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A THEORY OF SEMILINEAR OPERATOR EQUATIONS UNDER NONRESONANCE CONDITIONS

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Abstract. A general theory of semilinear operator equations under nonresonance conditions is developed in order to unify specific results. Existence of weak solutions (in the energetic space) is established by means of several fixed point principles.

1. Introduction and preliminaries

In this paper we present existence results for the semilinear operator equation

$$\begin{cases} Au = cu + F(u) \\ u \in H_A, \end{cases} \tag{1.1}$$

where H_A is the energetic space associated to a linear and positively defined operator A, defined on a subspace of a Hilbert space H, the constant c is not an eigenvalue of the operator A (nonresonance condition), F is a general operator defined from H to H and the growth of $F(u)$ is at most linear. To obtain our results we use a well known technique initiated by Mawhin and J. Ward Jr. ([5], [6], [7]) in the early 1980's. This technique has been extensively used for different classes of ordinary differential and partially differential equations (see, for example, [1], [3], [9], [10], [11], [12]). The aim of this paper is to present an abstract theory of semilinear operator equations which comprises, as particular cases, different specific results from the literature. The main tools are: energetic space, energetic norm, eigenvalues and eigenvectors, completely continuous operator and condensing operator.

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In what follows, we present basic results from the abstract variational theory (see [8]). Let H be a Hilbert space with the inner product denoted by $\langle .,. \rangle_H$ and $A : D(A) \to H$ a linear operator. We will suppose that $D(A)$ is dense in H. The operator A is said to be *symmetric* if $\langle Au, v \rangle_H = \langle u, Av \rangle_H$ for every $u, v \in D(A)$. The symmetric operator A is said to be *strictly positive* if for every $u \in D(A)$ we have that $\langle Au, u \rangle_H \geq 0$ and $\langle Au, u \rangle_H = 0$ if and only if $u = 0$. The symmetric operator A is said to be *positively defined* if there exists a constant $\gamma^2 > 0$ such that $\langle Au, u \rangle_H \geq \gamma^2 \|u\|_H^2$ for every $u \in D(A)$.

To each positively defined operator there is associated a particular Hilbert space which is called the *energetic space* of the given operator and will be denoted by H_A. The space $D(A)$ is endowed with the inner product

$$\langle u, v \rangle_{H_A} = \langle Au, v \rangle_H \tag{1.2}$$

and the energetic space H_A is the completion of $\left(D(A), \|\cdot\|_{H_A}\right)$, where the *energetic norm* is given by

$$\|u\|_{H_A} = \sqrt{\langle u, u \rangle_{H_A}}.$$

If $u \in D(A)$, since the given operator A is positively defined, we have

$$\|u\|_H \leq \frac{1}{\gamma} \|u\|_{H_A}. \tag{1.3}$$

We attach to the operator A the following problem:

$$\begin{cases} Au = f \\ u \in H_A \end{cases} \tag{1.4}$$

where $f \in H$. By a solution of problem (1.4) we mean an element $u \in H_A$ with $\langle u, v \rangle_{H_A} = \langle f, v \rangle_H$ for every $v \in H_A$.

Theorem 1.1. *For every $f \in H$ there exists a unique solution $u \in H_A$ of problem (1.4).*

We will denote by A^{-1} the inverse of the operator A. Thus:

$$A^{-1} : H \to H_A \subset H, \quad f \in H \longmapsto u_f \in H_A$$

where u_f is the unique solution of problem (1.4). The operator A^{-1} is well defined by the above theorem. Therefore, one has

$$\left\langle A^{-1} f, v \right\rangle_{H_A} = \langle f, v \rangle_H \tag{1.5}$$

for all $v \in H_A$ and $f \in H$. The properties of A^{-1} are given by the next

Lemma 1.2. *A^{-1} is a linear and symmetric operator. If, in addition, the embedding of H_A into H is completely continuous, then A^{-1} is a completely continuous operator from H to H.*

We suppose that the embedding of H_A into H is completely continuous. A constant $\mu \in \mathbb{R}$ is said to be an eigenvalue of a linear operator $T : D \to E$, $D \subset E$, if there exists an element $\phi \in D \backslash \{0\}$ such that

$$T(\phi) = \mu\phi.$$

In this case, ϕ is said to be an eigenvector of T. The general theory on the eigenvalues and eigenvectors of a linear, symmetric and completely continuous operator, see [2] and [13], guarantees for the operator A^{-1} defined above, the following properties:

Proposition 1.3. *i) the set of the eigenvalues of the operator A^{-1} is nonempty and at most countable.*
ii) zero is the only possible cluster point of the set of eigenvalues of A^{-1}.
iii) to each eigenvalue corresponds a finite number of linearly independent eigenvectors.

Proposition 1.4. *There exists an orthonormal sequence $(\phi_k)_{k\geq 1}$ of eigenvectors of A^{-1} which is at most countable and it is complete in the image of A^{-1}, i.e.*

$$A^{-1}u = \sum_{k\geq 1} \left\langle A^{-1}u, \phi_k \right\rangle_H \phi_k$$

for every $u \in H$.

We now consider the following problem for the operator A

$$\begin{cases} Au = \lambda u \\ u \in H_A \end{cases} \tag{1.6}$$

A constant $\lambda \in \mathbb{R}$ is said to be an *eigenvalue* of the operator A, if there exists a non null solution of the problem (1.6). If $\lambda \in \mathbb{R}$ is an eigenvalue of the operator A, then there exists a function (*eigenvector*) $u \in H_A \backslash \{0\}$ such that $\langle u, v\rangle_{H_A} = \lambda \langle u, v\rangle_H$ for every $v \in H_A$. The connection between the eigenvalues and eigenvectors of A^{-1} and the eigenvalues and eigenvectors of A is given in the following proposition.

