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CONTINUOUS SELECTIONS OF SET OF MILD SOLUTIONS OF SECOND ORDER DIFFERENTIAL INCLUSIONS

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Abstract. We prove the existence of continuous selections of set of mild solutions of second order differential inclusions of the form

$$x''(t) \in Ax(t) + F(t, x(t)), \quad t \in I = [0, T]$$
$$x(0) = \xi \ , \quad x'(0) = \eta$$

where $F : [0, T] \times X \to 2^X$ is a lower semi continuous, bounded closed, convex set valued map in a separable Banach space X, A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ of bounded linear operators from X to X and $\xi, \eta \in X$.

1. Introduction

Existence of solutions of differential inclusions and integrodifferential equations has been studied by many authors [1, 3, 4]. Existence of solution $x(\cdot, \xi)$ of the Cauchy problem $x'(t) \in F(t, x(t)),\ x(0) = \xi$ such that the map $\xi \to x(\cdot, \xi)$ is continuous from a compact subset of R^n into the space of absolutely continuous functions was proved first by Cellina [7] for F which is Lipchitzean with respect to x, defined on an open subset of $R \times R^n$ and taking compact, uniformly bounded values. Cellina proved that the map that associates the set of solutions $\mathcal{S}(\xi)$ of the above Cauchy problem to the initial point ξ, admits a selection continuous from R^n to the space of absolutely continuous functions.

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The boundedness assumption on the values of F has been avoided in [8] allowing F to take closed non empty values in R^n. An analog result for F defined on a closed subset of $R \times R^n$ has been proved in [9].

Extensions of Cellina's result to Lipchitzean maps with closed non empty values in a separable Banach space has been obtained in [6] and [10]. In [12] Staicu proved the existence of a continuous selection of the set valued map $\xi \to \mathcal{S}(\xi)$ where $\mathcal{S}(\xi)$ is the set of all mild solutions of the Cauchy problem

$$x'(t) \in Ax(t) + F(t, x(t)), \; x(0) = \xi$$

where A is the infinitesimal generator of a C_0 - semi group and F is Lipchitzean with respect to x. Staicu also proved the same result for the set of all weak solutions by considering that $-A$ is a maximal monotone map. In [11] the controllability of the second order differential inclusion in Banach spaces has been studied.

In [5] Benchohra.et.al proved the existence of mild solutions to the class of second order damped differential inclusions with nonlocal conditions by using fixed point theorems. In [2] Anguraj et al. proved the existence of mild solutions for first order integrodifferential inclusions in Banach spaces by using the successive approximation technique.

In this present work we prove the existence of a continuous selection of the set of mild solutions of the second order differential inclusions of the form

$$x''(t) \in Ax(t) + F(t, x(t)), \quad t \in I = [0, T], \quad x(0) = \xi, \quad x'(0) = \eta \tag{1.1}$$

where $F : [0, T] \times X \to 2^X$ is a lower semi continuous bounded convex set valued map, $\xi, \eta \in X$ and A is the linear infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ of bounded linear operators from a Banach space X to X, with norm $\| \cdot \|$.

2. Preliminaries

Let $T > 0$, $I = [0, T]$. Let $\mathcal{L}$ be the σ-algebra of all Lebesgue measurable subsets of I. Let X be a real separable Banach space with norm $\| \cdot \|$. Let 2^X be the family of all non empty subsets of X. Let $\mathcal{B}(X)$ be the family of Borel subsets of X.

If $x \in X$ and A is a subset of X, then we define

$$d(x, A) = \inf\{\|x - y\| : y \in A\}.$$

For any two closed and bounded non empty subsets A and B of X, we define *Housdorff distance* from A to B by

$$h(A, B) = \max\{\sup\{d(x, B) : x \in A\}, \sup\{d(y, A) : x \in B\}\}.$$

Let $C(I,X)$ denote the Banach space of all continuous functions $x : I \to X$ with norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in I\}.$$

A measurable function $x : I \to X$ is *Bochner integrable* if and only if $\|x\|$ is Lebesgue measurable. Let $L^1(I,X)$ denote the Banach space of all Bochner integrable functions $x : I \to X$ with norm $\|x\|_1 = \int_0^T \|x(t)\|dt$. Let $\mathcal{D}$ be the family of all decomposable closed non empty subsets of $L^1(I,X)$.

