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TRIGONOMETRIC GENERATING FUNCTION METHODS AND THE SIGN FUNCTION

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Abstract. The sign function $f(x) = \mathrm{sgn}(x)$ has many applications. It's approximations by polynomials and rational functions have been studied by several authors. Truncated Fourier series and trigonometric interpolants converge slowly for functions with jumps in value, the nonlinear Fourier approximants with better convergence based on trigonometric generating functions are developed. The convergence and error terms are obtained.

1. Introduction

The well-known sign function $f(x) = \mathrm{sgn}(x)$ has many applications. For example, Borici et. al. [2] and other references therein discussed the rational approximation and continued fraction expansion of the sign function to obtain the overlap lattice Dirac operator. Koyama et. al. [7] used an integral representation of the sign function to analyze the recalling processes of associative memory. Lai [8] used the sign function together with step function to establish the equation of the middle surface of a simply-supported truncated hip roof. In this paper, we discuss the Fourier approximants to the sign function, which we believe, may provide a good tool for the application of the function. Due to the periodicity, we modify $\mathrm{sgn}(x)$ to

$$s(x) = \begin{cases} -1 & -1 < x < 0 \\ 1 & 0 < x < 1 \\ 0 & x = 0, -1, 1. \end{cases} \tag{1.1}$$

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It is an odd function and its Fourier expansion is

$$s(x) = \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}[1-(-1)^k]\sin k\pi x, \quad x \in [1,\, 1]. \tag{1.2}$$

Although the series converges to $s(x)$ for any $x \in [-1,\, 1]$, the convergence rate is very slow. For example, at $x = 0.5$, the series requires over 522,000 terms to yield an approximate value to $s(0.5) = 1$, such that the error is not greater than 0.3×10^{-5}. We seek approximants for this function that converge faster. Due to the popularity of Fourier series, the nonlinear Fourier approximants based on trigonometric generating functions are chosen. Nonlinear approximation methods by using generating functions have been studied by many mathematicians and physicists, see for example [1, 3-6, 9, 10, 12]. For a detailed description of the generating function method, the reader is referred to [9]. In the most basic form, the approximants to a given function $f(x)$ are defined by

$$F(n;x) = \sum_{j=1}^{n} a_j v(x,t_j), \quad n = 1, 2, \ldots, \tag{1.3}$$

where

$$v(x,t) = \sum_{k=0}^{\infty} u_k(x)t^k \tag{1.4}$$

is a generating function for functions $\{u_k(x)\}$, $\{a_j\}$ and $\{t_j\}$ are parameters to be determined by the following agreement conditions

$$H_k[F(n;x)] = H_k[f(x)], \quad k = 0, 1, \ldots, 2n-1. \tag{1.5}$$

The operators $\{H_k[\cdot]\}$ satisfy

$$H_k[u_m(x)] = \delta_{mk}, \quad m, k = 0, 1, 2, \ldots,$$

where δ_{mk} is the Kronecker delta. That is, $\{u_k(x)\}$ and $\{H_k[\cdot]\}$ are orthonormal. Setting $f_k = H_k[f(x)]$ for $k = 0, 1, \ldots$, substituting (1.4) into (1.3) and exchanging the order of summation, the agreement conditions in (1.5) become

$$\sum_{j=1}^{n} a_j t_j^k = f_k, \quad k = 0, 1, \ldots, 2n-1. \tag{1.6}$$

In principle, one solves this system to obtain the values of the parameters for use in (1.3). The Prony's method described in [6] could be used. First one solves the system

$$c_0 f_k + c_1 f_{k+1} + \cdots + c_{n-1} f_{k+n-1} + f_{k+n} = 0, \quad k = 0, 1, \ldots, n-1, \tag{1.7}$$

for $c_0, c_1, \ldots, c_{n-1}$, then from solving all the roots of the polynomial

$$p(t) = c_0 + c_1 t + \cdots + c_{n-1} t^{n-1} + t^n,$$

one gets the nonlinear parameters $t_1, t_2, \dots, t_n$, and finally, the linear parameters $a_1, a_2, \dots, a_n$ could be solved from (1.6). If $\{t_j\}$ have multiplicities $\{r_j\}$, [9] modified the approximants $\{F(n;x)\}$ to take the form

$$F(n;x) = \sum_{j=1}^{s} \sum_{i=0}^{r_j-1} \frac{a_{ji}}{i!} \frac{\partial^i}{\partial t^i} v(x,t)\Bigg|_{t=t_j}, \quad n = 1, 2, \dots, \tag{1.8}$$

where $\sum_{j=1}^{s} r_j = n$.

