Nonlinear Functional Analysis and Applications
Vol. 13, No. 1 (2008), pp. 87-98

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

COMMON FIXED POINT THEOREMS FOR A FAMILY OF MAPPINGS IN METRICALLY CONVEX SPACES

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Abstract. The aim of this paper is to establish some coincidence and common fixed point theorems for a sequence of hybrid type nonself mappings defined on a closed subset of a metrically convex metric space using diametral δ-distance instead of Hausdorff distance. Our results generalize some earlier results due to Dhage [7], Dhage et al. [8], Huang and Cho [12], Imdad et al. [14], Khan [19], Ćirić and Ume [6], Rhoades [25] and several others. Some related results are also discussed.

1. Introduction

Several fixed point theorems for set-valued and hybrid pairs of mapping are proved using Hausdorff distances and by now there exists a spate of research article in this direction. To mention a few, one can cite Rhoades [25], Imdad and Ahmad [13], Pathak [24], Popa [22] and references cited therein. On the other hand, Assad and Kirk [4] gave a sufficient condition enunciating fixed point of set-valued mappings satisfying a specific boundary condition in metrically convex metric spaces. In the recent years the work due to Assad and Kirk [4] has inspired extensive activities which includes Itoh [15], Khan [19], Ahmad and Imdad [1,2], Imdad, Ahmad and Kumar [14] and others.

[0]Received June 15, 2006. Revised September 13, 2007.
[0]2000 Mathematics Subject Classification: 54H25, 47H10.
[0]Keywords: Metrically convex metric space, quasi-coincidentally commuting mappings, compatible mappings, pointwise R-weak commutativity, coincidentally idempotent.

Most recently, Dhage et al. [8] proved some fixed point theorems for a sequence of set-valued mappings which generalize several results due to Dhage [7], Huang and Cho [12] and others. The purpose of this paper is to prove some coincidence and common hybrid fixed point theorems for a sequence of set-valued and a pair of single valued nonself mappings using diametral distance (instead of Hausdorff distance) satisfying certain contraction type condition which is essentially patterned after Huang and Cho [12] or Dhage [7]. Our results either partially or completely generalize earlier results due to Itoh [15], Khan [19], Ȁirič and Ume [6], Rhoades [25], Imdad et al. [14], Huang and Cho [12], Dhage [7], Dhage et al. [8] and several others.

2. Preliminaries

Before proving our results, we collect the relevant notations and conventions. Let (X, d) be a metric space. Then following Nadler [21], we recall

(i) $CB(X) = \{A : A \text{ is nonempty closed and bounded subset of } X\}$,
(ii) $C(X) = \{A : A \text{ is nonempty compact subset of } X\}$.
(iii) For nonempty subsets A, B of X and $x \in X$,

$$\begin{aligned} d(x, A) &= \inf \{d(x, a) : a \in A\}, \\ D(A, B) &= \inf \{d(a, b) : a \in A, b \in B\}, \\ H(A, B) &= \max \left[\{\sup d(a, B) : a \in A\}, \{\sup d(A, b) : b \in B\}\right] \text{ and} \\ \delta(A, B) &= \sup \{d(a, b) : a \in A, b \in B\}. \end{aligned}$$

Notice that $D(A, B) \leq H(A, B) \leq \delta(A, B)$, it is well known (See [18]) that $CB(X)$ is a metric space with the distance H which is known as Hausdorff-Pompeiu metric on X.

The following definitions and lemmas will be frequently used in the sequel.

Definition 2.1. Let K be a nonempty subset of a metric space (X, d), $T : K \to X$ and $F : K \to CB(X)$. The pair (F, T) is said to be pointwise R-weakly commuting on K if for given $x \in K$ and $Tx \in K$, there exists some $R = R(x) > 0$ such that

$$d(Ty, FTx) \leq R\, d(Tx, Fx) \text{ for each } y \in K \cap Fx. \tag{2.1}$$

Moreover, the pair (F, T) will be called R-weakly commuting on K if (2.1) holds for each $x \in K,\ Tx \in K$ with some $R > 0$.

If $R = 1$, we get the definition of weak commutativity of (F, T) on K due to Hadžič [10]. For $K = X$, Definition 2.1 reduces to 'pointwise R-weak commutativity and R-weak commutativity' for single valued self mappings due to

Pant [23].

Definition 2.2.([10],[11]) Let K be a nonempty subset of a metric space (X, d), $T : K \to X$ and $F : K \to CB(X)$. The pair (F, T) is said to be weakly commuting (See [10]) if for every $x, y \in K$ with $x \in Fy$ and $Ty \in K$, we have

$$d(Tx, FTy) \leq d(Ty, Fy)$$

whereas the pair (F, T) is said to be compatible (See [11]) if for every sequence $\{x_n\} \subset K$, from the relation

$$\lim_{n\to\infty} d(Fx_n, Tx_n) = 0$$

and $Tx_n \in K$ (for every $n \in N$) it follows that $\lim_{n\to\infty} d(Ty_n, FTx_n) = 0$, for every sequence $\{y_n\} \subset K$ such that $y_n \in Fx_n, n \in N$.