A set valued map $\mathcal{G} : X \to 2^X$ is said to be *convex closed* if $G(x)$ is convex closed for all $x \in X$.

A set valued map $\mathcal{G} : S \to 2^X$ is said to be *lower semi continuous (l.s.c)* if for every closed subset C of X the set $\{s \in S : \mathcal{G}(s) \subset C\}$ is closed in S.

A function $g : S \to X$ such that $g(s) \in \mathcal{G}(s)$ for all $s \in S$ is called a *selection* of $\mathcal{G}(\cdot)$. Let $\{C(t) : t \in R\}$ be a family of bounded linear operators in a Banach space X . This family is called a strongly continuous cosine family if and only if

(1) $C(0) = I$ (the identity mapping of X onto X),
(2) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s, t \in R$,
(3) $C(t)x$ is continuous in t on R for every $x \in X$.

The strongly continuous sine family $\{S(t) : t \in R\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in R\}$, is defined by

$$S(t)x = \int_0^\infty C(s)x(s)ds, \quad x \in X, \quad t \in R$$

The infinitesimal generator A of a strongly continuous cosine family $\{C(t) : t \in R\}$, is an operator $A : X \to X$ defined by

$$Ax = \frac{d^2}{dt^2} C(t)x|_{t=0}.$$

We define $\left(\frac{d}{dt}\right) C(t)x = AS(t)x$ for all $x \in X$ and $t \in R$ and we have $\left(\frac{d^2}{dt^2}\right) C(t)x = AC(t)x = C(t)Ax$ for all $x \in D$ and $t \in R$.

Now we assume the following:

(H1) There exists a constant $M \geq 1$ such that $|C(t)| \leq M$ and $|S(t)| \leq MT$ for all $t \in I$
(H2) $F : I \times X \to 2^X$ is a lower semi continuous set valued map taking non empty closed bounded values.
(H3) F is $\mathcal{L} \otimes \mathcal{B}(X)$ measurable.

(H4) There exists a $k \in L^1(I, R)$ such that the Hausdorff distance satisfies $h(F(t, x(t)), F(t, y(t))) \leq k(t)\|x(t) - y(t)\|$ for all $x, y \in X$ and a.e. $t \in I$.

(H5) There exists a $\beta \in L^1(I, R)$ such that $d(0, F(t, 0)) \leq \beta(t)$ a.e. $t \in I$.

To prove our theorem we need the following two lemmas which can be easily proved from Lemma 2.1([10]) and Lemma 2.2 ([10]).

Lemma 2.1. *Let $F : I \times S \to 2^X, S \subseteq X$, be measurable with non empty closed values, and let $F(t, \cdot)$ be lower semi continuous for each $t \in I$. Then the map $(\xi, \eta) \to G_F(\xi, \eta)$ given by*

$$G_F(\xi, \eta) = \{v \in L^1(I, X) : v(t) \in F(t, \xi) \cap F(t, \eta) \quad \forall\ t \in I\}$$

is lower semi continuous from S into $\mathcal{D}$ if and only if there exists a continuous function $\beta : S \times S \to L^1(I, R)$ such that for all $\xi, \eta \in S$, we have $d(0, F(t, \xi) \cap F(t, \eta)) \leq \beta(\xi, \eta)(t)$ a.e. $t \in I$.

Lemma 2.2. *Let $\zeta : S \times S \to \mathcal{D}$ be a lower semi continuous set valued map and let $\varphi : S \times S \to L^1(I, X)$ and $\psi : S \times S \to L^1(I, X)$ be continuous maps. If for every $\xi, \eta \in S$ the set*

$$H(\xi, \eta) = cl\{v \in \zeta(\xi, \eta) : \|v(t) - \varphi(\xi, \eta)(t)\| < \psi(\xi, \eta)(t) \ \ a.e\ t \in I\}$$

is non empty, then the map $H : S \times S \to \mathcal{D}$ defined above admits a continuous selection.