In this paper, we develop the nonlinear approximants based on several trigonometric generating functions to the (modified) sign function $s(x)$ as defined in (1.1). In general, such approximants may not converge, and when they do the convergence may be very difficult to prove. But for our cases here, with the help of Gaussian type quadratures, the convergence and error terms are obtained.

2. Preliminaries

In this section, we introduce the trigonometric generating functions and the nonlinear approximants based on them.

Definition 2.1. *We define the following generating functions,*

$$\mathscr{G}_1(x,s) = \frac{(1-s^2)/2}{1-2s\cos(\pi x)+s^2} = \frac{1}{2} + \sum_{k=1}^{\infty} \cos(k\pi x) s^k, \quad |s|<1, \tag{2.1}$$

$$\mathscr{G}_2(x,t) = \frac{t\sin(\pi x)}{1-2t\cos(\pi x)+t^2} = \sum_{k=1}^{\infty} \sin(k\pi x) t^k, \quad |t|<1, \tag{2.2}$$

$$\mathscr{E}_1(x,s) = e^{s\cos(\pi x)}\cos(s\sin(\pi x)) - \frac{1}{2} = \frac{1}{2} + \sum_{k=1}^{\infty} \cos(k\pi x)\frac{s^k}{k!}, \tag{2.3}$$

$$\mathscr{E}_2(x,t) = e^{t\cos(\pi x)}\sin(t\sin(\pi x)) = \sum_{k=1}^{\infty} \sin(k\pi x)\frac{t^k}{k!}. \tag{2.4}$$

We call the first two geometric generating functions and the last two exponential generating functions.

Remark 2.2. The parameters s and t may be regarded as complex. If s and x are real, then $\mathscr{G}_1(x,s)$ and $\mathscr{G}_2(x,s)$ are the real and imaginary part of $\dfrac{1}{1-se^{i\pi x}} - \dfrac{1}{2}$, respectively, and $\mathscr{E}_1(x,s)$ and $\mathscr{E}_2(x,s)$ are the real and imaginary part of $e^{se^{i\pi x}} - \dfrac{1}{2}$, respectively.

The expressions that approximate a function $f(x)$ and use these generating functions take the form

$$Fg(n;x) = \sum_{j=1}^{n+1} a_j \mathit{vg}_1(x, s_j) + \sum_{j=1}^{n} b_j \mathit{vg}_2(x, t_j), \quad n = 1, 2, \ldots, \tag{2.5}$$

and

$$Fe(n;x) = \sum_{j=1}^{n+1} a_j \mathit{ve}_1(x, s_j) + \sum_{j=1}^{n} b_j \mathit{ve}_2(x, t_j), \quad n = 1, 2, \ldots, \tag{2.6}$$

where $\{a_j\}$, $\{b_j\}$, $\{s_j\}$ and $\{t_j\}$ are $4n+2$ parameters to be determined by the following agreement conditions. Let

$$\left.\begin{aligned} Hg_k(h(x)) &= \int_{-1}^{1} h(x)\cos(k\pi x)dx, \qquad He_k(h(x)) = k! \int_{-1}^{1} h(x)\cos(k\pi x)dx, \\ Jg_k(h(x)) &= \int_{-1}^{1} h(x)\sin(k\pi x)dx, \qquad Je_k(h(x)) = k! \int_{-1}^{1} h(x)\sin(k\pi x)dx, \end{aligned}\right\} \tag{2.7}$$

then the agreement conditions are defined by

$$Hg_k(f) = Hg_k(Fg) \ \ (\text{or } He_k(f) = He_k(Fe)), \quad k = 0, 1, \ldots, 2n+1,$$

and

$$Jg_k(f) = Jg_k(Fg) \ \ (\text{or } Je_k(f) = Je_k(Fe)), \quad k = 1, 2, \ldots, 2n.$$

Setting $f_k = Hg_k(f)$ (or $f_k = He_k(f)$) and $p_k = Jg_k(f)$ (or $p_k = Je_k(f)$), the above agreement conditions become

$$\left.\begin{aligned} \sum_{j=1}^{n+1} a_j s_j^k &= f_k, \quad k = 0, 1, \ldots, 2n+1, \\ \sum_{j=1}^{n} b_j t_j^k &= p_k, \quad k = 1, 2, \ldots, 2n. \end{aligned}\right\} \tag{2.8}$$

These two sets of nonlinear algebraic equations are solved for the parameters by Prony's method.

