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w-DISTANCES AND τ-DISTANCES

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Abstract. We study the difference between the notions of w-distance and τ-distance. Specially, we state useful examples of τ-distance which are not w-distances.

1. Introduction

Throughout this paper we denote by $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ by the set of positive integers, rational numbers and real numbers, respectively.

In 1996, Kada, Suzuki and Takahashi introduced the notion of w-distance. Using this notion, they improved Caristi's fixed point theorem [2, 3], Ekeland's ε-variational principle [4, 5], and the nonconvex minimization according to Takahashi [20].

Definition 1. (Kada, Suzuki and Takahashi [6]) Let X be a metric space with metric d. Then a function p from $X \times X$ into $[0, \infty)$ is called a *w-distance* on X if the following are satisfied:

- ($w1$) $p(x, z) \leq p(x, y) + p(y, z)$ for all $x, y, z \in X$;
- ($w2$) p is lower semicontinuous in its second variable;
- ($w3$) for any $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \varepsilon$.

The metric d is one of w-distances. Using w-distance, Suzuki and Takahashi in [19] improved the Banach contraction principle [1] and Nadler's fixed point theorem [9]. See also [10, 12, 22].

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In 2001, Suzuki introduced the notion of τ-distance, which is a generalized notion of w-distance. Suzuki also improved the results in Tataru [23], Zhong [25, 26] and others.

Definition 2. (Suzuki [13]) Let X be a metric space with metric d. Then a function p from $X \times X$ into $[0, \infty)$ is called a *τ-distance* on X if there exists a function η from $X \times [0, \infty)$ into $[0, \infty)$ and the following are satisfied:

(τ1) $p(x, z) \le p(x, y) + p(y, z)$ for all $x, y, z \in X$;

(τ2) $\eta(x, 0) = 0$ and $\eta(x, t) \ge t$ for all $x \in X$ and $t \in [0, \infty)$, and η is concave and continuous in its second variable;

(τ3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n, x_m)) : m \ge n\} = 0$ imply $p(w, x) \le \liminf_n p(w, x_n)$ for all $w \in X$;

(τ4) $\lim_n \sup\{p(x_n, y_m) : m \ge n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$ imply $\lim_n \eta(y_n, t_n) = 0$;

(τ5) $\lim_n \eta(z_n, p(z_n, x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n, y_n)) = 0$ imply $\lim_n d(x_n, y_n) = 0$.

We may replace (τ2) by the following (τ2)'

(τ2)' $\inf\{\eta(x, t) : t > 0\} = 0$ for all $x \in X$, and η is nondecreasing in its second variable.

See also [14–18] for many useful examples and properties.

The author thinks that it is meaningful to study both w-distances and τ-distances because we can consider w-distances much more easily than τ-distances. On the other hand, there are useful examples of τ-distance which are not w-distances. In this paper, we shall state such examples.

2. Preliminaries

In this section, we state two lemmas and one theorem which are used in this paper. The following lemmas were proved in [19].

Lemma 1. ([19]) *Let X be a metric space, let p be a w-distance on X, and let q be a function from $X \times X$ into $[0, \infty)$ satisfying (w1), (w2). Suppose that $q(x, y) \ge p(x, y)$ for every $x, y \in X$. Then, q is also a w-distance on X.*

Lemma 2. ([13]) *Let X be a metric space and let p be a τ-distance on X. Then $p(z, x) = 0$ and $p(z, y) = 0$ imply $x = y$.*

The following is the τ-distance version of Caristi's fixed point theorem.

Theorem 1. ([13]) *Let X be a complete metric space and let p be a τ-distance on X. Let T be a mapping on X and let f be a function from X into $(-\infty, +\infty]$ which is proper lower semicontinuous and bounded from below. Assume $f(Tx) + p(x, Tx) \le f(x)$ for all $x \in X$. Then there exists $x_0 \in X$ such that $Tx_0 = x_0$ and $p(x_0, x_0) = 0$.*

3. Tataru's distance

In this section, we consider Tataru's distance. Let X be a subset of a Banach space and let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on X, i.e.,

(sg1) For each $t \geq 0$, $T(t)$ is a nonexpansive mapping on X;
(sg2) $T(0)x = x$ for all $x \in X$;
(sg3) $T(s+t) = T(s) \circ T(t)$ for all $s, t \geq 0$;
(sg4) for each $x \in X$, the mapping $T(\cdot)x$ from $[0,\infty)$ into X is continuous.

