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ON THE SOLUTION OF GENERALIZED EQUATIONS UNDER HÖLDER CONTINUITY CONDITIONS

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Abstract. We provide a local convergence analysis of Newton's method for solving a certain class of generalized equations in a Banach space setting under Hölder and center-Hölder continuity conditions on the Fréchet-derivative of the operator involved. Using more precise estimates and under the same hypotheses/computational cost we provide a finer convergence analysis of Newton's method than before [5], [6], [9].

1. Introduction

In this study we are concerned with the problem of approximating a solution x of the generalized equation

$$0 \in f(x) + F(x), \quad x \in X, \tag{1.1}$$

where f is a Fréchet-differentiable operator defined on a Banach space X with values in a Banach space Y and $F\colon X \to 2^Y$ has a closed graph. If $F = \{0\}$, then (1.1) is an equation. When $F = R^i_+$ the positive orthant in R^i, then (1.1) is a system of inequalities. If F is a normal cone to a convex and closed set in X, then (1.1) represents a variational inequality.

The most popular method for generating a sequence approximating x is undoubtedly Newton's method

$$0 \in f(x_n) + f'(x_n)(x_{n+1} - x_n) + F(x_{n+1}) \quad (n \geq 0), \quad (x_0 \in X), \tag{1.2}$$

where $f'(x_n)$ is the Fréchet derivative of f at x_n and x_0 is the initial guess.

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A survey on local as well as semilocal convergence results on method (1.2) can be found in [1]–[11], and the references there. In particular we pay attention in the work by Piétrus in [9] who is using the K-Hölder condition

$$\|F'(v) - F'(w)\| \le K\|v - w\|^{\alpha} \quad \text{for all } v, w \in D \subseteq X, \quad \alpha \in [0,1]. \tag{1.3}$$

A local convergence analysis was provided in [9] for Newton's method using condition (1.3). Here we also introduce the (K_0, x) center-Hölder continuity condition for a given $x \in D$ by

$$\|F'(v) - F'(x)\| \le K_0\|v - x\|^{\alpha} \quad \text{for all } v \in D. \tag{1.4}$$

Note that

$$K_0 \le K \tag{1.5}$$

holds in general and $\dfrac{K}{K_0}$ can be arbitrarily large [2], [3].

Using a combination of (1.3) and (1.4) instead of only (1.3) we provide (in view of $K_0 < K$) a finer local convergence analysis of Newton's method (1.2).

2. Local Convergence Analysis of Method (1.2)

We need to introduce a certain type of continuity attributed to Aubin [1]. A set-valued map Γ from Y to the subsets of Z is said to be M-pseudo-Lipschitz around

$$(y_0, x_0) \in \text{Graph}\,\Gamma = \{(y, z) \in Y \times Z \mid z \in \Gamma(y)\}$$

if there exist neighborhoods V of y_0 and U of z_0 such that

$$\sup_{z \in \Gamma(y_1) \cap U} \text{dist}(z, \Gamma(y_2)) \le M\|y_1 - y_2\| \quad \text{for all } y_1 \text{ and } y_2 \text{ in } V. \tag{2.1}$$

Let A and C be subsets of X. If we denote by $e(C, A)$ the excess from the set A to the set C given by

$$e(C, A) = \sup\{\text{dist}(x, A),\ x \in C\} \tag{2.2}$$

then (2.1) is equivalent to a definition of a M-pseudo-Lipschitz set-valued replacing (2.1) by

$$e\big(\Gamma(y_1) \cap U, \Gamma(y_2)\big) \le M\|y_1 - y_2\|. \tag{2.3}$$

We need the following generalization of a fixed point theorem [6], [7]:

Lemma 2.1. *Let (X, ρ) be a Banach space, let T map X to the closed subsets of X; let $q_0 \in X$, and let $r > 0$ and $\lambda \in [0, 1)$ be such that:*

$$\textit{dist}(q_0, T(q_0)) < r(1 - \lambda) \tag{2.4}$$

and

$$e\big(T(v) \cap U(q_0, r), T(w)\big) \le \lambda\rho(v, w), \tag{2.5}$$

for all $v, w \in U(q_0, r) = \{x \in X \mid \|x - q_0\| \leq r\}$*. Then* T *has a fixed point in* $U(q_0, r)$*. Moreover, if* T *is single valued, then* X *is the unique fixed point of* T *in* $U(q_0, r)$*.*

Define

$$G(x) = f(x^*) + f'(x^*)(x - x^*) + F(x), \tag{2.6}$$

where x^* is a solution of (1.1). We show the map

$$x \to T_n(x) = G^{-1}\big(f(x^*) + f'(x^*)(x - x^*) - f(x_n) - f'(x_n)(x - x_n)\big) \tag{2.7}$$

has a fixed point x_{n+1}, which will satisfy

$$0 \in f(x_n) + f'(x_n)(x_{n+1} - x_n) + F(x_n).$$

That is x_{n+1} is a solution of (1.2).

We can show the local convergence theorem for Newton's method (1.2).

Theorem 2.2. *Let* x^* *be a solution of* (1.1)*, let* f *be a Fréchet differentiable operator in an open neighborhood* D *of* x^**, let* f' *be* K*-Hölder continuous and* (K, x^*) *center Hölder continuous in* D*. Let* F *be a set-valued map with closed graph and let* G^{-1} *given by* (2.6) *be* M*-pseudo Lipschitz around* $(0, x^*)$*. Then for any* $c > \frac{1}{\alpha+1}MK = c_0$*, there exists* $\delta > 0$ *such that for any initial guess* $x_0 \in U(x^*, \delta)$*, there exists a sequence* $\{x_n\}$ *generated by Newton's method* (1.2) *converging to* x^* *and satisfying*

$$\|x_{n+1} - x^*\| \leq c\|x_n - x^*\|^2 \quad (n \geq 0). \tag{2.8}$$

Moreover if x^* *is an isolated solution of* (1.1)*, then for every* $c > c_0$*, there exists* $\delta > 0$ *such that sequence* $\{x_n\}$ *with* $x_n \in U(x^*, \delta)$ $(n \geq 0)$ *satisfies* (2.8)*.*

To prove Theorem 2.2 we first show:

Proposition 2.3. *Under the hypotheses of Theorem* 2.2 *there exists* $\delta > 0$ *such that for all* $x_0 \neq x^*$ *and* $x_0 \in U(x^*, \delta)$ *the map* $T_0(x)$ *given by* (2.7) *has a fixed point* $x_1 \in U(x^*, \delta)$*.*

Proof. Let $c > c_0$, and let $a > 0$, $b > 0$ be fixed positive constants such that

$$e\big(G^{-1}(v) \cap U(x^*, a), G^{-1}(w)\big) \leq M\|v - w\| \quad \text{for all } v, w \in U(0, b). \tag{2.9}$$

Let $\delta > 0$ be fixed such that $U(x^*, \delta) \subseteq D$ and

$$\delta < \delta_0, \tag{2.10}$$

where,

$$\delta_0 = \min\left\{a, \left[\frac{(1+\alpha)b}{K + K_0(1+\alpha)}\right]^{\frac{1}{1+\alpha}}, \left(\frac{1}{c}\right)^{\frac{1}{\alpha}}, \left[\frac{1}{K_0}\left(\frac{1}{M} - \frac{1}{(1+\alpha)c}\right)\right]^{\frac{1}{\alpha}}\right\}. \tag{2.11}$$