The $\{s_j\}$ and $\{t_j\}$ are roots of polynomials. If any has multiplicity greater than one, the form of $\{Fg(n;x)\}$ or $\{Fe(n;x)\}$ must be modified to the form indicated in (1.8). The approximants $\{Fg(n;x)\}$ in (2.5) are based on the geometric generating functions defined in (2.1) and (2.2), and require that the magnitudes of $\{s_j\}$ and $\{t_j\}$ are less than one. If any of these $\{s_j\}$ and $\{t_j\}$ have magnitudes that violate this condition, we can use the magnitude reduction method described in [4] to reduce the magnitudes of these $\{s_j\}$ and $\{t_j\}$ until they are less than one.

3. Approximants to the Sign Function

In this section, we discuss the nonlinear Fourier approximants to the (modified) sign function $f(x) = s(x)$ defined in (1.1) and give the convergence and error terms. Such approximants are based on the generating functions introduced in the previous section and their modifications.

First, we consider nonlinear approximants using geometric generating functions. Since $s(x)$ is an odd function, its Fourier expansion contains only sine terms as shown in (1.2). If it is approximated by $\{Fg(n;x)\}$ in (2.5), the approximants take the form

$$Fg(n;x) = \sum_{j=1}^{n} b_j \mathcal{v}g_2(x, t_j), \quad n = 1, 2, \ldots,$$

and the agreement conditions (2.8) become

$$\sum_{j=1}^{n} b_j t_j^k = p_k, \quad k = 1, 2, \ldots, 2n.$$

where

$$p_k = Jg_k(f) = \frac{2}{k\pi}\left[1 - (-1)^k\right], \quad k = 1, 2, \ldots .$$

The Prony's method for solving this system encounters

$$c_0 p_k + c_1 p_{k+1} + \cdots + c_{n-1} p_{k+n-1} + p_{k+n} = 0, \quad k = 1, 2, \ldots, n.$$

When $n \geq 3$, the system is singular since the first and the third rows of the coefficient matrix are always the same. In order to conquer the problem, we introduce the following modified generating function for sine terms.

$$\begin{aligned} v_g(x,t) &= \frac{2}{\pi\, t} \mathcal{v}g_2(x,t) \\ &= \frac{2}{\pi} \frac{\sin \pi x}{1 - 2t\cos \pi x + t^2} \\ &= \frac{2}{\pi} \sum_{k=0}^{\infty} \sin{(k+1)\pi x}\, t^k, \quad |t| < 1, \end{aligned} \tag{3.1}$$

or if we regard t as real

$$v_g(x,t) = \frac{2}{\pi} \operatorname{Im}\left[\frac{1}{t(1 - te^{i\pi x})}\right]. \tag{3.2}$$

Throughout this section, we modify Jg_k in (2.7) to have k begin at zero.

We approximate $s(x)$ by the functions

$$F_g(n;x) = \sum_{j=1}^{n} b_j v_g(x, t_j), \quad n = 1, 2, \ldots . \tag{3.3}$$

The following two lemmas are needed when we prove the convergence of the approximants.