In [23], Tataru introduced the distance:

$$p(x,y) = \inf\{t + \|T(t)x - y\| : t \geq 0\}$$

for all $x, y \in X$, and studied Hamilton-Jacobi equations. See also [8]. We know Tataru's distances are also τ-distances.

Proposition 1. ([13]) *Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on a subset X of a Banach space. Then Tataru's distance p on X is also a τ-distance on X.*

We also know that Tataru's distances are also w-distances when X is compact.

Proposition 2. ([13]) *Let X be a compact subset of a Banach space. Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on X. Then Tataru's distance p on X is also a w-distance on X.*

As motivated by above, we shall characterize as follows.

Proposition 3. *Let $\{T(t) : t \geq 0\}$ be a strongly continuous semigroup of nonexpansive mappings on a subset X of a Banach space. Then the following are equivalent:*

(i) *Tataru's distance p on X is a w-distance on X;*
(ii) *for each $\varepsilon > 0$, there exists $\delta > 0$ such that $\|T(t)x - x\| \leq \varepsilon$ for all $t \in [0,\delta]$ and $x \in X$.*

Proof. We first show that (i) implies (ii). Fix $\varepsilon > 0$. Then since p is a w-distance on X, there exists $\delta > 0$ such that $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$ imply $\|x - y\| \leq \varepsilon$. For $t \in [0,\delta]$ and $x \in X$, we have $p(x,x) = 0$ and

$$p\big(x, T(t)x\big) = \inf\{s + \|T(s)x - T(t)x\| : s \geq 0\} \leq t \leq \delta.$$

Hence $\|T(t)x - x\| \leq \varepsilon$. This implies (ii). Conversely, we shall show that (ii) implies (i). We proved that p satisfies ($w1$) and ($w2$) in [13]. Thus, let us prove ($w3$). Fix $\varepsilon > 0$. Then from (ii), there exists $\delta' \in (0, \varepsilon/4)$ such that $\|T(t)x - x\| \leq \varepsilon/4$ for $t \in [0,\delta']$ and $x \in X$. We put $\delta = \delta'/2 > 0$ and fix $x, y, z \in X$ with $p(z,x) \leq \delta$ and $p(z,y) \leq \delta$. From $p(z,x) \leq \delta < \delta'$, there exists

$t \geq 0$ such that $t + \|T(t)z - x\| < \delta'$. Since $t < \delta'$, we have $\|T(t)z - z\| \leq \varepsilon/4$. Hence,

$$\|x - z\| \leq \|T(t)z - x\| + \|T(t)z - z\| < \delta' + \varepsilon/4 \leq \varepsilon/2.$$

Similarly we can prove $\|y - z\| < \varepsilon/2$. So we obtain

$$\|x - y\| \leq \|x - z\| + \|y - z\| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This implies $(w3)$. That is, p is a w-distance on X. This completes the proof. □

Example 1. Let X be the 2-dimensional real Hilbert space. Define a strongly continuous semigroup $\{T(t) : t \geq 0\}$ of nonexpansive mappings on X by

$$T(t) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \cos(t) - x_2 \sin(t) \\ x_1 \sin(t) + x_2 \cos(t) \end{bmatrix}$$

for $t \geq 0$ and $(x_1, x_2) \in X$. Then Tataru's distance p is a τ-distance, which is not a w-distance.

Proof. From Proposition 1, p is a τ-distance. We note that

$$\|T(t)x - x\| = 2\,\|x\|\,|\sin(t/2)|$$

for all $t \geq 0$ and $x \in X$. Hence,

$$\sup_{x \in X} \|T(t)x - x\| = \infty$$

for $t \in (0, 2\pi)$. So, by Proposition 3, p is not a w-distance. □

Example 2. Let E be the real Hilbert space consisting of all the real sequences $\{x(n)\}$ satisfying $\sum_{n=1}^{\infty} |x(n)|^2 < \infty$ with inner product $\langle x, y \rangle = \sum_{n=1}^{\infty} x(n)\, y(n)$ for all $x, y \in X$. Put $X = \{x \in E : \|x\| \leq 1\}$. Define a strongly continuous semigroup $\{T(t) : t \geq 0\}$ of nonexpansive mappings on X by

$$\big(T(t)x\big)\,(n) = \exp(-nt)\, x(n)$$

for all $t \geq 0$, $x \in E$ and $n \in \mathbb{N}$. Then Tataru's distance p is a τ-distance, which is not a w-distance.