Proposition 1.5. *i) the eigenvalues of A^{-1} are the inverses of the eigenvalues of the operator A and the eigenvectors are the same.*
ii) the eigenvalues of A^{-1} are positive.

Regarding the eigenvalues and eigenvectors of the operator A we have the following

Theorem 1.6. *Assume that H_A is infinite dimensional. Then there exists a sequence $(\lambda_k)_{k\geq 1}$ of eigenvalues of the operator A and correspondingly an orthonormat sequence (in H) $(\phi_k)_{k\geq 1}$ of eigenvectors such that*

$$0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k \leq ... , \lambda_k \to \infty \text{ as } k \to \infty.$$

Moreover, the sequence $(\phi_k)_{k\geq 1}$ *is an Hilbert base in* H *and the sequence* $(\lambda_k^{-1/2}\phi_k)_{k\geq 1}$ *is an Hilbert base in* H_A.

2. Semilinear nonresonance operator equations

In this section we present existence results for the following problem

$$\begin{cases} Au = cu + F(u) \\ u \in H_A \end{cases} \tag{2.1}$$

where A is a linear, positively defined operator, the constant c is not an eigenvalue of the operator A (nonresonance condition), F is a general operator defined from H to H and the growth of $F(u)$ is at most linear. More exactly, we will apply the fixed point theorems of Banach, Schauder and the Leray-Schauder principle in order to obtain solutions of (2.1), that is a function $u \in H_A$ with

$$\langle u, v\rangle_{H_A} = c\,\langle u, v\rangle_H + \langle F(u), v\rangle_H\,, \quad \text{for all } v \in H_A. \tag{2.2}$$

We suppose that the embedding of H_A into H is completely continuous and that H_A is infinite dimensional. We consider $F : H \to H$ to be a continuous operator and we define

$$L : H_A \to H, \quad Lu = Au - cu$$

Let $L^{-1} : H \to H_A \subset H$ be the inverse of L. If we look a priori for a solution u of the form $u = L^{-1}v$ with $v \in H$, then we have to solve the following fixed point problem in H :

$$(F \circ L^{-1})(v) = v. \tag{2.3}$$

At first we present an auxiliary result. Let $(\lambda_k)_{k\geq 1}$ be the sequence of all eigenvalues of the operator A and let $(\phi_k)_{k\geq 1}$ be the corresponding eigenfunctions, with $\|\phi_k\|_H = 1$.

Lemma 2.1. *Let* c *be any constant with* $c \neq \lambda_k$ *for* $k = 1, 2, ...$ *For each* $v \in H$, *there exists a unique solution* $u \in H_A$ *to the problem*

$$\begin{cases} Lu := Au - cu = v \\ u \in H_A \end{cases}$$

denoted by $L^{-1}v$, *and the following eigenvector expansion holds*

$$L^{-1}v = \sum_{k=1}^{\infty} (\lambda_k - c)^{-1} \langle v, \phi_k\rangle_H\, \phi_k \tag{2.4}$$

where the series converges in H_A. *In addition,*

$$\left\|L^{-1}v\right\|_H \leq \mu_c \|v\|_H \text{ for all } v \in H \tag{2.5}$$

where

$$\mu_c = \max\left\{ |\lambda_k - c|^{-1} ; k = 1, 2, ... \right\}.$$

Proof. We first prove the convergence of the series (2.4). Since $(\lambda_k^{-1/2}\phi_k)_{k\geq 1}$ is a Hilbert base in $(H_A, \|\cdot\|_{H_A})$, we have

$$\begin{aligned} \left\| \sum_{k=m+1}^{m+p} (\lambda_k - c)^{-1} \langle v, \phi_k \rangle_H \phi_k \right\|_{H_A}^2 &= \sum_{k=m+1}^{m+p} \langle v, \phi_k \rangle_H^2 \lambda_k / (\lambda_k - c)^2 \\ &\leq C \sum_{k=m+1}^{m+p} \langle v, \phi_k \rangle_H^2 \end{aligned}$$

where C is a constant such that $\lambda_k/(\lambda_k - c)^2 \leq C$ for all k. Thus the convergence of (2.4) follows from the convergence of the numerical series $\sum_{k=1}^{\infty} \langle v, \phi_k \rangle_H^2$ (Bessel's inequality). Let $u \in H_A$ be the sum of the series (2.4). Next we check that $Lu = v$, i.e.

$$\langle u, w \rangle_{H_A} - c \langle u, w \rangle_H = \langle v, w \rangle_H \quad \text{for all } w \in H_A.$$

Indeed, we have

$$\begin{aligned} \langle u, w \rangle_{H_A} &= \sum_{k=1}^{\infty} (\lambda_k - c)^{-1} \langle v, \phi_k \rangle_H \langle \phi_k, v \rangle_{H_A} \\ &= \sum_{k=1}^{\infty} \lambda_k (\lambda_k - c)^{-1} \langle v, \phi_k \rangle_H \langle \phi_k, v \rangle_H \end{aligned}$$

and

$$\langle u, w \rangle_H = \sum_{k=1}^{\infty} (\lambda_k - c)^{-1} \langle v, \phi_k \rangle_H \langle \phi_k, v \rangle_H .$$

Hence,

$$\begin{aligned} \langle u, w \rangle_{H_A} - c \langle u, w \rangle_H &= \sum_{k=1}^{\infty} \langle v, \phi_k \rangle_H \langle \phi_k, w \rangle_H \\ &= \left\langle \sum_{k=1}^{\infty} \langle v, \phi_k \rangle_H \phi_k, w \right\rangle_H = \langle v, w \rangle_H \end{aligned}$$

as desired.