Lemma 3.1. ([11]) *If* $w(x) \geq 0$ *on* $[a, b]$ *(which may be an infinite interval), then a Gauss formula*

$$\int_a^b w(x)f(x)dx \simeq \sum_{j=1}^{n} A_j f(t_j)$$

which has degree $2n-1$ *is a Riemann-Stieltjes sum.*

Lemma 3.2. ([9]) *Let* $t_1,\ t_2,\ \ldots,\ t_N$ *be real but not necessarily distinct values. Let* C *be a contour in the complex* z*-plane that encloses the* $\{t_j\}$ *in the counterclockwise sense, but does not surround any nonanalytic points of* $v(z)$ *and let* $P(z) = \prod_{j=1}^{N}(z-t_j)$*. There exists a real number* τ *with* $\min(t_1, \ldots, t_N) \leq \tau \leq \max(t_1, \ldots, t_N)$ *such that*

$$\frac{1}{2\pi i}\oint_C \frac{v(z)}{P(z)}dz = \frac{1}{(N-1)!}\left.\frac{d^{N-1}v(t)}{dt^{N-1}}\right|_{t=\tau}. \tag{3.4}$$

The following theorem gives the convergence and error terms of the approximants $\{F_g(n; x)\}$.

Theorem 3.3. *The nonlinear approximants* $\{F_g(n; x)\}$ *to* $s(x)$ *on* $[-1,\ 1]$ *are the corresponding Gaussian-Legendre quadratures applied to the integral*

$$s_g(x) = \int_{-1}^{1} v_g(x,t)dt. \tag{3.5}$$

Moreover,

$$s_g(x) = s(x) \quad \text{and} \quad \lim_{n\to\infty} F_g(n; x) = s(x), \quad x \in [-1,\ 1]. \tag{3.6}$$

Furthermore, for each positive integer n *and each* $x \in [-1,\ 1]$*, there exists a point* $\xi \in (-1,\ 1)$*, such that*

$$E_n(x) = s(x) - F_g(n; x) = \frac{1}{n!}\int_{-1}^{1} Q_n(t)\left.\frac{\partial^n}{\partial z^n}v_g(x,z)\right|_{z=\xi} dt,$$

where $Q_n(t) = \prod_{j=1}^{n}(t-t_j)$ *and* $\{t_j\}$ *are the* n *roots of the Legendre polynomial of order* n.

Proof. The conditions of agreement are

$$\sum_{j=1}^{n} b_j t_j^k = J_{gk}(f) = p_k, \quad k = 0, 1, \ldots, 2n-1,$$

where the operators $\{J_{gk}\}$ are defined by

$$J_{gk}(h(x)) = \frac{\pi}{2}\int_{-1}^{1} h(x)\sin{(k+1)\pi x}dx, \quad k = 0, 1, 2, \ldots,$$

and satisfy

$$J_{gk}(v_g(x,t)) = t^k, \quad k = 0, 1, 2, \ldots,$$

while

$$\begin{aligned} p_k &= J_{gk}(s(x)) = \pi\int_0^1 \sin{(k+1)\pi x}dx \\ &= \frac{1}{k+1}[1-(-1)^{k+1}] \\ &= \int_{-1}^{1} t^k dt, \qquad k = 0, 1, 2, \ldots . \end{aligned}$$

This gives the first part of the theorem. As for the second part, since the series (3.1) is uniformly convergent in t on any interval $[a,\, b] \subset (-1,\, 1)$, by the well-known convergence theorem for Fourier series, we have

$$\begin{aligned} \int_{-1}^{1} v_g(x,t)dt &= \frac{2}{\pi}\sum_{k=0}^{\infty}\sin{(k+1)\pi x}\int_{-1}^{1} t^k dt \\ &= \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1-(-1)^k}{k}\sin k\pi x \\ &= s(x), \quad x \in [-1,\, 1]. \end{aligned}$$

Together with Lemma 3.1, we get (3.6).

We turn to prove the error expression. From the equality $s(x) = s_g(x),\ x \in [-1, 1]$ and (21) in [9]], it follows that

$$E_n(x) = s_g(x) - F_g(n;x) = \frac{1}{2\pi i}\oint_C \int_0^1 \frac{v_g(x,z)Q_n(t)}{(z-t)Q_n(z)}dtdz,$$

where C is a contour in the complex z-plane which encloses, in the counter-clockwise sense, $t, t_1, \ldots, t_n$, and $Q_n(t) = \prod_{j=1}^{n}(t-t_j)$. Replacing f_k in (1.7) by p_k and substituting $p_k = \int_{-1}^{1} t^k dt$, $k = 0, 1, \ldots, n-1$ into (1.7), we obtain

$$\int_{-1}^{1} Q_n(t)t^k dt = 0, \quad k = 0, 1, \ldots, n-1,$$

which states that $\{t_j\}$ are the n roots of the Legendre polynomial of degree n. Therefore, all the parameters $\{t_j\}$ are real, distinct from each other and all lie in $[-1,\ 1]$. When accompanied by Lemma 3.2, we get the expression for $E_n(x)$. The proof is complete. □

