Nonlinear Functional Analysis and Applications
Vol. 13, No. 1 (2008), 29-41

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

ON COMMUTATIVITY AND ITS GENERALIZATIONS IN HYBRID FIXED POINT THEORY

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Abstract. Coincidence and common fixed points of non-linear hybrid mappings have been obtained, without using the commutativity conditions or any of its generalizations. Our results improve many well known results in the context of Hybrid fixed point theory.

1. Introduction

Nadlers contraction principle has led to a good fixed point theory in non-linear analysis. Coincidence and common fixed points of non-linear hybrid contractions (i.e. contractions involving single valued and multi-valued mappings) have been recently studied by many authors. The concept of commutativity of single valued mappings was extended by [5] to the setting of a single valued mapping and a multi-valued mapping on a metric space. This concept of

[0]Received May 18, 2005. Revised June 3, 2007.
[0]2000 Mathematics Subject Classification: 47H10.
[0]Key words and phrases: Coincidence and common fixed points, metrically convex spaces.
[0]The first author was supported by University Grants Commission C. R. O, Bhopal, India (Grant No. 4-126(8)/98(MRP/CRO)).

commutativity has been further generalized by many authors, viz weakly commuting [6], compatible [14], weak compatible [9]. It is interesting to note that in all the results obtained so far, concerning common fixed points of hybrid mappings, the mappings (single valued and multi-valued) under consideration satisfy either the commutativity condition or any of its generalizations (For instance see [2],[7],[9],[10],[11]). In this note we have shown the existence of fixed points of hybrid contractions which do not satisfy any of the commutativity conditions or its generalizations. Our results extends and improves many well known results in the field of hybrid fixed point theory.

2. Preliminaries

For a metric space (X, d), let (CB(X), H) and (CL(X), H) denote respectively the hyper-spaces of nonempty closed bounded and non-empty closed subsets of X, where H is the Hausdorff metric induced by d. For $f : X \longrightarrow X$ and $T : X \longrightarrow CL(X)$ we shall use the following notations.

$$L(x,y) = \max\{d(fx,fy), d(fx,Tx), d(fy,Ty), \frac{1}{2}(d(fx,Ty)+d(fy,Tx))\}$$

$$N(x,y) = (\max\{d^2(fx,fy), d(fx,Tx)d(fy,Ty), d(fx,Ty)d(fy,Tx), \frac{1}{2}d(fx,Tx)d(fy,Tx)\frac{1}{2}d(fx,Ty)d(fy,Ty)\})^{1/2}$$

Definition 2.1. ([5]) Mappings f and T are said to be commuting at a point $x \in X$ if $fTx \subseteq Tfx$. The mappings f and T are said to be commuting on X if $fTx \subseteq Tfx$ for all $x \in X$.

Definition 2.2. ([6]) Mappings f and T are said to be weakly commuting at a point $x \in X$ if $H(fTx, Tfx) \leq D(fx, Tx)$. The mappings f and T are said to be weakly commuting on X if $H(fTx, Tfx) \leq D(fx, Tx)$ for all $x \in X$.

Definition 2.3. ([14]) The mappings f and T are said to be compatible iff $fTx \in CB(X)$ for all $x \in X$ and $H(Tfx_n, fTx_n) \to 0$, whenever $\{x_n\}$ is a sequence in X, such that $Tx_n \to M \in CB(X)$ and $fx_n \to t \in M$.

Definition 2.4. ([9]) The mappings f and T are said to be f-weak compatible iff $fTx \in CB(X)$ for all $x \in X$ and the following limits exists and satisfy the inequalities:

(i) $\lim_{n\to\infty} H(Tfx_n, fTx_n) \leq \lim_{n\to\infty} H(Tfx_n, Tx_n)$,

(ii) $\lim_{n\to\infty} d(fTx_n, fx_n) \leq \lim_{n\to\infty} H(Tfx_n, Tx_n)$,

whenever $\{x_n\}$ is a sequence in X, such that $Tx_n \to M \in CB(X)$ and $fx_n \to t \in M$ as $n \to \infty$.

Definition 2.5. The mappings f and T are said to be coincidentally commuting iff they commute at their coincidence points.