3. Second order differential inclusions

Definition 3.1. A function $x(\cdot, \xi, \eta) : I \to X$ is called *a mild solution* of (1.1) if there exists a function $f(\cdot, \xi, \eta) \in L^1(I, X)$ such that

(i) $f(t, \xi, \eta) \in F(t, x(t, \xi, \eta))$ for almost all $t \in I$

(ii) $x(t, \xi, \eta) = C(t)\xi + S(t)\eta + \int_0^t S(t - s)f(s, \xi, \eta)ds$ for each $t \in I$.

Theorem 3.2. *Let A be the linear infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in R\}$ of bounded linear operators of X into X and the hypotheses (H1)–(H5) be satisfied. Then there exists a function $x(\cdot, \cdot, \cdot) : I \times X \times X \to X$ such that*

(i) *$x(\cdot, \xi, \eta) \in \mathcal{S}(\xi)$,the set of all mild solutions* (1.1) *for every $\xi \in X$ and*

(ii) *$(\xi, \eta) \to x(\cdot, \xi, \eta)$ is continuous from $X \times X$ into $C(I, X)$.*

Proof. Let $\epsilon > 0$ be given. For $n \in N$ let $\epsilon_n = \frac{1}{\epsilon^{n+1}}$. Let $M = \sup\{|C(t)| : t \in I\}$. For every $(\xi, \eta) \in X \times X$ define $x_0(\cdot, \xi) : I \to X$ by

$$x_0(t, \xi, \eta) = C(t)\xi + S(t)\eta \tag{3.1}$$

Now

$$\begin{aligned}\|x_0(t,\xi_1,\eta_1)-x_0(t,\xi_2,\eta_2)\| &\le |C(t)|\|\xi_1-\xi_2\|+|S(t)|\|\eta_1-\eta_2\|\\ &\le M\|\xi_1-\xi_2\|+MT\|\eta_1-\eta_2\|\end{aligned}$$

i.e. The map $(\xi,\eta)\to x_0(\cdot,\xi,\eta)$ is continuous from $X\times X$ into $C(I,X)$. For each $(\xi,\eta)\in X$ define $\alpha(\xi,\eta): I\to R$ by

$$\alpha(\xi,\eta)(t)=\beta(t)+k(t)\|x_0(t,\xi,\eta)\|. \tag{3.2}$$

Now

$$\begin{aligned}|\alpha(\xi_1,\eta_1)(t)-\alpha(\xi_2,\eta_2)(t)| &= k(t)\big(\|x_0(t,\xi_1,\eta_1)\|-\|x_0(t,\xi_2,\eta_2)\|\big)\\ &< k(t)\|x_0(t,\xi_1,\eta_1)-x_0(t,\xi_2,\eta_2)\|\\ &< M\|\xi_1-\xi_2\|+MT\|\eta_1-\eta_2\|\end{aligned}$$

i.e. $\alpha(\cdot,\cdot)$ is continuous from $X\times X$ to $L^1(I,R)$. By (H4) and (3.2) we have

$$d(0,F(t,x_0(t,\xi,\eta))<\beta(t)+k(t)\|x_0(t,\xi,\eta)\|$$

and so

$$d(0,F(t,x_0(t,\xi\eta))<\alpha(\xi,\eta)(t)\text{ for a.e. } t\in I. \tag{3.3}$$

Define set valued maps $G_0: X\times X\to 2^{L^1(I,X)}$ and $H_0: X\times X\to 2^{L^1(I,X)}$ by

$$\begin{aligned}G_0(\xi,\eta)&=\{v\in L^1(I,X): v(t)\in F(t,x_0(t,\xi,\eta))\quad\text{a.e. } t\in I\},\\ H_0(\xi,\eta)&=\mathrm{cl}\{v\in G_0(\xi,\eta):\|v(t)\|<\alpha(\xi,\eta)(t)+\epsilon_0\}.\end{aligned}$$

By Lemma 2.2 and (3.3) there exists a continuous selection $h_0: X\times X\to L^1(I,X)$ of $H_0(\cdot,\cdot)$.

Define $m(t)=\displaystyle\int_0^t k(s)ds$.