Proof. (sg1), (sg2) and (sg3) clearly hold. Let $x \in X$ be fixed and let $\{t_k\}$ be a sequence in $[0, \infty)$ converging to 0. For each $\varepsilon > 0$, there exist $n_0, k_0 \in \mathbb{N}$ such that

$$\sum_{n=n_0+1}^{\infty} \big|x(n)\big|^2 < \frac{\varepsilon^2}{2} \quad \text{and} \quad \sum_{n=1}^{n_0} \big(1 - \exp(-nt_k)\big)^2 < \frac{\varepsilon^2}{2}$$

for all $k \in \mathbb{N}$ with $k \geq k_0$. Hence, we have

$$\begin{aligned}
&\|T(t_k)x - x\|^2 \\
&= \sum_{n=1}^{n_0} \big(1 - \exp(-nt_k)\big)^2 \left|x(n)\right|^2 + \sum_{n=n_0+1}^{\infty} \big(1 - \exp(-nt_k)\big)^2 \left|x(n)\right|^2 \\
&\leq \sum_{n=1}^{n_0} \big(1 - \exp(-nt_k)\big)^2 + \sum_{n=n_0+1}^{\infty} \left|x(n)\right|^2 \\
&\leq \varepsilon^2.
\end{aligned}$$

That is, $\|T(t_k)x - x\| \leq \varepsilon$ for all $k \in \mathbb{N}$ with $k \geq k_0$. Therefore

$$\lim_{k\to\infty} \|T(t_k)x - x\| = 0$$

for all $x \in X$. Let $\{t_k\}$ be a sequence in $[0, \infty)$ converging to some $t \in [0, \infty)$. Then we have

$$\lim_{n\to\infty} \|T(t_k)x - T(t)x\| \leq \lim_{n\to\infty} \|T(|t_k - t|)x - x\| = 0$$

for all $x \in X$. This implies (sg4). Define a sequence $\{e_k\}$ in X by

$$e_k(n) = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

Then we have

$$\sup_{k\in\mathbb{N}} \|T(t)e_k - e_k\| = \sup_{k\in\mathbb{N}} \big(1 - \exp(-kt)\big) = 1$$

for every $t > 0$. From Proposition 3, we obtain the desired result. □

4. Zhong's function

In this section, we let h be a nondecreasing function from $[0, \infty)$ into $[0, \infty)$ satisfying

$$\int_0^\infty \frac{1}{1 + h(t)} dt = \infty.$$

We also let X be a Banach space and $z_0 \in X$. Zhong [25, 26] considered the function g from $X \times X$ into $[0, \infty)$ defined by

$$g(x, y) = \frac{\|x - y\|}{1 + h\big(\|z_0 - x\|\big)}$$

for $x, y \in X$, and studied the relation between weak Palais-Smale condition and coercivity. Suzuki in [13, 18] considered the functions p and q from $X \times X$

into $[0, \infty)$ defined by

$$p(x, y) = \int_{\|z_0 - x\|}^{\|z_0 - x\| + \|x - y\|} \frac{1}{1 + h(t)} dt \tag{4.1}$$

and

$$q(x, y) = \int_{\|z_0 - x\|}^{\|z_0 - x\| + \|x - y\|} \frac{1}{1 + h(t)} dt + \int_{\|z_0 - y\|}^{\|z_0 - y\| + \|x - y\|} \frac{1}{1 + h(t)} dt \tag{4.2}$$

for $x, y \in X$, and gave simpler proofs of Zhong's results. We know the following.

Proposition 4. ([13, 18]) *p and q are τ-distances on X.*

We now prove the following.

Proposition 5. *The following are equivalent:*

(i) *p is a w-distance on X;*
(ii) *q is a w-distance on X;*
(iii) $\lim_{t \to \infty} h(t) < \infty$.
(iv) *there exist $c_1, c_2 > 0$ such that*

$$c_1 \|x - y\| \le p(x, y) \le c_2 \|x - y\|$$

for $x, y \in X$;

(v) *there exist $c_3, c_4 > 0$ such that*

$$c_3 \|x - y\| \le q(x, y) \le c_4 \|x - y\|$$

for $x, y \in X$.