Definition 2.6. Mappings f and T are said to be coincidentally idempotent iff f is idempotent at the coincidence points of f and T.

Let C(T, f) denote the set of all coincidence points of the mappings f and T, that is $C(T, f) = \{u : fu \in Tu\}$.

We introduce the following.

Definition 2.7. Mappings f and T are said to be weakly coincidentally idempotent iff $ffu = fu$ for some $u \in C(T, f)$.

We remark that coincidentally idempotent pair of mappings are weakly coincidentally idempotent, but the converse is not necessarily true as shown in Example 3.9 of this note.

3. Main results

Let Γ denote the family of maps ϕ from the set $R^+$of nonnegative real numbers to itself such that ϕ is upper semi-continuous, non decreasing and $\phi(t) < t$ for all $t > 0$.

We appeal to the following :

Lemma 3.1. ([10]) *Let $T : X \longrightarrow CB(X)$ and $f : X \longrightarrow X$ be f-weak compatible. If $fw \in Tw$ for some $w \in X$ and $H(Tx, Ty) \leq h(aL(x, y) + (1-a)N(x, y))$ for all x, y in Y, $0 \leq a \leq 1$ and $0 < h < 1$, then $fTw = Tfw$.*

The above lemma has been used in [9], [10], and [11], to prove the existence of fixed points of hybrid mappings. However, we have found that the above lemma admits some counter example. We note that the usage of the incorrect inequality $d(f^2w, fw) \leq d(f^2w, fTw) + d(fTw, fw)$ has led to this error. We give the following counter example.

Example 3.2. ([10]) Let $X = [0, \infty)$ be endowed by the Euclidean metric. Let $f(x) = 3/2(x^2 + x)$ and $Tx = [0, x^2 + 2]$. We see that f and T satisfies all conditions of the above lemma, and $f_0 \in T_0$, but $fT_0 \neq Tf_0$.

Now we present our main results as follows:

Theorem 3.3. *Let Y be an arbitrary nonempty set, (X, d) be a metric space, $f : Y \longrightarrow X$ and $T : Y \longrightarrow CL(X)$ be such that*

$$T(Y) \subseteq f(Y), \tag{3.1}$$

$$H(Tx, Ty) \leq \phi(aL(x, y) + (1-a)N(x, y)) \text{ for all } x, y \in Y,\ 0 \leq a \leq 1, \tag{3.2}$$

$$\phi(t) \leq qt \text{ for all } t > 0 \text{ and for some } q \in (0, 1), \tag{3.3}$$

$$f(Y) \text{ or } T(Y) \text{ is complete,} \tag{3.4}$$

$$\text{there exists a point } x_0 \text{in } Y \text{ such that } T \text{ is asymptotically regular at } x_0. \tag{3.5}$$

Then f and T has a coincidence point. Further if, f and T are weakly coincidentally idempotent, then f and T has a common fixed point.

Proof. In view of (3.1), for a point $x_0 \in Y$, we can construct sequences$\{x_n\} \in Y$ and $y_n \in X$ such that, for each $n \in N$, $y_n = fx_n \in Tx_{n-1}$ and $d(y_n, y_{n+1}) \leq q^{-1/2}H(Tx_{n-1}, Tx_n)$.

By (3.5), we have $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

We will claim that $\{y_n\}$ is a Cauchy sequence. Suppose $\{y_n\}$ is not a Cauchy sequence,then there exists a positive number ϵ such, that for each positive integer k, there exists integers n(k) and m(k) such that

$$k \leq n(k) < m(k) \tag{3.6}$$

and

$$d(y_{n(k)}, y_{m(k)}) \geq \epsilon. \tag{3.7}$$

Then for each integer k, we have

$$\epsilon \leq d(y_{n(k)}, y_{m(k)}) \leq d(y_{n(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}). \tag{3.8}$$