Since the Fourier expansion of $s(x)$ contains only the terms with odd k, it would then appear that one could be much more efficient if the generating function contained only terms of the correct parity. For this purpose, we introduce the following generating function for

$$\left\{\frac{2}{\pi}\sin\pi x, \frac{2}{\pi}\sin 3\pi x, \ldots, \frac{2}{\pi}\sin(2k+1)\pi x, \ldots\right\}.$$

$$\begin{aligned} \mathscr{w}_g(x,t) &= \frac{1}{2}[v_g(x,\sqrt{t}) + v_g(x,-\sqrt{t})] \\ &= \frac{2}{\pi}\frac{(1+t)\sin\pi x}{1-2t\cos 2\pi x + t^2} \\ &= \frac{2}{\pi}\sum_{k=0}^{\infty}\sin(2k+1)\pi x\, t^k, \quad |t|<1, \end{aligned} \tag{3.7}$$

or if we regard t as real

$$\mathscr{w}_g(x,t) = \frac{2}{\pi}\,\mathrm{Im}\left(\frac{1}{e^{-i\pi x} - te^{i\pi x}}\right). \tag{3.8}$$

We now consider the following approximants to $s(x)$,

$$Fo_g(n;x) = \sum_{j=1}^{n} b_j \mathscr{w}_g(x,t_j), \quad n = 1, 2, \ldots. \tag{3.9}$$

We have

Theorem 3.4. *The nonlinear approximants* $\{Fo_g(n;x)\}$ *to* $s(x)$ *on* $[-1,\,1]$ *are the corresponding Gaussian-Jacobi quadratures, with the weight* $w(t) = t^{-\frac{1}{2}}$*, applied to the integral*

$$\mathscr{s}_g(x) = \int_0^1 w(t)\mathscr{w}_g(x,t)dt. \tag{3.10}$$

Moreover,

$$\mathscr{s}_g(x) = s(x), \quad x \in [-1,\,1], \tag{3.11}$$

and

$$Fo_g(n;x) = F_g(2n;x), \quad x\in[-1,\,1], n = 1, 2, \ldots. \tag{3.12}$$

Hence

$$\lim_{n\to\infty} Fo_g(n;x) = s(x), \quad x \in [-1,\,1]. \tag{3.13}$$

Furthermore, for each positive integer n *and each* $x \in [-1,\,1]$*, there exists a point* $\xi \in (-1,\,1)$*, such that*

$$E_n(x) = s(x) - Fo_g(n;x) = \frac{1}{n!}\int_0^1 Q_n(t)\frac{\partial^n}{\partial z^n}\mathscr{w}_g(x,z)\bigg|_{z=\xi} dt,$$

where $Q_n(t) = \prod_{j=1}^{n}(t - t_j)$ *and* $\{t_j\}$ *are the* n *roots of the Jacobi polynomial of order* n *with weight* $t^{-\frac{1}{2}}$.

Proof. The conditions of agreement are

$$\sum_{j=1}^{n} b_j t_j^k = p_{2k}, \quad k = 0, 1, \dots, 2n-1,$$

where the data p_k are the same as in the proof of Theorem 3.3. Since

$$p_{2k} = \frac{2}{2k+1} = \int_0^1 w(t) t^k dt, \quad k = 0, 1, 2, \dots,$$

the first part of the theorem is proved. As to the second part, (3.11) can be obtained by a similar proof to that of Theorem 3.3. By noticing (3.7), we can get (3.12) by a similar proof to that described in [12]. Therefore $\{Fo_g(n;x)\}$ is a subsequence of $\{F_g(n;x)\}$, (3.13) follows from (3.6). The error expression can be obtained in a similar fashion as we did in the proof of Theorem 3.3. This completes the proof. □

Remark 3.5. We now compare our nonlinear approximants $\{Fo_g(n;x)\}$ with the linear ones by an example. As we mentioned in the first section, at x=0.5, we need over 522,000 terms of the series (1.2) to get an approximate value to $s(0.5) = 1$, such that the error is not greater than 0.3×10^{-5}. While our nonlinear approximants need only the first 8 nonzero terms of the series to get a better approximate value, $Fo_g(4, 0.5) = F_g(8, 0.5) \simeq 0.9999987814$. The error is less than 0.13×10^{-5}.