The uniqueness follows from $c \neq \lambda_k, \; k = 1, 2, ...$

To prove (2.5), observe that

$$\left\| \sum_{k=1}^{m} (\lambda_k - c)^{-1} \langle v, \phi_k \rangle_H \phi_k \right\|_H^2 \longrightarrow \left\| L^{-1} v \right\|_H^2 \text{ as } m \to \infty$$

and, on the other hand,

$$\begin{aligned}\left\|\sum_{k=1}^{\infty}(\lambda_k-c)^{-1}\langle v,\phi_k\rangle_H\,\phi_k\right\|_H^2 &= \sum_{k=1}^{\infty}(\lambda_k-c)^{-2}\langle v,\phi_k\rangle_H^2\\ &\le \mu_c^2\sum_{k=1}^{\infty}\langle v,\phi_k\rangle_H^2 \longrightarrow \mu_c^2\,\|v\|_H^2\,.\end{aligned}$$

This completes the proof. □

In what follows we will give existence results for the problem (2.1). We first show how the fixed point theorems of Banach and Schauder can be used to obtain existence results for problem (2.1).

Theorem 2.2. *Suppose*

$$\lambda_j < c < \lambda_{j+1} \text{ for some } j\in\mathbb{N},\ j\ge 1, \text{ or } 0\le c<\lambda_1 \tag{2.6}$$

Also assume that

$$\|F(v_1)-F(v_2)\|_H \le a\,\|v_1-v_2\|_H \tag{2.7}$$

for all $v_1, v_2 \in H$, *where* a *is a nonnegative constant such that*

$$a\mu_c < 1. \tag{2.8}$$

Then (2.1) has a unique solution $u\in H_A$. *In addition*

$$(F\circ L^{-1})^n(v_0)\to v \text{ in } H \text{ as } n\to\infty$$

for any $v_0\in H$, *where* $v = Lu$.

Proof. We will show that $F\circ L^{-1}$ is a contraction on H. For this, let $v_1, v_2 \in H$. Using (2.7) and (2.5) we have

$$\left\|F(L^{-1}(v_1))-F(L^{-1}(v_2))\right\|_H \le a\left\|L^{-1}(v_1-v_2)\right\|_H \le a\mu_c\,\|v_1-v_2\|_H$$

This together with (2.8) shows that $F\circ L^{-1}$ is a contraction. The conclusion follows from Banach's fixed point theorem. □

Theorem 2.3. *Suppose that (2.6) holds, F is continuous and satisfies the growth condition*

$$\|F(u)\|_H \le a\,\|u\|_H + h \tag{2.9}$$

for all $u\in H$, *where* $h\in\mathbb{R}_+$ *and* $a\in\mathbb{R}_+$ *is as in (2.8). Then (2.1) has at least one solution* $u\in H_A$.

Proof. We have $F\circ L^{-1} = F\circ J\circ L_0^{-1}$ where

$$\begin{cases} L_0^{-1}: H\to H_A,\ L_0^{-1}u = L^{-1}u \text{ and}\\ J: H_A\to H,\ Ju = u.\end{cases}$$

The operator L_0^{-1} is linear and continuous, the operator J was supposed to be completely continuous, while F is continuous and by (2.9) is bounded. Thus,

$F \circ L^{-1}$ is a completely continuous operator. On the other hand, from (2.9) and (2.5) we have

$$\left\|F(L^{-1}(v))\right\|_H \le a \left\|L^{-1}(v)\right\|_H + h \le a\mu_c \|v\|_H + h.$$

Now (2.8) guarantees that $F \circ L^{-1}$ is a self-map of a sufficiently large closed ball of H. Thus we may apply Schauder's fixed point theorem. □

Better results can be obtained if we use the Leray-Schauder principle (see [12]).

Theorem 2.4. *Suppose that F is continuous and has the decomposition*

$$F(u) = G(u) + F_0(u) + F_1(u)$$

Also assume that $0 \le c \le \beta < \lambda_1$ and

$$\|F_0(u)\|_H \le a \|u\|_H + h_0 \tag{2.10}$$

$$\|F_1(u)\|_H \le b \|u\|_H + h_1 \tag{2.11}$$

$$\langle u, F_1(u)\rangle_H \le 0 \tag{2.12}$$

$$\langle G(u), u\rangle_H \le (\beta - c) \|u\|_H^2 \tag{2.13}$$

for all $u \in H$, where $a, b, h_0, h_1, \beta \in \mathbb{R}_+$. In addition, assume that

$$a/\lambda_1 < 1 - \beta/\lambda_1. \tag{2.14}$$

Then (2.1) has at least one solution $u \in H_A$.