For each integer k, let m(k) denote the smallest integer satisfying (3.6) and (3.7) for some n(k). Then we have $d(y_{n(k)}, y_{m(k)-1}) < \epsilon$ and it follows from (3.8) that

$$\lim_{k\to\infty} d(y_{n(k)}, y_{m(k)}) = \epsilon. \tag{3.9}$$

Using the triangle inequality, we get

$$| \, d(y_{n(k)}, y_{m(k)-1}) - d(y_{n(k)}, y_{m(k)}) \, | \leq d(y_{m(k)-1}, y_{m(k)})$$

and

$$| \, d(y_{n(k)+1}, y_{m(k)-1}) - d(y_{n(k)}, y_{m(k)}) \, | \leq d(y_{n(k)}, y_{n(k)+1}) + d(y_{m(k)-1}, y_{m(k)}),$$

which yield

$$\lim_{k\to\infty} d(y_{n(k)}, y_{m(k)-1}) = \lim_{k\to\infty} d(y_{n(k)+1}, y_{m(k)-1}) = \epsilon.$$

Now

$$\begin{aligned} d(y_{n(k)}, y_{m(k)}) &\leq d(y_{n(k)}, y_{n(k)+1}) + d(y_{n(k)+1}, y_{m(k)}) \\ &\leq d(y_{n(k)}, y_{n(k)+1}) + q^{-1/2}H(Tx_{m(k)-1}, Tx_{n(k)}) \\ &\leq d(y_{n(k)}, y_{n(k)+1}) + q^{-1/2}\phi(a.L(x_{m(k)-1}, x_{n(k)}) \\ &\quad + (1-a)N(x_{m(k)-1}, x_{n(k)}) \end{aligned} \tag{3.10}$$

$$\begin{aligned}
&L(x_{m(k)-1}, x_{n(k)}) \\
&= \max\{d(fx_{m(k)-1}, fx_{n(k)}), d(fx_{m(k)-1}, Tx_{m(k)-1}), d(fx_{n(k)}, Tx_{n(k)}), \\
&\quad \frac{1}{2}(d(fx_{m(k)-1}, Tx_{n(k)}) + d(fx_{n(k)}, Tx_{m(k)-1}))\} \\
&\leq \max\{d(fx_{m(k)-1}, fx_{n(k)}), d(fx_{m(k)-1}, fx_{m(k)}), d(fx_{n(k)}, fx_{n(k)+1}), \\
&\quad \frac{1}{2}(d(fx_{m(k)-1}, fx_{n(k)+1}) + d(fx_{n(k)}, fx_{m(k)}))\} \\
&N(x_{m(k)-1}, x_{n(k)}) \\
&= [\max\{d^2(fx_{m(k)-1}, fx_{n(k)}), d(fx_{m(k)-1}, Tx_{m(k)-1})d(fx_{n(k)}, Tx_{n(k)}), \\
&\quad d(fx_{m(k)-1}, Tx_{n(k)})d(fx_{n(k)}, Tx_{m(k)-1}), \\
&\quad \frac{1}{2}d(fx_{m(k)-1}, Tx_{m(k)-1})d(fx_{n(k)}, Tx_{m(k)-1}), \\
&\quad \frac{1}{2}d(fx_{m(k)-1}, Tx_{n(k)})d(fx_{n(k)}, Tx_{n(k)})\}]^{1/2} \\
&\leq [\max\{d^2(fx_{m(k)-1}, fx_{n(k)}), d(fx_{m(k)-1}, fx_{m(k)})d(fx_{n(k)}, fx_{n(k)+1}), \\
&\quad d(fx_{m(k)-1}, fx_{n(k)+1})d(fx_{n(k)}, fx_{m(k)}), \\
&\quad \frac{1}{2}d(fx_{m(k)-1}, fx_{m(k)}).d(fx_{n(k)}, fx_{m(k)}) \\
&\quad \frac{1}{2}d(fx_{m(k)-1}, fx_{n(k)+1})d(fx_{n(k)}, fx_{n(k)+1})\}]^{1/2}
\end{aligned}$$

Using the upper semicontinuity of f, and letting $k \to \infty$, we get using (3.10) $\epsilon \leq q^{-1/2}.\phi(\epsilon) \leq q^{-1/2}.q.\epsilon < \epsilon$, which is a contradiction to the choice of ϵ and so sequence $\{y_n\}$ is a Cauchy sequence.