For $n\ge 1$, define $\beta_n(\xi,\eta)(t)$ by

$$\begin{aligned}\beta_n(\xi,\eta)(t)&=M^nT^n\int_0^t\alpha(\xi,\eta)(s)\frac{[m(t)-m(s)]^{n-1}}{(n-1)\,!}ds\\ &\quad+M^nT^{n+1}\Big(\sum_{i=0}^n\epsilon_i\Big)\frac{[m(t)]^{n-1}}{(n-1)\,!}\end{aligned} \tag{3.4}$$

Set $f_0(t,\xi,\eta)=h_0(\xi,\eta)(t)$. By the definition of h_0 we see that $f_0(t,\xi,\eta)\in F(t,x_0(t,\xi,\eta))$. Define

$$x_1(t,\xi,\eta)=C(t)\xi+S(t)\eta+\int_0^t S(t-s)f_0(s,\xi,\eta)ds\qquad\forall\ t\in I\backslash\{0\} \tag{3.5}$$

By the definition of $H_0(\cdot,\cdot)$ we see that $\|f_0(t,\xi,\eta)\|<\alpha(\xi,\eta)(t)+\epsilon_0$ for all $t\in I\backslash\{0\}$.

From (3.1) and (3.5) we have

$$\begin{aligned}\|x_1(t,\xi,\eta)-x_0(t,\xi,\eta)\| &\le \int_0^t |S(t-s)|\|f_0(s,\xi,\eta)\|ds\\ &< MT\int_0^t \Big(\alpha(\xi,\eta)(s)+\epsilon_0\Big)ds\\ &< MT\int_0^t \alpha(\xi,\eta)(s)ds+MT^2\Big(\sum_{i=0}^{1}\epsilon_i\Big)\\ &< \beta_1(\xi,\eta)(t).\end{aligned}$$

We claim that there are two sequences $\{f_n(\cdot,\xi,\eta)\}$ and $\{x_n(\cdot,\xi,\eta)\}$ such that for $n\ge 1$ the following properties are satisfied:

(a) the map $(\xi,\eta)\to f_n(\cdot,\xi)$ is continuous from $X\times X$ into $L^1(I,X)$.
(b) $f_n(t,\xi,\eta)\in F(t,x_n(t,\xi,\eta))$ for each $(\xi,\eta)\in X\times X$ a.e. $t\in I$.
(c) $\|f_n(t,\xi,\eta)-f_{n-1}(t,\xi,\eta)\|\le k(t)\beta_n(\xi,\eta)(t)$ for a.e. $t\in I$.
(d) $x_{n+1}(t,\xi,\eta)=C(t)\xi+S(t)\eta+\int_0^t S(t-s)f_n(s,\xi,\eta)ds$, for all $t\in I$.

We shall claim the above by induction on n.We assume that already there exist functions $f_1,f_2,\dots,f_n$ and $x_1,x_2\dots,x_n$ satisfying (a)–(d).
Define $x_{n+1}(\cdot,\xi,\eta):I\to X$ by

$$x_{n+1}(t,\xi,\eta)=C(t)\xi+S(t)\eta\int_0^t S(t-s)f_n(s,\xi,\eta)ds,\quad \forall t\in I.$$

Then by (c) and (d), for $t\in I\backslash\{0\}$, we have
$\|x_{n+1}(t,\xi,\eta)-x_n(t,\xi,\eta)\|$

$$\begin{aligned}&=\Big\|\int_0^t S(t-s)\{f_n(s,\xi,\eta)-f_{n-1}(s,\xi,\eta)\}ds\Big\|\\ &\le \int_0^t |S(t-s)|\ \|f_n(s,\xi,\eta)-f_{n-1}(s,\xi,\eta)\|\,ds\\ &\le MT\int_0^t \|f_n(s,\xi,\eta)-f_{n-1}(s,\xi,\eta)\|ds \qquad (3.6)\\ &\le MT\int_0^t k(s)\beta_n(\xi,\eta)(s)ds\\ &\le MT\int_0^t k(s)\Big\{M^nT^n\int_0^s \alpha(\xi,\eta)(u)\frac{[m(s)-m(u)]^{n-1}}{(n-1)\ !}du\\ &\quad+M^nT^{n+1}\Big(\sum_{i=0}^{n}\epsilon_i\Big)\frac{[m(s)]^{n-1}}{(n-1)\ !}\Big\}ds \qquad (3.7)\end{aligned}$$

$$\begin{aligned}
&< M^{n+1}T^{n+1}\int_0^t \alpha(\xi,\eta)(u)\frac{[m(t)-m(u)]^n}{n\,!}du \\
&\quad + M^{n+1}T^{n+2}\Big(\sum_{i=0}^{n+1}\epsilon_i\Big)\frac{[m(t)]^n}{n\,!} \qquad (3.8)\\
&< \beta_{n+1}(\xi,\eta)(t).
\end{aligned}$$