Proof. We look for a fixed point $v \in H$ of $F \circ L^{-1}$. As above, $F \circ L^{-1}$ is a completely continuous operator. We will show that the set of all solutions to

$$v = \lambda(F \circ L^{-1})(v), \tag{2.15}$$

when $\lambda \in [0, 1]$, is bounded in H. Let $v \in H$ be any solution of (2.15). Let $u = L^{-1}v$. It is clear that u solves

$$\begin{cases} Au - cu = \lambda F(u) \\ u \in H_A \end{cases} \tag{2.16}$$

Since u is a weak solution of (2.16), we have

$$\|u\|_{H_A}^2 = \langle cu + \lambda F(u), u\rangle_H .$$

From (2.13) we deduce

$$\langle cu + \lambda G(u), u\rangle_H \le \beta \|u\|_H^2 . \tag{2.17}$$

We define

$$R(u) := \|u\|_{H_A}^2 - \beta \|u\|_H^2 \tag{2.18}$$

and using (2.12) and (2.17) we obtain

$$\begin{aligned} R(u) &\leq \langle cu + \lambda G(u), u \rangle_H + \lambda \langle F_0(u), u \rangle_H + \lambda \langle F_1(u), u \rangle_H - \beta \|u\|_H^2 \\ &\leq |\langle F_0(u), u \rangle_H| \end{aligned}$$

On the other hand, if we denote $c_k = \langle u, \phi_k \rangle_H = \langle u, \phi_k \rangle_{H_A} / \lambda_k$, we see that

$$\begin{aligned} R(u) &= \sum_{k=1}^{\infty} (\lambda_k - \beta) c_k^2 \geq \sum_{k=1}^{\infty} \lambda_k (1 - \beta/\lambda_1) c_k^2 \\ &\geq (1 - \beta/\lambda_1) \|u\|_{H_A}^2 . \end{aligned} \tag{2.19}$$

Recall that

$$\lambda_1 = \inf \left\{ \|u\|_{H_A}^2 / \|u\|_H^2 ; u \in H_A \setminus \{0\} \right\}$$

and using (2.19), (2.18), (2.10) and the fact that A is a positively defined operator, we obtain

$$\begin{aligned} (1 - \beta/\lambda_1) \|u\|_{H_A}^2 &\leq |\langle F_0(u), u \rangle_H| \leq \|F_0(u)\|_H \|u\|_H \leq a \|u\|_H^2 + h_0 \|u\|_H \\ &\leq \frac{a}{\lambda_1} \|u\|_{H_A}^2 + C \|u\|_{H_A} \end{aligned}$$

for some constant $C > 0$. Thus (2.14) guarantees that there is a constant $r > 0$ independent of λ with $\|u\|_{H_A} \leq r$. Finally, a bound for $\|v\|_H$ can be immediately derived from $u = L^{-1} v$. The conclusion now follows from the Leray-Schauder principle. □

When $G = F_1 = 0$, Theorem 2.4 reduces to Theorem 2.3 for $j = 1$. Indeed, we have $\beta = c < \lambda_1$, $\mu_c = 1/(\lambda_1 - c)$ and it is easy to check that (2.14) is equivalent to (2.8).

Theorem 2.5. *Suppose that F is continuous and has the decomposition*

$$F(u) = G(u) + F_0(u) + F_1(u).$$

Also assume that $0 \leq c \leq \beta < \lambda_1$ *and*

$$\|F_0(u) - F_0(\overline{u})\|_H \leq a \|u - \overline{u}\|_H \tag{2.20}$$

$$\|F_1(u)\|_H \leq a_1 \|u\|_H + h \tag{2.21}$$

$$\langle u, F_1(u) \rangle \leq 0 \tag{2.22}$$

$$\langle G(u), u \rangle_H \leq (\beta - c) \|u\|_H^2 \tag{2.23}$$

for all $u, \overline{u} \in H$ *where* $a, a_1, \beta, h \in \mathbb{R}_+$. *In addition, assume that (2.14) holds. Then (2.1) has at least one solution* $u \in H_A$.

Proof. Let $F \circ L^{-1} = T_0 + T_1$, where

$$T_0(v) = (F_0 \circ L^{-1})(v)$$

and

$$T_1(v) = (G \circ L^{-1})(v) + (F_1 \circ L^{-1})(v)$$

for $v \in H$. Then T_1 is a completely continuous map, while T_0 is a contraction since (2.14) implies (2.8). Hence $F \circ L^{-1}$ is a set-contraction on H. Next, the a priori bound of solutions is obtained by essentially the same reasoning as in Theorem 2.4. □

Notice that when $G = F_1 = 0$, Theorem 2.5 reduces to Theorem 2.2 for $0 \le c < \lambda_1$.

Theorem 2.6. *Suppose that F is continuous and has the decomposition*

$$F(u) = G(u) + F_0(u).$$

Also assume that the following conditions are satisfied:

$$\|F_0(u)\|_H \le a \|u\|_H + h \tag{2.24}$$

$$\langle G(u), z - y\rangle_H \le (c - \beta_1) \|y\|_H^2 + (\beta_2 - c) \|z\|_H^2 \tag{2.25}$$

for all $u \in H$, $y = \sum_{k=1}^{j} c_k\phi_k$, $z = \sum_{k=j+1}^{\infty} c_k\phi_k$, where $c_k = \langle u, \phi_k\rangle_H$, and $a, \beta_1, \beta_2 \in \mathbb{R}_+$, $j \ge 1$. In addition, assume that $\lambda_j < \beta_1 \le c \le \beta_2 < \lambda_{j+1}$ and

$$a/\lambda_1 < \min\{\beta_1/\lambda_j - 1,\ 1 - \beta_2/\lambda_{j+1}\}. \tag{2.26}$$

Then (2.1) has at least one solution $u \in H_A$.