If f(Y) is complete, then sequence $\{fx_n\}$ has a limit in f(Y), say u. Let $w \in f^{-1}(u)$. By (3.2) we have

$$\begin{aligned}
d(fx_{n+1}, Tw) &\leq H(Tx_n, Tw) \\
&\leq \phi(aL(x_n, w) + (1-a)N(x_n, w)
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&L(x_n, w) \\
&= \max\{d(fx_n, fw), d(fx_n, Tx_n), d(fw, Tw), \frac{1}{2}(d(fx_n, Tw) + d(fw, Tx_n))\} \\
&\leq \max\{d(fx_n, fw), d(fx_n, fx_{n+1}), d(fw, Tw), \frac{1}{2}(d(fx_n, Tw) + d(fw, fx_{n+1}))\} \\
&N(x_n, w) \\
&= [\max\{d^2(fx_n, fw), d(fx_n, Tx_n).d(fw, Tw), d(fx_n, Tw)d(fw, Tx_n), \\
&\quad \frac{1}{2}d(fx_n, Tx_n)d(fw, Tx_n), \frac{1}{2}d(fx_n, Tw)d(fw, Tw)\}]^{1/2}
\end{aligned}$$

$$\leq [\max\{d^2(fx_n, fw), d(fx_n, fx_{n+1})d(fw, Tw), d(fx_n, Tw)d(fw, fx_{n+1}), \frac{1}{2}d(fx_n, fx_{n+1})d(fw, fx_{n+1}), \frac{1}{2}d(fx_n, Tw)d(fw, Tw)\}]^{1/2}$$

Passing the limits as $n \to \infty$ and using (3.11), we get

$$d(fw, Tw) \leq \phi(ad(fw, Tw) + (\frac{1-a}{\sqrt{2}})d(fw, Tw)$$

If $fw \notin Tw$, we get, $d(fw, Tw) \leq q(ad(fw, Tw) + (\frac{1-a}{\sqrt{2}})d(fw, Tw)$, a contradiction. Hence $fw \in Tw$. If T(Y) is complete, then since $T(Y) \subseteq f(Y)$, this case pertains to the previous case.

Now if f and T are weakly coincidentally idempotent then $ffw = fw$ for some $w \in C(T, f)$. Then we have

$$H(Tfw, Tw) \leq \phi(aL(fw, w) + (1-a)N(fw, w)) \tag{3.12}$$

$$\begin{aligned} &L(fw, w) \\ &= \max\{d(ffw, fw), d(ffw, Tfw), d(fw, Tw), \frac{1}{2}(d(ffw, Tw) + d(fw, Tfw))\} \\ &\leq \max\{d(fw, fw), d(fw, Tfw), d(fw, Tw), \frac{1}{2}(d(fw, Tw) + d(fw, Tfw))\} \\ &\leq \max\{d(fw, fw), H(Tw, Tfw), d(fw, Tw), \frac{1}{2}(d(fw, Tw) + H(Tw, Tfw))\} \\ &= H(Tw, Tfw), \end{aligned}$$

$$\begin{aligned} &N(fw, w) \\ &= [\max\{d^2(ffw, fw), d(ffw, Tfw)d(fw, Tw), d(ffw, Tw)d(fw, Tfw), \\ &\quad \frac{1}{2}d(ffw, Tfw)d(fw, Tfw), \frac{1}{2}d(ffw, Tw)d(fw, Tw)]^{1/2} \\ &\leq [\max\{d^2(fw, fw), d(fw, Tfw)d(fw, Tw), d(fw, Tw)d(fw, Tfw), \\ &\quad \frac{1}{2}d(fw, Tfw)d(fw, Tfw), \frac{1}{2}d(fw, Tw)d(fw, Tw)]^{1/2} \\ &\leq [\max\{d^2(fw, fw), H(Tw, Tfw)d(fw, Tw), d(fw, Tw)H(Tw, Tfw), \\ &\quad \frac{1}{2}H(Tw, Tfw)H(Tw, Tfw), \frac{1}{2}d(fw, Tw)d(fw, Tw)]^{1/2} \\ &= H(Tw, Tfw). \end{aligned}$$

Hence from (3.12), we have

$$H(Tfw, Tw) \leq \phi(H(Tfw, Tw)).$$

If $Tfw \neq Tw$, we get $H(Tfw, Tw) \leq q(H(Tfw, Tw))$, a contradiction. Hence Tfw = Tw. Thus we have fw = ffw and $fw \in Tw = Tfw$. That is fw is a common fixed point of f and T. □