Proof. We note that p and q satisfy $(w1)$ and $(w2)$, and that $p(x, y) \le q(x, y)$ for $x, y \in X$. We also note that the metric d defined by $d(x, y) = \|x - y\|$ for $x, y \in X$ is a w-distance on X. So by Lemma 1, we can prove (i) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (i), and (v) $\Rightarrow$ (ii). We next prove (ii) $\Rightarrow$ (iii). We assume (iii) does not hold, i.e., $\lim_{t \to \infty} h(t) = \infty$. We fix $v \in X$ with $\|v\| = 1$. For each $\delta > 0$, there exists $s > 0$ such that $1/\big(1 + h(s)\big) \le \delta/2$. Putting $x = z_0 + sv$ and $y = z_0 + (s + 1)v$, we have

$$\begin{aligned} q(x, y) &= \int_{s}^{s+1} \frac{1}{1 + h(t)} dt + \int_{s+1}^{s+2} \frac{1}{1 + h(t)} dt \\ &\le \frac{2}{1 + h(s)} \le \delta, \end{aligned}$$

$q(x, x) = 0$ and $\|x - y\| = 1$. Hence, q does not satisfy $(w3)$. Therefore (ii) implies (iii). Let us prove (iii) $\Rightarrow$ (iv). We assume (iii). Then it is obvious that

$$\frac{1}{1 + \lim_{t \to \infty} h(t)} \|x - y\| \le p(x, y) \le \frac{1}{1 + h(0)} \|x - y\|$$

for all $x, y \in X$. This is (iv). We finally prove (iv) $\Rightarrow$ (v). We assume (iv). Then putting $c_3 = c_1$ and $c_4 = 2\,c_2$, we have

$$\begin{aligned} c_3\,\|x-y\| &= c_1\,\|x-y\| \le p(x,y) \\ &\le q(x,y) = p(x,y) + p(y,x) \\ &\le c_2\,\|x-y\| + c_2\,\|y-x\| \\ &= c_4\,\|x-y\| \end{aligned}$$

for $x, y \in X$. This is (v). This completes the proof. □

Example 3. Define a function h by $h(t) = t$ for $t \in [0, \infty)$. Then functions p and q defined by (4.1) and (4.2) are τ-distances, which are not w-distances.

5. τ-Distances on $\mathbb{R}$

In this section, put $X = \mathbb{R}$ and let f and g be continuous functions from X into $[0, \infty)$. Define a function p from $X \times X$ into $[0, \infty)$ by

$$p(x,y) = \begin{cases} \int_x^y f(t)\,dt, & \text{if } x \le y, \\ \int_y^x g(t)\,dt, & \text{if } x \ge y \end{cases} \tag{5.1}$$

for $x, y \in X$. We also define nondecreasing continuous functions F and G from X into $\mathbb{R}$ by

$$F(x) = \int_0^x f(t)\,dt \quad \text{and} \quad G(x) = \int_0^x g(t)\,dt \tag{5.2}$$

for $x \in X$. It is obvious that

$$p(x,y) = \begin{cases} F(y) - F(x), & \text{if } x \le y, \\ G(x) - G(y), & \text{if } x \ge y \end{cases}$$

for $x, y \in X$.

Proposition 6. *The following are equivalent:*

(i) *p is a τ-distance on X;*

(ii) *F and G are strictly increasing, and*

$$\int_0^{+\infty} f(t)\,dt = \infty \quad \textit{and} \quad \int_{-\infty}^0 g(t)\,dt = \infty.$$

Remark. F and G are strictly increasing if and only if

$$\int_x^{x+\varepsilon} f(t)\,dt > 0 \quad \text{and} \quad \int_x^{x+\varepsilon} g(t)\,dt > 0$$

for all $x \in X$ and $\varepsilon > 0$. Compare this condition with (ii) in Proposition 7.

Proof. We first prove (ii) implies (i). Assume (ii). It is obvious p is continuous. Hence p satisfies (τ3). It is also obvious that the following are equivalent:

- A sequence $\{x_n\}$ in X converges to z;
- $\lim_n p(z, x_n) = 0$;
- $\lim_n p(x_n, z) = 0$.