Proof. Let $v \in H$ any solution to (2.15) and $u = L^{-1}v$. Since $(\lambda_k^{-1/2}\phi_k)_{k\ge1}$ is a Hilbert base for H_A, we may decompose H_A as follows:

$$H_A = X_1 \oplus X_2,$$

where X_1 is the subspace generated by the first j eigenvectors $\phi_1, \phi_2, \dots, \phi_j$ and $X_2 = X_1^{\perp}$. Let $u = y + z$ with $y \in X_1$ and $z \in X_2$. Then

$$y = \sum_{k=1}^{j} c_k\phi_k,\ z = \sum_{k=j+1}^{\infty} c_k\phi_k,$$

where

$$c_k = \langle u, \phi_k\rangle_H = \langle u, \phi_k\rangle_{H_A}/\lambda_k. \tag{2.27}$$

Since u is a solution to (2.16), we have

$$\langle u, z - y\rangle_{H_A} = \langle cu + \lambda F(u), z - y\rangle_H.$$

Hence,

$$\begin{gathered}\|z\|_{H_A}^2 - c\|z\|_H^2 - \lambda\langle G(u), z\rangle_H - \|y\|_{H_A}^2 + c\|y\|_H^2 + \lambda\langle G(u), y\rangle_H = \\ = \|z\|_{H_A}^2 - c\|z\|_H^2 - \|y\|_{H_A}^2 + c\|y\|_H^2 - \lambda\langle G(u), z-y\rangle_H = \\ = \lambda\langle F_0(u), z-y\rangle_H\end{gathered}$$

Furthermore, if we denote by

$$R(u) := \|z\|_{H_A}^2 - \beta_2\|z\|_H^2 - \|y\|_{H_A}^2 + \beta_1\|y\|_H^2,$$

by (2.25) we deduce

$$R(u) \le |\langle F_0(u), z-y\rangle_H|. \tag{2.28}$$

Using (2.27), we find that

$$\begin{aligned} R(u) &= \sum_{k=j+1}^{\infty}(\lambda_k - \beta_2)\,c_k^2 + \sum_{k=1}^{j}(\beta_1 - \lambda_k)c_k^2 \\ &= \sum_{k=j+1}^{\infty}(1 - \beta_2/\lambda_k)\,\lambda_k c_k^2 + \sum_{k=1}^{j}(\beta_1/\lambda_k - 1)\,\lambda_k c_k^2 \\ &\ge \min\{\beta_1/\lambda_j - 1,\ 1 - \beta_2/\lambda_{j+1}\}\,\|u\|_{H_A}^2 \end{aligned}$$

On the other side, from (2.24),

$$|\langle F_0(u), z-y\rangle_H| \le (a\|u\|_H + h)\,\|z-y\|_H$$

and since $\|z-y\|_H = \|z+y\|_H = \|u\|_H$, this yields

$$|\langle F_0(u), z-y\rangle_H| \le a\|u\|_H^2 + h\|u\|_H \le \frac{a}{\lambda_1}\|u\|_{H_A}^2 + C\|u\|_{H_A}$$

for some constant $C > 0$. Thus, (2.28) implies that

$$\begin{aligned}&\min\{\beta_1/\lambda_j - 1,\ 1 - \beta_2/\lambda_{j+1}\}\,\|u\|_{H_A}^2 \\ &\le \frac{a}{\lambda_1}\|u\|_{H_A}^2 + C\|u\|_{H_A}.\end{aligned}$$

This, together with (2.26) guarantees that there exists $r > 0$ independent of λ with $\|u\|_{H_A} \le r$. Next, as usual, we obtain a bound of $\|v\|_H$ and we apply the Leray-Schauder principle. □

Remark 2.7. In Theorem 2.6 we can replace condition (2.25) with the following conditions:

$$\langle G(u), z\rangle_H \le (\beta_2 - c)\,\|z\|_H^2 \tag{2.29}$$

and

$$\langle G(u), y\rangle_H \ge (\beta_1 - y)\,\|y\|_H^2.$$

Theorem 2.8. *Suppose that F is continuous and has the decomposition*

$$F(u) = G(u) + F_0(u). \tag{2.30}$$

Also assume that the following condition is satisfied:

$$\|F_0(u) - F_0(\overline{u})\|_H \le a \|u - \overline{u}\|_H \tag{2.31}$$

for all $u \in H$. In addition, assume that (2.8), (2.25), (2.26) hold, where $y = \sum\limits_{k=1}^{j} c_k\phi_k$, $z = \sum\limits_{k=j+1}^{\infty} c_k\phi_k$, $c_k = \langle u, \phi_k\rangle_H$, $a, \beta_1, \beta_2 \in \mathbb{R}_+$, $j \ge 1$, and $\lambda_j < \beta_1 \le c \le \beta_2 < \lambda_{j+1}$. Then (2.1) has at least one solution $u \in H_A$.

Proof. Condition (2.8) implies that $F \circ L^{-1}$ is a set-contraction, while (2.26) ensures the a priori boundedness of the solutions. □

3. Application to semilinear elliptic equations

In this section several known results are obtained as consequences of our abstract theory from Section2.