Another comparison of the generating function method and linear approximation using the same data is illustrated in Figure 1.

We turn to study approximants to $s(x)$ based on the exponential generating functions. Due to the analysis we did for the geometric generating function cases, we would like to introduce the following modifications of $\varpi_2(x,t)$ in (2.4):

$$v_e(x,t) = \frac{1}{\pi t}\varpi_2(x,t), \tag{3.14}$$

and

$$\varpi_e(x,t) = \frac{1}{2}[v_e(x,\sqrt{t}) + v_e(x,-\sqrt{t})]. \tag{3.15}$$

We first discuss the approximants

$$Fo_e(n;x) = \sum_{j=1}^{n} b_j \varpi_e(x,t_j), \quad n = 1, 2, \dots. \tag{3.16}$$

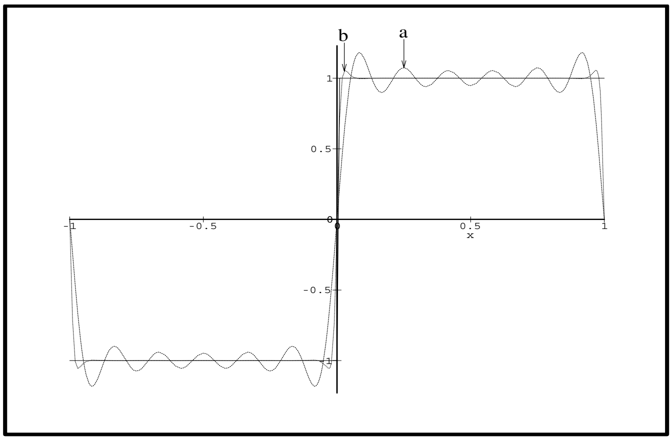

FIGURE 1. Approximation of the sign function by Fourier series
a: 12 term linear approximant
b: Nonlinear approximant with $\mathscr{w}_g$, n=6

Our result is

Theorem 3.6. *The nonlinear approximants* $\{Fo_e(n;x)\}$ *to* $s(x)$ *on* $[-1,\, 1]$ *are the corresponding Gaussian quadratures, with the weight* $w(t) = 2t^{-\frac{1}{2}}e^{-\sqrt{t}}$, *applied to the integral*

$$\mathscr{s}_e(x) = \int_0^\infty w(t)\mathscr{w}_e(x,t)dt. \tag{3.17}$$

Moreover,

$$\lim_{n\to\infty} Fo_e(n;x) = \mathscr{s}_e(x) = s(x), \quad x \in [-1,\, 1]. \tag{3.18}$$

Furthermore, for each positive integer n *and each* $x \in [-1,\, 1]$, *there exists a point* $\xi \in (-1,\, 1)$, *such that*

$$E_n(x) = s(x) - Fo_e(n;x) = \frac{1}{n!}\int_0^\infty Q_n(t)\frac{\partial^n}{\partial z^n}\mathscr{w}_e(x,z)\bigg|_{z=\xi} dt,$$

where $Q_n(t) = \prod_{j=1}^{n}(t-t_j)$ *and* $\{t_j\}$ *are the* n *roots of the orthogonal polynomial* $p_n(x)$ *of order* n *with weight* $t^{-\frac{1}{2}}$ *on* $[0,\infty)$, *that is,* p_n *has degree* n *and*

$$\int_0^\infty w(x)p_m(x)p_n(x) = \delta_{mn}, \quad m,n = 0,1,2,\ldots,$$

where δ_{mn} *is the Kronecker delta.*

The proofs of the first part and (3.18) are similar to that of Theorem 3.4 after we notice that the conditions of agreement are

$$\sum_{j=1}^{n} b_j t_j^k = 4(2k)! = \int_0^\infty w(t) t^k dt, \quad k = 0, 1, \ldots, 2n-1.$$

The error expressions can be obtained similarly as we did for Theorem 3.4.