In the case of $x \leq y \leq z$, we have

$$p(x,z) = \int_x^z f(t)\ dt = \int_x^y f(t)\ dt + \int_y^z f(t)\ dt = p(x,y) + p(y,z).$$

In the case of $x \leq z \leq y$, we have

$$p(x,z) = \int_x^z f(t)\ dt \leq \int_x^y f(t)\ dt = p(x,y) \leq p(x,y) + p(y,z).$$

In the case of $z \leq x \leq y$, we have

$$p(x,z) = \int_z^x g(t)\ dt \leq \int_z^y g(t)\ dt = p(y,z) \leq p(x,y) + p(y,z).$$

Similarly we can prove $p(x,z) \leq p(x,y) + p(y,z)$ in the other cases. This is (τ1). Define a function η from $X \times [0,\infty)$ into $[0,\infty)$ by

$$\eta(x,t) = t + \sup\big\{|x-y| : p(x,y) \leq t\big\}$$

for $x \in X$ and $t \in [0,\infty)$. We shall show that η satisfies (τ2)'. It is obvious that η is nondecreasing in its second variable. Fix $x \in X$ and define a function h from X into $[0,\infty)$ by $h(y) = p(x,y)$ for $y \in X$. Then we have h is continuous, h is strictly decreasing on $(-\infty, x]$, h is strictly increasing on $[x,+\infty)$, and $h(x) = 0$. Therefore $\inf\{\eta(x,t) : t > 0\} = 0$. We next show ($\tau$4). We assume that $\lim_n \sup\{p(x_n, y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n, t_n) = 0$. From $\int_0^{+\infty} f(t)\ dt = \infty$ and $\int_{-\infty}^0 g(t)\ dt = \infty$, we have $\{y_n\}$ is bounded. Let $z_1, z_2 \in X$ be cluster points of $\{y_n\}$. Then

$$\lim_{n\to\infty} p(x_n, z_1) \leq \lim_{n\to\infty} \sup_{m \geq n} p(x_n, y_m) = 0$$

and hence $\{x_n\}$ converges to z_1. Similarly we obtain $\{x_n\}$ converges to z_2. That is, $\{x_n\}$ converges to some number $z \in X$. Since

$$\lim_{n\to\infty} p(z, y_n) \leq \lim_{n\to\infty} \big(p(z,x_n) + p(x_n,y_n)\big) = 0,$$

$\{y_n\}$ also converges to z. We have

$$\begin{aligned}
&\lim_{n\to\infty} \eta(y_n, t_n) \\
&= \lim_{n\to\infty} \Big(t_n + \sup\big\{|y_n - w| : p(y_n, w) \leq t_n\big\}\Big) \\
&\leq \lim_{n\to\infty} \Big(t_n + \sup\big\{|y_n - z| + |z - w| : p(y_n, w) \leq t_n\big\}\Big)
\end{aligned}$$

$$\begin{aligned}
&\le \lim_{n\to\infty} \Big(t_n + |y_n - z| + \sup\big\{ |z-w| : p(z,w) \le p(z,y_n) + t_n \big\} \Big) \\
&\le \lim_{n\to\infty} \Big(t_n + |y_n - z| + \eta\big(z, p(z,y_n) + t_n\big) \Big) \\
&= 0
\end{aligned}$$

because of $\lim_n \big(p(z,y_n)+t_n\big) = 0$. This implies ($\tau$4). Let us prove ($\tau$5). We assume that $\lim_n \eta(z_n, p(z_n,x_n)) = 0$ and $\lim_n \eta(z_n, p(z_n,y_n)) = 0$. Since

$$\eta\big(z, p(z,x)\big) \ge \sup\big\{ |z-y| : p(z,y) \le p(z,x) \big\} \ge |z-x|$$

for all $z, x \in X$, we have $\lim_n |z_n - x_n| = 0$ and $\lim_n |z_n - y_n| = 0$. Hence we have $\lim_n |x_n - y_n| = 0$. Therefore we obtain p is a τ-distance on X. We next prove (i) implies (ii). Assume that p is a τ-distance. If F is not strictly increasing, then there exists $x, y \in X$ with $x < y$ and $F(x) = F(y)$. So we have

$$p(x,x) = 0 \quad \text{and} \quad p(x,y) = F(y) - F(x) = 0.$$

By Lemma 2, we have $x = y$. This is a contradiction. Hence, F is strictly increasing. Similarly we can prove that G is strictly increasing. If $\int_0^{+\infty} f(t)\,dt < \infty$, then $-F$ is continuous, strictly decreasing function from X into $\mathbb{R}$ satisfying

$$\inf_{x\in X} -F(x) = \lim_{x\to\infty} -F(x) = -\int_0^{+\infty} f(t)\,dt > -\infty.$$