Let $\Omega \subset \mathbb{R}^n$ be an open, bounded subset of $\mathbb{R}^n$. We consider H as being the Hilbert space $L^2(\Omega)$, the operator $A = (-\Delta)^{-1}$ and $D(A) = C_0^2(\overline{\Omega}) = \left\{u \in C^2\left(\overline{\Omega}\right) : u = 0 \text{ on } \partial\Omega\right\}$. It is well known that $(-\Delta)^{-1}$ is a symmetric, positively defined operator (by Poincare's inequality) on $C_0^2(\overline{\Omega})$. We endow $C_0^2(\overline{\Omega})$ with the inner product

$$\langle u, v\rangle_{H_0^1} = \langle -\Delta u, v\rangle_{L^2} = \langle \nabla u, \nabla v\rangle_{L^2},$$

for every $u, v \in C_0^2(\overline{\Omega})$. Thus, $C_0^2(\overline{\Omega})$ endowed with the inner product $\langle \cdot, \cdot\rangle_{H_0^1}$ is a pre-hilbertian space. The completion of $\left(C_0^2(\overline{\Omega}), \langle \cdot, \cdot\rangle_{H_0^1}\right)$is $H_0^1(\Omega)$. Therefore, the operator $(-\Delta)^{-1}$ can be exetended to $H_0^1(\Omega)$ and (see [13]) it is completely continuous as an operator from $L^2(\Omega)$ into $L^2(\Omega)$ since the embedding of $H_0^1(\Omega)$ in $L^2(\Omega)$ is compact (see [4]). In what follows, $\langle \cdot, \cdot\rangle_{H_0^1}$ will also stay for the inner product of $H_0^1(\Omega)$ and the corresponding energetic norm will be denoted by $\|\cdot\|_{H_0^1}$ and is given by

$$\|u\|_{H_0^1} = \sqrt{\langle u, u\rangle_{H_0^1}}.$$

Let us consider the semiliniar Dirichlet problem

$$\begin{cases} -\Delta u = cu + f(t, u), & \text{on } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

under the assuption that the constant c is not an eigenvalue of $-\Delta$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function which satisfies the *Caratheodory conditions, i.e.*

$f(\cdot, w)$ is measurable for each $w \in \mathbb{R}$ and $f(t, \cdot)$ is continuous for a.e. $t \in \Omega$, and the growth condition

$$|f(t,u)| \le a\,|u| + h(t), \tag{3.2}$$

for all $u \in \mathbb{R}$ and a.e. $t \in \Omega$, where a and b are nonnegative constants and $h \in L^2(\Omega, \mathbb{R}_+)$.

We look for a *weak solution* to (3.1), that is a function $u \in H_0^1(\Omega)$ with

$$\langle u, v\rangle_{H_0^1} = c\,\langle u, v\rangle_{L^2} + \langle f(t,u), v\rangle_{L^2}\,,$$

for all $v \in H_0^1(\Omega)$. Note that in this case, the general operator F from the abstract theory is the usual Nemitskii superposition operator

$$F(u)(t) = f(t, u(t)).$$

We define the operator $L : H_0^1(\Omega) \to L^2(\Omega)$ given by $Lu = -\Delta u - cu$. If we look a priori for a solution u of the form $u = L^{-1}v$ with $v \in L^2(\Omega)$, hence in the subspace $(-\Delta)^{-1}(L^2(\Omega))$, then we have to solve a fixed point problem on $L^2(\Omega):\ T(v) = v$, where

$$T : L^2(\Omega) \to L^2(\Omega), \quad T(v) = f(\cdot, L^{-1}v) \tag{3.3}$$

Theorems 2.2 to 2.8 yield, in particular, the following existence results to problem (3.1). The first two theorems are also given in [14] and [12].

Theorem 3.1. *Assume that*

$$\lambda_j < c < \lambda_{j+1} \text{ for some } j \in \mathbb{N},\ j \ge 1,\ \text{ or } 0 \le c < \lambda_1 \tag{3.4}$$

Also assume that f satisfies the Caratheodory conditions, $f(\cdot, 0) \in L^2(\Omega)$ and that f satisfies the Lipschitz condition

$$|f(t, v_1) - f(t, v_2)| \le a\,|v_1 - v_2| \tag{3.5}$$

for every $v_1, v_2 \in \mathbb{R}$, a.e. $t \in \Omega$,where a is a nonnegative constant with

$$a\mu_c < 1. \tag{3.6}$$

Then (3.1) has a unique solution $u \in H_0^1(\Omega)$. In addition

$$T^k(w) \to v,\ \text{ in } L^2(\Omega) \text{ as } k \to \infty$$

for all $w \in L^2(\Omega)$, where $u = L^{-1}v$.

Proof. From (3.5) we deduce

$$|f(t,u)| \le |f(t,u) - f(t,0)| + |f(t,0)| \le a\,|u| + |f(t,0)|$$

for every $u \in \mathbb{R}$ and a.e. $t \in \Omega$. Moreover, f being a Caratheodory function, we have the Nemitskii operator

$$u \longmapsto f(\cdot, u(\cdot))$$

well defined, bounded and continuous from $L^2(\Omega)$ into $L^2(\Omega)$ (see [15]). Using again (3.5) we obtain

$$\int_\Omega |f(t,v_1(t)) - f(t,v_2(t))|^2 \, dt \le a^2 \int_\Omega |v_1(t) - v_2(t)|^2 \, dt.$$

Consequently,

$$\|F(v_1) - F(v_2)\|_{L^2} \le a \|v_1 - v_2\|_{L^2}.$$

Thus, condition (2.7) in Theorem 2.2 is satisfied. The conclusion follows now by applying Theorem 2.2. □

Theorem 3.2. *Suppose that (3.4) holds and f satisfies the Caratheodory conditions and the growth condition (3.2) with a as in (3.6). Then (3.1) has at least one solution $u \in H_0^1(\Omega)$.*