Theorem 3.4. *Let Y be an arbitrary nonempty set, (X, d) be a metric space,* $f : Y \longrightarrow X$ *and* $T : Y \longrightarrow CL(X)$ *be such that*

$$T(Y) \subseteq f(Y), \tag{3.13}$$

$$H(Tx, Ty) \leq h(aL(x, y) + (1 - a)N(x, y)) \quad \text{for all } x, y \text{ in } Y,\ 0 \leq a \leq 1, \quad \text{and } 0 < h < 1. \tag{3.14}$$

$$f(Y) \text{ or } T(Y) \text{ is complete.} \tag{3.15}$$

Then f and T has a coincidence point. Further if, f and T are weakly coincidentally idempotent, then f and T has a common fixed point.

Proof. A proper blend of proof of Theorem 3.3 and [[11],Theorem 2] will complete the proof. □

Taking a = 1 in Theorem 3.3, we have the following :

Corollary 3.5. *Let Y be an arbitrary nonempty set, (X, d) be a metric space,* $f : Y \longrightarrow X$ *and* $T : Y \longrightarrow CL(X)$ *be such that:*

$$T(Y) \subseteq f(Y), \tag{3.16}$$

$$H(Tx, Ty) \leq \phi(L(x, y)) \text{ for all } x\ ,\ y \text{ in } Y, \tag{3.17}$$

$$\phi(t) \leq qt \text{ for all } \ t > 0 \text{ and for some } q \in (0, 1), \tag{3.18}$$

$$f(Y) \text{ or } T(Y) \text{ is orbitally complete,} \tag{3.19}$$

$$\text{there exists a point } x_0 \text{in } Y \text{ such that } T \text{ is asymptotically regular at } x_0. \tag{3.20}$$

Then f and T has a coincidence point. Further if f and T are weakly coincidentally idempotent, then f and T has a common fixed point.

Taking a = 0 in Theorem 3.3, we have the following :

Corollary 3.6. *Let Y be an arbitrary nonempty set, (X, d) be a metric space,* $f : Y \longrightarrow X$ *and* $T : Y \longrightarrow CL(X)$ *be such that*

$$T(Y) \subseteq f(Y), \tag{3.21}$$

$$H(Tx, Ty) \leq \phi(N(x, y)) \text{ for all } x,\ y \text{ in } Y, \tag{3.22}$$

$$\phi(t) \leq qt \text{ for all } t > 0 \text{ and for some} q \in (0, 1). \tag{3.23}$$

$$f(Y) \text{ or } T(Y) \text{ is orbitally complete.} \tag{3.24}$$

$$\text{there exists a point } x_0 \text{ in } Y \text{ such that } T \text{ is asymptotically regular at } x_0. \tag{3.25}$$

Then f and T has a coincidence point. Further if f and T are weakly coincidentally idempotent, then f and T has a common fixed point.

Taking a = 1 in Theorem 3.4, we have the following :

Corollary 3.7. *Let Y be an arbitrary non empty set , (X, d) be a metric space, $f : Y \longrightarrow X$ and $T : Y \longrightarrow CL(X)$ be such that*

$$T(Y) \subseteq f(Y) \tag{3.26}$$

$$H(Tx, Ty) \leq h.(L(x, y)) \text{ for all } x \text{ , } y \text{ in } Y \tag{3.27}$$

$$f(Y) \text{ or } T(Y) \text{ is orbitally complete} \tag{3.28}$$

Then f and T has a coincidence point. Further if , f and T are weakly coincidentally idempotent, then f and T has a common fixed point.

Taking a = 0 in Theorem 3.4, we have the following:

Corollary 3.8. *Let Y be an arbitrary non empty set, (X, d) be a metric space, $f : Y \longrightarrow X$ and $T : Y \longrightarrow CL(X)$ be such that*

$$T(Y) \subseteq f(Y) \tag{3.29}$$

$$H(Tx, Ty) \leq h(N(x, y)) \text{ for all } x, \text{ } y \text{ in } Y \tag{3.30}$$

$$f(Y) \text{ or } T(Y) \text{ is orbitally complete} \tag{3.31}$$

Then f and T has a coincidence point. Further if f and T are weakly coincidentally idempotent, then f and T has a common fixed point.