That is, $-F$ is bounded from below. Define a mapping T on X by $Tx = x+1$ for all $x \in X$. Then we have

$$-F(Tx) + p(x,Tx) = -F(Tx) + F(Tx) - F(x) = -F(x)$$

for all $x \in X$. So, by Theorem 1, there exists a fixed point of T. This is a contradiction. Hence, we have $\int_0^{+\infty} f(t)\,dt = \infty$. Similarly we can prove $\int_{-\infty}^0 g(t)\,dt = \infty$. This completes the proof. □

We also obtain the following.

Proposition 7. *The following are equivalent:*

(i) *p is a w-distance on X;*

(ii) *f and g satisfy*

$$\inf_{x\in X} \int_x^{x+\varepsilon} f(t)\,dt > 0 \quad \textit{and} \quad \inf_{x\in X} \int_x^{x+\varepsilon} g(t)\,dt > 0$$

for every $\varepsilon > 0$.

Proof. Note that p satisfies (w1) and (w2). We also know that (ii) implies (i); see [6, 11]. Thus, we shall prove that (i) implies (ii). From (w3), for each $\varepsilon > 0$, there exists $\delta > 0$ such that $p(z,x) \le \delta$ and $p(z,y) \le \delta$ imply $|x-y| \le \varepsilon$.

Since $p(x,x) = 0$ for all $x \in X$, we have $p(x,y) \le \delta$ implies $|x-y| \le \varepsilon$. That is, $|x-y| > \varepsilon$ implies $p(x,y) > \delta$. So,

$$0 < \delta \le \inf_{x \in X} p(x, x+2\varepsilon) = \inf_{x \in X} \int_x^{x+2\varepsilon} f(t)\, dt.$$

Similarly, we obtain

$$0 < \delta \le \inf_{x \in X} p(x+2\varepsilon, x) = \inf_{x \in X} \int_x^{x+2\varepsilon} g(t)\, dt.$$

This completes the proof. □

From Proposition 6 and 7, we obtain the following

Example 4. Define functions f and g by

$$f(x) = g(x) = \min\{1, 1/|x|\}$$

for $x \in X$. Then a function p defined by (5.1) is a τ-distance, which is not a w-distance.

6. Other Examples

Examples 1 – 4 do not satisfy $(w3)$. In this section, we give examples which do not satisfy $(w2)$.

We know the following. See also Kim, Kim and Shin [7], Takahashi [21], and Ume [24].

Proposition 8. ([6]) *Let X be a metric space with metric d and let T be a continuous mapping on X. Then a function p from $X \times X$ into $[0,\infty)$ defined by*

$$p(x,y) = \max\{d(Tx,Ty), d(Tx,y)\}$$

for $x, y \in X$ is a w-distance on X.

Proposition 9. ([13]) *Let X be a metric space and let p be a τ-distance on X. Let T be a mapping on X satisfying that $\lim_n x_n = y$ and $\lim_n Tx_n = y$ imply $Ty = y$. Then a function q from $X \times X$ into $[0,\infty)$ defined by*

$$q(x,y) = \max\{p(Tx,Ty), p(Tx,y)\}$$

for all $x, y \in X$ is also a τ-distance.

As motivated by above, we give an example.

Example 5. Put $X = \mathbb{R}$ and define a function $X \times X$ into $[0,\infty)$ by

$$p(x,y) = \max\left\{ \Big| [x] - [y] \Big|,\ \Big| [x] + 1/2 - y \Big| \right\}$$

for $x, y \in X$, where $[x]$ is denoted by the maximum integer not exceeding x. Then p is a τ-distance, which is not a w-distance.