Proof. Since f satisfies the the Caratheodory conditions and the growth condition (3.2) we deduce that the Nemitskii superposition operator

$$u \longmapsto f(\cdot, u(\cdot)) = F(u)$$

is well defined, bounded and continuous from $L^2(\Omega)$ into $L^2(\Omega)$. Let now $u \in L^2(\Omega)$. Using (3.2) we have

$$f(t,u(t))^2 \le a^2 u^2(t) + 2a\,|u(t)|\,h(t) + h^2(t).$$

Furthermore, by integration on Ω and applying Holder's inequality we obtain

$$\begin{aligned} \|F(u)\|_{L^2}^2 &\le a^2 \|u\|_{L^2}^2 + 2a \|u\|_{L^2} \|h\|_{L^2} + \|h\|_{L^2}^2 \\ &\le \left(a \|u\|_{L^2} + \|h\|_{L^2}^2\right)^2. \end{aligned}$$

Hence,

$$\|F(u)\|_{L^2} \le a \|u\|_{L^2} + C,$$

where $C = \|h\|_{L^2}$, so condition (2.9) in Theorem 2.3 is fulfilled. The conclusion follows now by applying Theorem 2.3. □

In what follows, by F, F_0, F_1, we shall mean the Nemitskii superposition operators associated to the functions f, f_0 and f_1 respectively, i.e.

$$F(u) = f(\cdot, u(\cdot)), \ F_0(u) = f_0(\cdot, u(\cdot)), \ F_1(u) = f_1(\cdot, u(\cdot)).$$

Theorem 3.3. *Suppose that f has the decomposition*

$$f(t,u) = g(t,u) + f_0(t,u) + f_1(t,u)$$

where g, f_0, f_1 satisfy the Caratheodory conditions. Also assume that $0 \le c \le \beta < \lambda_1$ and

$$|f_0(t,u)| \le a\,|u| + h_0(t) \tag{3.7}$$

$$|f_1(t,u)| \le b\,|u| + h_1(t) \tag{3.8}$$

$$u f_1(t,u) \le 0 \tag{3.9}$$

$$g(t,u) \le (\beta - c)u \tag{3.10}$$

for all $u \in \mathbb{R}$, *a.e.* $t \in \Omega$, *where* $a, b, \beta \in \mathbb{R}_+$, $h_0, h_1 \in L^2(\Omega; \mathbb{R}_+)$. *In addition, we assume that*

$$a/\lambda_1 < 1 - \beta/\lambda_1. \tag{3.11}$$

Then (3.1) has at least one solution $u \in H_0^1(\Omega)$.

Proof. Let $G(u) = g(\cdot, u(\cdot))$. As above, by the fact that g, f_0, f_1 satisfy the Caratheodory conditions and by (3.7), (3.8), (3.10) we deduce that F is continuous. Also (3.7) and (3.8) imply (2.10) respectively (2.11). Integrating (3.9) on Ω we obtain (2.12). Let now $u \in L^2(\Omega)$. From (3.10) we have

$$g(t, u(t)) \le (\beta - c)u(t)$$

a.e. $t \in \Omega$. Hence, multiplying by $u(t)$ and integrating we obtain (2.13). The conclusion now follows from Theorem 2.4. □

The following result is also given [12]. Here it is a direct consequence of Theorem 2.4

Theorem 3.4. *Suppose that* f *has the decomposition*

$$f(t,u) = g(t,u)u + f_0(t,u) + f_1(t,u)$$

where g, f_0, f_1 *satisfy the Caratheodory conditions. Also assume that* $0 \le c \le \beta < \lambda_1$ *and that conditions (3.7), (3.8), (3.9) and*

$$g(t,u) + c \le \beta < \lambda_1 \tag{3.12}$$

are satisfied for all $u \in \mathbb{R}$, *a.e.* $t \in \Omega$, *where* $a, b, \beta \in \mathbb{R}_+$, $h_0, h_1 \in L^2(\Omega; \mathbb{R}_+)$. *In addition, assume that (3.11) is satisfied. Then (3.1) has at least one solution* $u \in H_0^1(\Omega)$.

Proof. Considering $G(u) = g(\cdot, u(\cdot))u(\cdot)$, the proof follows from Theorem 2.4, similarly to the proof of Theorem 3.3. □

Theorem 3.5. *Suppose that* f *has the decomposition*

$$f(t,u) = g(t,u) + f_0(t,u) + f_1(t,u)$$

where g, f_0, f_1 *satisfy the Caratheodory conditions and* $f_0(\cdot, 0) \in L^2(\Omega)$. *We also assume that* $0 \le c \le \beta < \lambda_1$*and*

$$|f_0(t, v_1) - f_0(t, v_2)| \le a\,|v_1 - v_2| \tag{3.13}$$

$$|f_1(t,u)| \le a_1\,|u| + h(t) \tag{3.14}$$

$$u f_1(t,u) \le 0 \tag{3.15}$$

$$g(t,u) \le (\beta - c)u \tag{3.16}$$

for all u,$v_1, v_2 \in \mathbb{R}$, *a.e.* $t \in \Omega$, *where* $a, a_1, \beta \in \mathbb{R}_+$, *and* $h \in L^2(\Omega; \mathbb{R}_+)$. *In addition, we assume that (3.11) holds. Then (3.1) has at least one solution* $u \in H_0^1(\Omega)$.