In view of Lemma 2.1 and [3], we have in turn

$$\begin{aligned}\operatorname{dist}(x^*, T_0(x^*)) &\leq e\big(G^{-1}(0) \cap U(x^*, \delta), G^{-1}(f(x^*) - f(x_0) \\ &\quad - f'(x_0)(x^* - x_0))\big) \\ &\leq M\|f(x^*) - f(x_0) - f'(x_0)(x^* - x_0)\| \\ &\leq \frac{MK}{1+\alpha}\|x^* - x_0\|^{1+\alpha} \\ &\leq c\|x^* - x_0\|^{1+\alpha}(1 - MK_0\delta^\alpha) \qquad (2.12)\end{aligned}$$

by the choice of δ and c. Let

$$q_0 = x^*, \quad \lambda = MK_0\delta^\alpha, \quad r = r_0 = c\|x_0 - x^*\|^{1+\alpha}.$$

In view of (2.10) it follows $\lambda \in (0,1)$, and $r_0 \leq a$, which shows (2.4). Set

$$y = f(x^*) + f'(x^*)(x - x^*) - f'(x_0)(x - x_0) \quad \text{for } x \in U(x^*, \delta).$$

Using (1.3) and (1.4) we get

$$\begin{aligned}\|y\| &\leq \|f(x^*) - f(x_0) - f'(x_0)(x^* - x_0)\| + \|(f'(x^*) - f'(x_0))(x - x^*)\| \\ &\leq \frac{1}{1+\alpha}K\|x^* - x_0\|^{1+\alpha} + K_0\|x_0 - x^*\|^\alpha\|x - x^*\| \\ &\leq \left(\frac{1}{1+\alpha}K + K_0\right)\delta^{1+\alpha} < b \qquad (2.13)\end{aligned}$$

by the choice of δ, which implies

$$f(x^*) + f'(x^*)(x - x^*) - f(x_0) - f'(x_0)(x - x_0) \in U(0, b). \qquad (2.14)$$

Therefore by (2.9), (2.13) and (2.14) we get for all $v, w \in U(x^*, r_0)$:

$$\begin{aligned}&e\big(T_0(v) \cap U(x^*, r_0), T_0(w)\big) \\ &\quad \leq e\big(T_0(v) \cap U(x^*, \delta), T_0(w)\big) \leq M\|(f'(x^*) - f'(x_0))(v - w)\| \\ &\quad \leq MK_0\delta^\alpha\|v - w\|,\end{aligned}$$

which shows (2.5). It follows from Lemma 2.1 that there exists a fixed point $x_1 \in U(x^*, r_0)$ for T_0. The proof of Proposition 2.3 is now complete since $\delta > r_0$. □

Proof of Theorem 2.2. Note $x_1 \in U(x^*, r_0)$ implies $\|x_1 - x^*\| \leq r_0 = c\|x_0 - x^*\|^{1+\alpha}$, so that x_1 satisfies (2.8). Using induction we assume $x_k \in U(x^*, r_{k-1})$ and apply Lemma 2.1 with

$$q_0 = x^*, \quad \lambda = MK_0\delta^\alpha, \quad r = r_k = c\|x_k - x^*\|^{1+\alpha}$$

to the map T_k given by (2.7) to obtain the existence of a fixed point $x_{k+1} \in U(x^*, r_k)$ for T_k (by Proposition 2.3).

Let x^* be an isolated solution of (1.1). Note that we can find a sufficiently small neighborhood U inside which x^* is the unique solution of (1.1). Let us choose a fixed $\delta > 0$ satisfying (2.10), such that $U(x^*, j\delta) \subset U$ for $j \in N - \{0\}$. It suffices to use $j = 4$. Let $x_k \in U(x^*, 4\delta)$, then

$$x_{k+1} \in G^{-1}\big(f(x^*) + f'(x^*)(x_{k+1} - x^*) - f(x_k) - f'(x_k)(x_{k+1} - x_k)\big).$$