Proof. Define a mapping T on X by $Tx = [x] + 1/2$ for $x \in X$. Then $\lim_n x_n = y$ and $\lim_n Tx_n = y$ imply $Ty = y$. So, by Proposition 9, p is a τ-distance. We also have

$$p(0, y) = \max \left\{ \left| \, [y] \, \right|, \; \left| \, 1/2 - y \, \right| \right\}.$$

So $y \mapsto p(0, y)$ is not lower semicontinuous because

$$\lim_{n \to \infty} p(0, 1 - 1/n) = \lim_{n \to \infty} (1 - 1/n - 1/2) = 1/2 < 1 = p(0, 1).$$

Therefore p is not a w-distance. □

A mapping T on a metric space X is called a *contractive mapping with respect to a τ-distance* p if there exists a τ-distance p and $r \in [0, 1)$ such that

$$p(Tx, Ty) \leq \, rp(x, y)$$

for all $x, y \in X$. In [10], Shioji, Suzuki and Takahashi discussed the relation between contractive and Kannan mappings with respect to w-distances. In [17], Suzuki did for τ-distances. Using the results in [10, 17], we can give the following.

Example 6. Put $X = \mathbb{R}$. Let C be a subset of $\mathbb{R}$ such that $\operatorname{cl} C = \mathbb{R}$ and $\bigsqcup_{q \in \mathbb{Q}} (q + C) = \mathbb{R} \setminus \mathbb{Q}$, where $\operatorname{cl} C$ is the closure of C and $\bigsqcup$ represents disjoint union. Define a mapping T on X by

$$Tx = \begin{cases} 0, & \text{if } x \in \mathbb{Q}, \\ q, & \text{if } x \in (q + C) \text{ for some } q \in \mathbb{Q}. \end{cases}$$

Then for every w-distance p, T is not a contractive mapping with respect to p. However, T is a contractive mapping with respect to a τ-distance q defined by

$$q(x, y) = \begin{cases} 0, & \text{if } (x, y) = (0, 0), \\ 1, & \text{if } (x, y) \in (\mathbb{Q} \times \mathbb{Q}) \setminus \{(0, 0)\}, \\ 2, & \text{if } (x, y) \in (\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{Q} \times \mathbb{Q}) \end{cases}$$

for $x, y \in \mathbb{R}$.

Proof. We first show q is a τ-distance. Let $x, y, z \in X$ be fixed. In the case of $(x, z) = (0, 0)$, we have

$$q(x, z) = 0 \leq q(x, y) + q(y, z).$$

In the case of $(x, z) \in (\mathbb{Q} \times \mathbb{Q}) \setminus \{(0, 0)\}$, since either $(x, y) \neq (0, 0)$ or $(y, z) \neq (0, 0)$ holds, we have

$$q(x, z) = 1 \leq \max\{q(x, y), q(y, z)\} \leq q(x, y) + q(y, z).$$

In the case of $(x, z) \in (\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{Q} \times \mathbb{Q})$, since either $x \in \mathbb{R} \setminus \mathbb{Q}$ or $z \in \mathbb{R} \setminus \mathbb{Q}$ holds, we have

$$q(x, z) = 2 = \max\{q(x, y), q(y, z)\} \leq q(x, y) + q(y, z).$$

Therefore $(\tau 1)_q$ holds. Define a function η from $X \times [0, \infty)$ into $[0, \infty)$ by $\eta(x, t) = t$. Then η satisfies $(\tau 2)$. Also, we can easily prove $(\tau 3)_q$ – $(\tau 5)_q$. Thus, q is a τ-distance. We next fix $x, y \in X$. In the case of $(x, y) \in \mathbb{Q} \times \mathbb{Q}$, since $Tx = 0$ and $Ty = 0$, we have

$$q(Tx, Ty) = 0 \leq (1/2)\, q(x, y).$$

In the case of $(x, y) \in (\mathbb{R} \times \mathbb{R}) \setminus (\mathbb{Q} \times \mathbb{Q})$, since $Tx \in \mathbb{Q}$ and $Ty \in \mathbb{Q}$, we have

$$q(Tx, Ty) \leq 1 = (1/2)\, q(x, y).$$

Therefore $q(Tx, Ty) \leq (1/2)\, q(x, y)$ holds for every $x, y \in X$. That is, T is a contractive mapping with respect to q. The remain is proved in [10]. $\square$

References

[1] S. Banach, *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*, Fund. Math., **3** (1922), 133–181.