As to the approximants

$$F_e(n;x) = \sum_{j=1}^{n} b_j v_e(x, t_j), \tag{3.19}$$

the first result is

Theorem 3.7.

$$F_e(2n;x) = Fo_e(n;x), \quad x \in [-1,\, 1], n = 1, 2, \ldots \,. \tag{3.20}$$

The proof is similar to that of Theorem 3.4.

And the second result is

Theorem 3.8. *Let* $T_1, T_2, \ldots, T_n$ *be the zeroes of the orthogonal polynomial of order* n*, with weight* $w(t) = t^{\frac{1}{2}} e^{-\sqrt{t}}$ *on* $[0, \infty)$*. Then the parameters* b_j *and* t_j *of* $F_e(2n+1;x)$ *in (3.19) are*

$$t_{n+1} = 0, t_j = -t_{2n+2-j} = \sqrt{T_j} \quad j = 1, 2, \ldots, n,$$

$$b_j = a_{2n+2-j} = \frac{1}{T_j} \int_0^\infty w(t) \frac{P(t)}{(t - T_j) P'(T_j)} dt, \quad j = 1, 2, \ldots, n,$$

$$a_{n+1} = 4 - 2 \sum_{j=1}^{n} b_j,$$

where

$$P(t) = \prod_{j=1}^{n} (t - T_j).$$

Proof. The conditions of agreement are

$$\sum_{j=1}^{2n+1} b_j t_j^k = J_{ek}(f) = p_k, \quad k = 0, 1, \ldots, 4n+1, \tag{3.21}$$

where the operators $\{J_{ek}\}$ are defined by

$$J_{ek}(h(x)) = \pi (k+1)! \int_{-1}^{1} h(x) \sin{(k+1)\pi x} dx, \quad k = 0, 1, 2, \ldots,$$

and satisfy

$$J_{ek}(v_e(x,t)) = t^k, \quad k = 0, 1, 2, \ldots,$$

while

$$\begin{aligned} p_k &= J_{ek}(s(x)) = 2\pi(k+1)! \int_0^1 \sin{(k+1)\pi x dx} \\ &= 2k![1-(-1)^{k+1}], \quad k = 0, 1, 2, \ldots . \end{aligned}$$

It is clear that the system (3.21) is satisfied for odd k and $k = 0$. We need only verify

$$\sum_{j=1}^{2n+1} b_j t_j^{2k+2} = p_{2k+2}, \quad k = 0, 1, \ldots, 2n-1.$$

From the theory of Gaussian quadrature we have

$$\sum_{j=1}^{n} (b_j T_j) T_j^k = \int_0^\infty w(t) t^k dt, \quad k = 0, 1, \ldots, 2n-1.$$

The right hand side is $2(2k+2)!$ for $k = 0, 1, \ldots, 2n-1$, which is $\frac{1}{2} p_{2k+2}$, and this completes the proof. □

The comparison of the generating function method and linear approximation using the same data is illustrated in Figure 2.

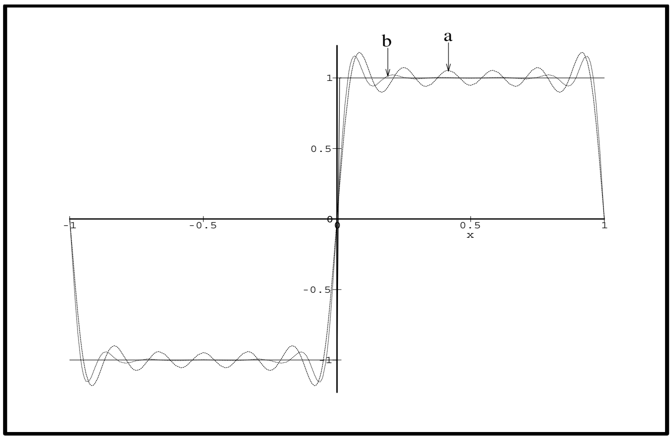

Figure 2. Approximation of the sign function by Fourier series
a: 12 term linear approximant
b: Nonlinear approximant with w_e, n=6

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