The following example shows that Theorem 3.3 is a proper generalization of the fixed point results of [7], [9] , [10], [11].

Example 3.9. Let $X = [0, \infty)$ be endowed with the Euclidean metric, $f : X \longrightarrow X$ and $T : X \longrightarrow CL(X)$ be defined by $fx = 3.(x^2 + x)$ and $Tx = [0, x^2 + 5]$. Then mappings f and T are not commuting and also does not satisfy any of its generalizations, viz weakly commuting, compatible, weak compatible. Also the mappings f and T are not coincidentally commuting. Note that $f1 \in T1$, but $ff1 \neq f1$ and so f and T are not coincidentally idempotent, but $f0 \in T0$ and $ff0 = f0$ and so f and T are weakly coincidentally idempotent. For all x and y in X, we have

$$\begin{aligned}
H(Tx, Ty) &= \mid x^2 - y^2 \mid \\
&= (\frac{(x + y}{3})(x + y + 1))(3 \mid x - y \mid (x + y + 1)) \\
&= (\frac{x + y}{3})(x + y + 1)(3 \mid x^2 - y^2 + x - y \mid) \\
&\leq (1/3)d(fx, fy)
\end{aligned}$$

Thus f and T satisfy all conditions of Theorem 3.1, and 0 is a common fixed point of f and T. But we see that the results of [7], [9], [10], and [11] cannot be applied to the mappings f and T.

Let $\psi : (0, \infty) \longrightarrow [0, 1)$ be a function having the following property (cf. [2],[3]):

(P) For $t > 0$, there exists $\delta(t) > 0,\ s(t) < 1$ such that

$$0 \leq r - t < \delta(t) \textit{ implies } \psi(r) \leq s(t).$$

The following theorem is a generalization of Hu [3, Theorem 2], Jungck [5], Kaneko [8], Nadler [12, Theorem 5] and Beg and Azam [2, Theorem 5.4 and Corollary 5.5.

Theorem 3.10. *Let Y be an arbitrary nonempty set, (X, d) be a metric space, $f : Y \longrightarrow X$ and $T : Y \longrightarrow CL(X)$ be such that*

$$T(Y) \subseteq f(Y) \tag{3.32}$$

$$H^r(Tx, Ty) < \psi(d(fx, Tx))d^r(fx, fy) \quad \textit{for all } x,\ y \textit{ in } Y, \tag{3.33}$$

where r is some positive real number.

If f(Y) is complete, then

(i) *there exists an asymptotically T-regular sequence $\{x_n\}$ with respect to f in Y*

(ii) *f and T has a coincidence point.*

Further if , f and T are weakly coincidentally idempotent, then f and T has a common fixed point.

Proof. For some x_0 in Y, let $y_0 = fx_0$ and choose x_1 in Y such that $y_1 = fx_1 \in Tx_0$. Then by (3.7.2) we have

$$H(Tx_0, Tx_1) < \psi(d(fx_0, fx_1))d^r(fx_0, fx_1).$$

Using (3.32) and Lemma 3.1, we can choose $x_2 \in Y$ such that $y_2 = fx_2 \in Tx_1$ and

$$\begin{aligned} d^r(y_1, y_2) &= d^(fx_1, fx_2) \\ &< \psi(d(fx_0, fx_1))d^r(fx_0, fx_1) \\ &< d^r(fx_0, fx_1). \end{aligned}$$

By induction we construct sequence $\{x_n\}$ in Y and $\{y_n\}$ in f(Y) such that $y_n = fx_n \in Tx_{n-1}$. Also we have,

$$\begin{aligned} d^r(y_{n+1}, y_{n+2}) &= d^r(fx_{n+1}, fx_{n+2}) \\ &< \psi(d(fx_n, fx_{n+1}))d^r(fx_n, fx_{n+1}) \\ &< d^r(fx_n, fx_{n+1}) \\ &= d^r(y_n, y_{n+1}). \end{aligned}$$