By (H3) we now have

$$\begin{aligned}
&d(f_n(t,\xi,\eta),F_n(t,x_{n+1}(t,\xi,\eta)))\\
&\qquad \le k(t)\|x_{n+1}(t,\xi,\eta)-x_n(t,\xi,\eta)\|\\
&\qquad < k(t)\beta_{n+1}(\xi,\eta)(t) \qquad (3.9)
\end{aligned}$$

Define a set valued map $G_{n+1}: X\times X\to 2^{L^1(I,X)}$ by

$$G_{n+1}(\xi,\eta)=\{v\in L^1(I,X): v(t)\in F(t,x_{n+1}(t,\xi,\eta)) \quad \text{a.e. } t\in I\}$$

By Lemma 2.1 and (3.9), G_{n+1} is lower semi continuous from $X\times X$ into $\mathcal{D}$. Define a set valued map $H_{n+1}: X\times X\to 2^{L^1(I,X)}$ by

$$\begin{aligned}
&H_{n+1}(\xi,\eta)=\\
&\quad \mathrm{cl}\{v\in G_{n+1}(\xi,\eta): \|v(t)-f_n(t,\xi,\eta)\|<k(t)\beta_{n-1}(\xi,\eta)(t) \text{ a.e.} t\in I\} \qquad (3.10)
\end{aligned}$$

Therefore, $H_{n+1}(\xi,\eta)$ is non empty for each $(\xi,\eta)\in X\times X$. By Lemma 2.2 and (3.10) we see that there exists a continuous selection

$$h_{n+1}: X\times X\to L^1(I,X) \quad \text{of } H_{n+1}(\cdot\cdot).$$

Then $f_{n+1}(t,\xi,\eta)=h_{n+1}(\xi\eta)(t)$ for each $\xi,\eta\in X$ and each $t\in I$ satisfies the properties (a)–(c) of our claim. By the property (c) and (3.6)–(3.9) we have

$$\begin{aligned}
\|x_{n+1}(\cdot,\xi,\eta)-x_n(\cdot,\xi,\eta)\|_\infty &\le MT\|f_n(\cdot,\xi,\eta)-f_{n-1}(\cdot,\xi,\eta)\|_1\\
&\le \frac{(MT\|k\|_1)^n}{n!}\big\{MT\|\alpha(\xi,\eta)\|_1+MT\epsilon\big\}
\end{aligned}$$

Therefore, the sequence $\{f_n(\cdot,\xi,\eta)\}$ is a Cauchy sequence in $L^1(I,X)$ and the sequence $\{x_n(\cdot),\xi,\eta)\}$ is a Cauchy sequence in $C(I,X)$. Let $f(\cdot,\xi,\eta)\in L^1(I,X)$ be the limit of the Cauchy sequence $\{f_n(\cdot,\xi,\eta)\}$ and $x(\cdot,\xi,\eta)\in C(I,X)$ be the limit of the Cauchy sequence $\{x_n(\cdot),\xi,\eta)\}$.

Now we can easily show that the map $(\xi,\eta)\to f(\cdot,\xi,\eta)$ is continuous from X into $L^1(I,X)$ and the map $(\xi,\eta)\to x(\cdot,\xi,\eta)$ is continuous from $X\times X$ into $C(I,X)$ and for all $(\xi,\eta)\in X\times X$ and almost all $t\in I$, $f(t,\xi,\eta)\in F(t,x(t,\xi,\eta))$. Taking limit in (d) we obtain

$$x(t,\xi,\eta)=C(t)\xi+S(t)\eta\int_0^t S(t-s)f(s,\xi,\eta)ds \qquad \forall\ t\in I.$$

This completes the proof. □

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