In view of $x^* = G^{-1}(0) \cap U(x^*, 4\delta)$, we obtain in turn

$$\begin{aligned} \|x_{k+1} - x^*\| &= \operatorname{dist}\big(x_{k+1}, G^{-1}(0) \cap U(x^*, 4\delta)\big) \\ &= \operatorname{dist}\big(x_{k+1}, G^{-1}(0)\big) \le e\big(G^{-1}(T_k) \cap U(x^*, 4\delta), G^{-1}(0)\big) \\ &\le M\|f(x^*) - f(x_k) - f'(x_k)(x^* - x_k)\| \\ &\quad + M\|\big(f'(x_k) - f'(x^*)\big)(x_{k+1} - x^*)\| \\ &\le \frac{1}{1+\alpha} MK\|x^* - x_k\|^{1+\alpha} + MK_0\delta^\alpha\|x_{k+1} - x^*\|, \end{aligned} \tag{2.15}$$

which leads to

$$\|x_{k+1} - x^*\| \le \frac{MK\|x_k - x^*\|^{1+\alpha}}{(1+\alpha)(1 - MK_0\delta^\alpha)} \le c\|x_k - x^*\|^{1+\alpha}, \tag{2.16}$$

by the choice of δ and c. That completes the proof of Theorem 2.2. □

Remark 2.4. If equality holds in (1.5) our results reduce to the corresponding ones in [9]. Moreover if $\alpha = 1$ then they reduce to the ones in [5], [6]. Otherwise they constitute an improvement over these results. Indeed define

$$\delta_1 = \min\left\{a\ ,\ \sqrt{\frac{2b}{3K}}\ ,\ \frac{1}{c}\ ,\ \frac{1}{MK} - \frac{1}{2c}\right\} \tag{2.17}$$

and

$$\delta_2 = \min\left\{a, \left[\frac{(1+\alpha)b}{(2+a)K}\right]^{\frac{1}{1+\alpha}}, \left(\frac{1}{c}\right)^{\frac{1}{\alpha}}, \left[\frac{1}{MK} - \frac{1}{(\alpha+1)c}\right]^{\frac{1}{\alpha}}\right\}. \tag{2.18}$$

The constants δ_1 and δ_2 were used in [9] and [6] respectively instead of our δ_0 given by (2.11). In view of (1.5), (2.11), (2.17) and (2.18) we get

$$\delta_1 \le \delta_0$$

and

$$\delta_2 \le \delta_0.$$

Therefore our results can provide a wider choice of initial guesses x_0 than the corresponding ones in [6], [9]. Note that the choice of δ influences the choice of c, which c can be smaller in our case. These observations are important in computational mathematics [3], [8].

References

[1] J. P. Aubin, *Lipschitz behavior of solutions to convex minimization problems*, Math. Oper. Res., **9** (1984), 87–111.

[2] I. K. Argyros, *A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space*, J. Math. Anal. Appl., **298** (2004), 374–397.

[3] I. K. Argyros, *Newton Methods*, Nova Science Publ. Inc., New York, 2005

[4] I. K. Argyros, *On the secant method for solving nonsmooth equations*, J. Math. Anal. Appl., **322** (2006), 146–157.

[5] A. L. Dontchev, W. W. Hager, *An inverse function theorem for set-valued maps*, Proc. Amer. Math. Soc., **121** (1994), 481–484.

[6] A. L. Dontchev, *Local convergence of the Newton method for generalized equations*, C. R. Acad. Sci. Paris, Série I, **332** (1996), 327–331.

[7] A. D. Ioffe and V. M. Tikhomirov, *Theory of Extremal Problems*, North Holland, Amsterdam, 1979.

[8] L. V. Kantorovich and G. P. Akilov, *Functional Analysis in Normed Spaces*, Pergamon Press, Oxford, 1982.

[9] A. Piétrus, *Generalized equations under mild differentiability conditions*, Rev. R. Acad. Cienc. Exact. Fis. Nat. Mathematicas, **94** (2000), 15–18.

[10] R. T. Rockafellar, *Lipschitz properties of multifunctions*, *Nonlinear Analysis*, **9** (1985), 867–885.

[11] B. Xiao and P. T. Harker, *A nonsmooth Newton method for variational inequalities I: Theory*, Math. Progr., **65** (1994), 151–194.