[2] J. Caristi, *Fixed point theorems for mappings satisfying inwardness conditions*, Trans. Amer. Math. Soc., **215** (1976), 241–251.

[3] J. Caristi and W. A. Kirk, *Geometric fixed point theory and inwardness conditions*, Lecture Notes in Math., Vol. 490, pp. 74–83, Springer, Berlin, 1975.

[4] I. Ekeland, *On the variational principle*, J. Math. Anal. Appl., **47** (1974), 324–353.

[5] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc., **1** (1979), 443–474.

[6] O. Kada, T. Suzuki and W. Takahashi, *Nonconvex minimization theorems and fixed point theorems in complete metric spaces*, Math. Japon., **44** (1996), 381–391.

[7] T. H. Kim, E. S. Kim and S. S. Shin, *Minimization theorems relating to fixed point theorems on complete metric spaces*, Math. Japon., **45** (1997), 97–102.

[8] M. Kocan and A. Święch, *Perturbed optimization on product spaces*, Nonlinear Anal., **26** (1996), 81–90.

[9] S. B. Nadler, Jr., *Multi-valued contraction mappings*, Pacific J. Math., **30** (1969), 475–488.

[10] N. Shioji, T. Suzuki and W. Takahashi, *Contractive mappings, Kannan mappings and metric completeness*, Proc. Amer. Math. Soc., **126** (1998), 3117–3124.

[11] T. Suzuki, *Fixed point theorems in complete metric spaces*, in Nonlinear Analysis and Convex Analysis (W. Takahashi Ed.), RIMS Kokyuroku, 939 (1996), pp 173–182.

[12] T. Suzuki, *Several fixed point theorems in complete metric spaces*, Yokohama Math. J., **44** (1997), 61–72.

[13] T. Suzuki, *Generalized distance and existence theorems in complete metric spaces*, J. Math. Anal. Appl., **253** (2001), 440–458.

[14] T. Suzuki, *On Downing-Kirk's theorem*, J. Math. Anal. Appl., **286** (2003), 453–458.

[15] T. Suzuki, *Several fixed point theorems concerning τ-distance*, Fixed Point Theory Appl., **2004** (2004), 195–209.

[16] T. Suzuki, *Generalized Caristi's fixed point theorems by Bae and others*, J. Math. Anal. Appl., **302** (2005), 502–508.

[17] T. Suzuki, *Contractive mappings are Kannan mappings, and Kannan mappings are contractive mappings in some sense*, Comment. Math. Prace Mat., **45** (2005), 45–58.

[18] T. Suzuki, *On the relation between the weak Palais-Smale condition and coercivity by Zhong*, Nonlinear Anal. (2007), doi:10.1016/j.na.2007.01.071.

[19] T. Suzuki and W. Takahashi, *Fixed point theorems and characterizations of metric completeness*, Topol. Methods Nonlinear Anal., **8** (1996), 371–382.

[20] W. Takahashi, *Existence theorems generalizing fixed point theorems for multivalued mappings*, in Fixed Point Theory and Applications (M. A. Théra and J. B. Baillon Eds.), Pitman Research Notes in Mathematics Series 252, 1991, pp. 397–406, Longman Sci. Tech., Harlow, 1991.

[21] W. Takahashi, *"Minimization theorems and fixed point theorems"* (in Japanese), in Nonlinear Analysis and Mathematical Economics (T. Maruyama Ed.), RIMS Kokyuroku 829, 1993, pp. 175–191.

[22] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.

[23] D. Tataru, *Viscosity solutions of Hamilton-Jacobi equations with unbounded nonlinear terms*, J. Math. Anal. Appl., **163** (1992), 345–392.

[24] J. S. Ume, *Some existence theorems generalizing fixed point theorems on complete metric spaces*, Math. Japon., **40** (1994), 109–114.

[25] C. -K. Zhong, *On Ekeland's variational principle and a minimax theorem*, J. Math. Anal. Appl., **205** (1997), 239–250.

[26] C. -K. Zhong, *A generalization of Ekeland's variational principle and application to the study of the relation between the weak P.S. condition and coercivity*, Nonlinear Anal., **29** (1997), 1421–1431.