$ such that $x_0 \preceq Tx_0$ and 0 is a fixed point of $T$.

**Example 3.7.** Let $X = \{0, 1, 2\}$ endowed with the partial metric $p$ given by $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. It is clear that $(X, p)$ is a complete partial metric space. We give the partial order on $X$ by

$$x \preceq y \Longleftrightarrow p(x,x) = p(x,y) \Longleftrightarrow x = \max\{x,y\} \Longleftarrow y \le x.$$

It is clear that $(X, \preceq)$ is ordered. Define $T0 = T1 = 0$, $T2 = 1$, $\alpha = \frac{1}{8}$, $\beta = \frac{1}{4}$ and $\gamma = \delta = \frac{1}{7}$ we have: $\alpha + \beta + 2\gamma + 2\delta = \frac{53}{56} < 1$.

Any $x, y \in X$ are comparable, so for example we take $y \preceq x$, then $p(x,x)=x$, $p(x,y)=y$, so $0 \leq x < y$. We have

$$\begin{aligned}
p(T0,T1)=0 &\leq \tfrac{1}{8}\big(\tfrac{p(0,T0).p(1,T1)}{p(0,1)}\big)+\tfrac{1}{4}p(0,1)\\
&\quad+\tfrac{1}{7}\big[p(0,T0)+p(1,T1)\big]+\tfrac{1}{7}\big[p(0,T1)+p(1,T0)\big]\\
&= \tfrac{1}{4}+\tfrac{1}{7}+\tfrac{1}{7}\\
&= \tfrac{15}{28},\\
p(T0,T2)=1 &\leq \tfrac{1}{8}\big(\tfrac{p(0,T0).p(2,T2)}{p(0,2)}\big)+\tfrac{1}{4}p(0,2)\\
&\quad+\tfrac{1}{7}\big[p(0,T0)+p(2,T2)\big]+\tfrac{1}{7}\big[p(0,T2)+p(2,T0)\big]\\
&= \tfrac{2}{4}+\tfrac{2}{7}+\tfrac{3}{7}\\
&= \tfrac{17}{14},\\
p(T1,T2)=1 &\leq \tfrac{1}{8}\big(\tfrac{p(1,T1).p(2,T2)}{p(1,2)}\big)+\tfrac{1}{4}p(1,2)\\
&\quad+\tfrac{1}{7}\big[p(1,T1)+p(2,T2)\big]+\tfrac{1}{7}\big[p(1,T2)+p(2,T1)\big]\\
&= \tfrac{1}{8}+\tfrac{2}{4}+\tfrac{3}{7}+\tfrac{3}{7}\\
&= \tfrac{83}{56}.
\end{aligned}$$

Hence the inequality holds. On the other hand, it is obvious that $T$ is a trivially continuous and nondecreasing mapping with respect to $\preceq$ and there exists $x_0 = 0$ such that $x_0 \preceq Tx_0$ and 0 is a fixed point of $T$.

## 4. Application

The aim of this section is to apply our new results to mappings involving contractions of integral type. For this purpose, denote by $\Lambda$ the set of function $\gamma : [0,+\infty) \longrightarrow [0,+\infty)$ satisfying the following hypotheses:

$(h_1)$ $\gamma$ is a Lebesgue-integrable mapping on each compact of $[0,+\infty)$.

$(h_2)$ For every $\varepsilon > 0$, we have

$$\int_0^{\varepsilon} \gamma(s)ds > 0.$$

We have the following result.

**Corollary 4.1.** *Let $(X,\preceq)$ be a partially ordered set and let $p$ be a partial metric on $X$ such that $(X,p)$ is complete. Let $T : X \longrightarrow X$ be a continuous and nondecreasing mapping such that*

$$\int_0^{p(Tx,Ty)} \psi(s)\,ds \;\leq\; \alpha \int_0^{\frac{p(x,Tx)p(y,Ty)}{p(x,y)}} \psi(s)\,ds + \beta \int_0^{p(x,y)} \psi(s)\,ds + \gamma \int_0^{p(x,Tx)+p(y,Ty)} \psi(s)\,ds + \delta \int_0^{p(x,Ty)+p(y,Tx)} \psi(s)\,ds$$

*for all $x, y \in X$ with $x \succeq y$, $x \neq y$, $\psi \in \Lambda$ and for some $\alpha, \beta, \gamma, \delta \in [0, 1)$ with $\alpha + \beta + 2\gamma + 2\delta < 1$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then $T$ has a fixed point.*

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