Proof. Let $G(u) = g(\cdot, u(\cdot))$. As above, F is continuous and (3.13), (3.14), (3.15) and (3.16) imply conditions (2.20), (2.21), (2.22) and (2.23) respectively. The conclusion follows from Theorem 2.5. □

For the remainder of this paragraph, we will consider $G(u) = g(\cdot, u(\cdot))u(\cdot)$. The following results are also given in [12].

Theorem 3.6. *Suppose that f has the decomposition*

$$f(t,u) = g(t,u)u + f_0(t,u) + f_1(t,u)$$

where g, f_0, f_1 satisfy the Caratheodory conditions and $f_0(\cdot, 0) \in L^2(\Omega)$. We also assume that $0 \le c \le \beta < \lambda_1$ and that conditions (3.13), (3.14), (3.15) and (3.12) are satisfied for all $u, v_1, v_2 \in \mathbb{R}$, a.e. $t \in \Omega$, where $a, a_1, \beta \in \mathbb{R}_+$, $h \in L^2(\Omega; \mathbb{R}_+)$. In addition, we assume that (3.11) is satisfied. Then (3.1) has at least one solution $u \in H_0^1(\Omega)$.

Proof. The proof follows similarly from Theorem 2.5. □

Theorem 3.7. *Suppose that f has the decomposition*

$$f(t,u) = g(t,u)u + f_0(t,u)$$

where g and f_0 satisfy the Caratheodory conditions. Also assume that the following conditions are satisfied:

$$|f_0(t,u)| \le a\,|u| + h(t) \tag{3.17}$$

$$\lambda_j < \beta_1 \le g(t,u) + c \le \beta_2 < \lambda_{j+1} \tag{3.18}$$

for all $u \in \mathbb{R}$, where $a, \beta_1, \beta_2 \in \mathbb{R}_+$, $h \in L^2(\Omega, \mathbb{R}_+)$, $j \ge 1$ and $\beta_1 \le c \le \beta_2$. If

$$a/\lambda_1 < \min\{\beta_1/\lambda_j - 1, 1 - \beta_2/\lambda_{j+1}\}, \tag{3.19}$$

then (3.1) has at least one solution $u \in H_0^1(\Omega)$.

Proof. As above, condition (3.17) implies (2.24). We will show that condition (3.18) implies condition (2.25). Let $u \in L^2(\Omega)$, $y = \sum_{k=1}^{j} c_k\phi_k$, $z = \sum_{k=j+1}^{\infty} c_k\phi_k$, where $c_k = \langle u, \phi_k\rangle_{L^2}$. We have that $u = y + z$. Let $\lambda \in (0,1)$. From (3.18) we have that

$$c + \lambda g = \lambda(c+g) + (1-\lambda)c \ge \lambda\beta_1 + (1-\lambda)\beta_1 = \beta_1$$

and

$$c + \lambda g = \lambda(c+g) + (1-\lambda)c \le \lambda\beta_2 + (1-\lambda)\beta_2 = \beta_2.$$

Furthermore,

$$c\,\|y\|_{L^2}^2 + \lambda\,\langle g(t,u)y, y\rangle_{L^2} = \langle (c+\lambda g)\,y, y\rangle_{L^2} \ge \beta_1\,\|y\|_{L^2}^2\,,$$

therefore

$$\langle g(t,u)y,y\rangle_{L^2} \geq (\beta_1 - c)\,\|y\|^2_{L^2}\,.$$

Similarly,

$$c\,\|z\|^2_{L^2} + \lambda\,\langle g(t,u)z,z\rangle_{L^2} = \langle (c+\lambda g)\,z,z\rangle_{L^2} \leq \beta_2\,\|z\|^2_{L^2}\,.$$

Hence,

$$\langle g(t,u)z,z\rangle_{L^2} \leq (\beta_2 - c)\,\|z\|^2_{L^2}\,.$$

Consequently,

$$\begin{aligned}\langle g(t,u)z,z\rangle_{L^2} - \langle g(t,u)y,y\rangle_{L^2} &= \int_\Omega g(t,u)(z^2-y^2)dt = \\ &= \langle g(t,u)u, z-y\rangle \leq \\ &\leq (\beta_2 - c)\,\|z\|^2_{L^2} - (\beta_1 - c)\,\|y\|^2_{L^2}\,.\end{aligned}$$

Thus, condition (2.25) in Theorem 2.6 is satisfied. The conclusion follows applying Theorem 2.6. □

Theorem 3.8. *Suppose that f has the decomposition*

$$f(t,u) = g(t,u)u + f_0(t,u)$$

where g and f_0 satisfy the Caratheodory conditions and $f_0(\cdot,0) \in L^2(\Omega)$. Also assume that the following conditions are satisfied:

$$|f_0(t,v_1) - f_0(t,v_2)| \leq a\,|v_1 - v_2| \qquad (3.20)$$

$$\lambda_j < \beta_1 \leq g(t,u) + c \leq \beta_2 < \lambda_{j+1} \qquad (3.21)$$

for all $u, v_1, v_2 \in \mathbb{R}$, a.e. $t \in \Omega$, where $a, \beta_1, \beta_2 \in \mathbb{R}_+$, $j \geq 1$ and $\beta_1 \leq c \leq \beta_2$. In addition, assume that (3.6) and (3.19) hold. Then (3.1) has at least one solution $u \in H^1_0(\Omega)$.

Proof. As above, conditions (3.20) and (3.21) imply conditions (2.31) and (2.25) from Theorem 2.8 respectively. The conclusion follows from Theorem 2.8. □

Some other applications of the abstract theory developed in Section 2 will be presented in a forthcoming paper.

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