It follows that the sequence $\{d(y_n, y_{n+1}\}$ is decreasing and converges to its greatest lower bound say t. Clearly $t \geq 0$. We will claim that $t = 0$.. For if $t > 0$ then by property (P) of ψ, there will exist $\delta(t) > 0,\ s(t) < 1$ such that

$$0 \leq r - t < \delta(t) \text{ implies } \psi(r) \leq s(t).$$

For this $\delta(t) > 0$ there exist a natural number N such that,

$$0 \leq d(y_n, y_{n+1}) - t < \delta(t), \text{ whenever } n \geq N.$$

Hence $\psi(dy_n, y_{n+1}) \leq s(t)$, whenever $n \geq N$.

Let $K = \max\{\psi(d(y_0, y_1)), \psi(d(y_1, y_2)), \cdots, \psi(d(y_{N-1}, y_N)), s(t)\}$.
Then for $n = 1, 2, 3, \cdots$,

$$\begin{aligned} d^r(y_n, y_{n+1}) &< \psi(d(y_{n-1}, y_n))d^r(y_{n-1}, y_n) \\ &\leq K d^r(y_{n-1}, y_n) \\ &\leq K^n d^r(y_0, y_1) \to 0 \text{ as } n \to \infty, \end{aligned}$$

which contradicts the assumption that $t > 0$. Hence

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0,$$

i.e

$$d(fx_n, Tx_n) \to 0.$$

Hence the sequence $\{x_n\}$ is asymptotically T-regular with respect to f.
We claim that $\{fx_n\}$ is a Cauchy sequence. Suppose not. Then there exists a positive number t^* and subsequences $\{n(i)\},\ \{m(i)\}$ of natural numbers with $n(i) < m(i)$ and such that $d(y_{n(i)}, y_{m(i)}) \geq t^*,\ d(y_{n(i)-1}, y_{m(i)-1}) < t^*$ for $i = 1, 2, 3, \cdots$. Then we have

$$\begin{aligned} t^* &\leq d(y_{n(i)}, y_{m(i)}) \\ &\leq d(y_{n(i)}, y_{m(i)-1})d(y_{m(i)-1}, y_{m(i)}). \end{aligned}$$

Letting $i \to \infty$ we get

$$\lim_{n\to\infty} d(y_{n(i)}, y_{m(i)}) = t^*.$$

For this $t^* > 0$ there exists $\delta(t^*) > 0,\ s(T^*) < 1$ such that

$$0 \leq r - t^* < \delta(t^*) \ implies\ \psi(r) \leq s(t^*).$$

For this $\delta(t^*) > 0$ there exists a natural number N_0 such that

$$i \geq N_0 \ implies\ 0 \leq d(y_{n(i)}, y_{m(i)}) - t^* < \delta(t^*).$$

Hence $\psi(d(y_{n(i)}, y_{m(i)})) \leq s(t^*)\ for\, \imath \geq N_0$. Thus we have
$d^r(y_{n(i)}, y_{m(i)} \leq [d(y_{n(i)}, y_{n(i)+1}) + d(y_{n(i)+1}, y_{m(i)+1}) + d(y_{m(i)+1}, y_{m(i)})]^r.$

Expanding binomially we get

$$\begin{aligned} d^r(y_{n(i)}, y_{m(i)} &\leq d^r(y_{n(i)+1}, y_{m(i)+1}) \\ &\quad + \textit{ terms containing } dr(y_{n(i)}, y_{n(i)+1}) \textit{ and } d(y_{m(i)}, y_{m(i)+1}) \\ &\leq \psi(d(y_{n(i)}, y_{m(i)}))d^r(y_{n(i)}, y_{m(i)}) \\ &\quad + \textit{ terms containing } d(y_{n(i)}, y_{n(i)+1}) \textit{ and } d(y_{m(i)}, y_{m(i)+1}) \\ &\leq s(t^*).d^r(y_{n(i)}, y_{m(i)}) \\ &\quad + \textit{ terms containing } d(y_{n(i)}, y_{n(i)+1}) \textit{ and } d(y_{m(i)}, y_{m(i)+1}). \end{aligned}$$

Letting $i \to \infty$ we get $t^* \leq s(t^*)t^* < t^*$, a contradiction. Hence $\{fx_n\}$ is a Cauchy sequence in f(Y). Since f(Y) is complete $\{fx_n\}$ converges to some p in f(Y). Let $z \in f^{-1}(p)$. Then $fz = p$. Then we have
$d^r(fz, Tz) \leq [d(fz, fx_{n+1}) + d(fx_{n+1}, Tz)]^r$. Expanding binomially we get,

$$\begin{aligned} d^r(fz, Tz) &\leq d^r(fx_{n+1}, Tz) \ + \textit{ terms containg } d(fz, fx_{n+1}) \\ &\leq H^r(Tx_n, Tz) \ + \textit{ terms containg } d(fz, fx_{n+1}) \\ &\leq \psi(d(fx_n, fz))d^r(fx_n, fz) \ + \textit{ terms containg } d(fz, fx_{n+1}). \end{aligned}$$

Letting $n \to \infty$ we get

$$d^r(fz, Tz) \leq 0.$$

Hence $fz \in Tz$.

Now if f and T are weakly coincidentally idempotent then $ffw = fw$ for some $w \in C(T, f)$. Then we have

$$H^r(Tfw, Tw) \leq \psi(d(ffw, fw))d^r(ffw, fw) = 0.$$

Hence Tfw = Tw. Thus we have $ffw = fw \in Tw = Tfw$. Hence fw is a common fixed point of T and f. □

Now we state some fixed point theorems for Kannan type multivalued mappings which extends and generalizes the corresponding results of Shiau, Tan and Wang [11] and Beg and Azam [1,2]. A proper blend of proof of Theorem 3.1 and those of [11, Th.6, Th.7, Th.8 respectively] will complete the proof. Hence we omit the proof of these theorems.

Theorem 3.11. *Let Y be an arbitrary nonempty set, (X, d) be a metric space, $f : Y \longrightarrow X$ and $T : Y \longrightarrow CL(X)$ be such that*

$$T(Y) \subseteq f(Y), \tag{3.34}$$

$$H^r(Tx, Ty) \leq \alpha_1(d(fx, Tx))d^r(fx, Tx) + \alpha_1(d(fy, Ty))d^r(fy, Ty), \tag{3.35}$$

for all $x,\ y \in Y$, where $\alpha_i : R \longrightarrow [0, 1)$ $(i = 1, 2)$ and r is some fixed positive real number. If one of f(Y) or T(Y) is complete, and if there exists *an asymptotically T-regular sequence* $\{x_n\}$ *with respect to* f *in* Y *(i.e, If there exist a sequence* $\{x_n\}$ *in* Y *such that*

$$\lim_{n\to\infty} d(fx_n, Tx_n) = 0,$$

then $\{x_n\}$ *is said to be asymptotically T-regular with respect to f.), then* f *and* T *has a coincidence point. Further, if* f *and* T *are weakly coincidentally idempotent, then* f *and* T *has a common fixed point.*

Theorem 3.12. *Let* Y *be an arbitrary nonempty set, (X, d) be a metric space,* $f : Y \longrightarrow X$ *and* $T : Y \longrightarrow CL(X)$ *be such that (3.8.1) and (3.8.2) are satisfied. If one of f(Y) or T(Y) is complete, and if there exists an asymptotically T-regular sequence* $\{x_n\}$ *with respect to* f *in Y, and* Tx_n *is compact, for all* $n \in N$*, then* f *and* T *has a coincidence point. Further, if* f *and* T *are weakly coincidentally idempotent, then* f *and* T *has a common fixed point.*

Theorem 3.13. *Let* Y *be an arbitrary nonempty set, (X, d) be a metric space,* $f : Y \longrightarrow X$ *and* $T : Y \longrightarrow CL(X)$ *be such that (3.7.1) and (3.7.2) are satisfied for all x,y in Y. If one of f(Y) or T(Y) is complete, and if there exists an asymptotically T-regular sequence* $\{x_n\}$ *with respect to* f *in Y, and* Tx_n *is compact, for all* $n \in N$*, then* f *and* T *has a coincidence point. Further, if* f *and* T *are weakly coincidentally idempotent, then* f *and* T *has a common fixed point.*

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