For hybrid pairs of self type mappings these definitions were introduced by Kaneko and Sessa [17].

Definition 2.3.([14]) Let K be a nonempty subset of a metric space (X, d), $T : K \to X$ and $F : K \to CB(X)$. The pair (F, T) is said to be quasi-coincidentally commuting if for all coincidence points 'x' of (F, T), $TFx \subset FTx$ whenever $Fx \subset K$ and $Tx \in K$ for all $x \in K$.

Definition 2.4.([14]) A mapping $T : K \to X$ is said to be coincidentally idempotent w.r.t mapping $F : K \to CB(X)$, if T is idempotent at the coincidence points of the pair (F, T) *i.e.* $Tx \in F(x)$ implies $T^2x = Tx$.

Definition 2.5.([4]) A metric space (X, d) is said to be metrically convex if for any $x, y \in X$ with $x \neq y$ there exists a point $z \in X, x \neq z \neq y$ such that

$$d(x, z) + d(z, y) = d(x, y).$$

Lemma 2.6.([4]) *Let K be a nonempty closed subset of a metrically convex metric space (X, d). If $x \in K$ and $y \notin K$ then there exists a point $z \in \delta K$ (the boundary of K) such that $d(x, z) + d(z, y) = d(x, y)$.*

Lemma 2.7.([9]) *Let $\{A_n\}$ and $\{B_n\}$ be two sequences in $CB(X)$ and converging in $CB(X)$ to the sets A and respectively B. Then*

$$\lim_{n\to\infty} \delta(A_n, B_n) = \delta(A, B).$$

3. Results

Our main result runs as follows.

Theorem 3.1. *Let (X,d) be a complete metrically convex metric space and K a nonempty closed subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $S, T : K \to X$ which satisfy:*

(i) $\partial K \subseteq SK \cap TK, F_i(K) \cap K \subseteq SK, F_j(K) \cap K \subseteq TK$,

(ii) $Tx \in \partial K \Rightarrow F_i(x) \subseteq K, Sx \in \partial K \Rightarrow F_j(x) \subseteq K$, *and*

$$\delta(F_i(x), F_j(y)) \leq a\ max\{d(Tx,Sy), d(Tx,F_i(x)), d(Sy,F_j(y))\}$$

$$+b\ [d(Tx,F_j(y)) + d(Sy,F_i(x))],\ (3.1)$$

where $i = 2n-1,\ j = 2n,\ (n \in N),\ i \neq j$ *for all* $x, y \in K$ *with* $x \neq y,\ a, b \geq 0,$ *such that* $2a + 3b < 1$,

(iii) (F_i, T) *and* (F_j, S) *are compatible pairs,*

(iv) $\{F_n\}, S$ *and* T *are continuous on* K.

Then $\{F_n\}, S$ *and* T *have a common coincidence point.*

Proof. Firstly, we proceed to construct two sequences $\{x_n\}$ and $\{y_n\}$ in the following way.

Let $x \in \partial K$. Then (due to $\partial K \subseteq TK$) there exists a point $x_0 \in K$ such that $x = Tx_0$. Since $Tx \in \partial K \Rightarrow F_i(x) \subseteq K$ for every odd integer $(i \in N)$, one concludes that $F_1(x_0) \subseteq F_1(K) \cap K \subseteq SK$. Let $x_1 \in K$ be such that $y_1 = Sx_1 \in F_1(x_0) \subseteq K$. Since $y_1 \in F_1(x_0)$, there exists a point $y_2 \in F_2(x_1)$. Suppose $y_2 \in K$. Then $y_2 \in F_2(K) \cap K \subseteq TK$, implies that there exists a point $x_2 \in K$ such that $y_2 = Tx_2$. Otherwise, if $y_2 \notin K$ then there exists a point $p \in \partial K$ such that

$$d(Sx_1, p) + d(p, y_2) = d(Sx_1, y_2).$$

Since $p \in \partial K \subseteq TK$, there exists a point $x_2 \in K$ with $p = Tx_2$ so that

$$d(Sx_1, Tx_2) + d(Tx_2, y_2) = d(Sx_1, y_2).$$

Let $y_3 \in F_3(x_2)$. If $y_3 \in K$ then $y_3 \in F_3(K) \cap K \subseteq SK$ which implies that there exists a point $x_3 \in K$ such that $y_3 = Sx_3$. Otherwise if $y_3 \notin K$ there exists a point $p_1 \in \partial K$ such that

$$d(Tx_2, p_1) + d(p_1, y_3) = d(Tx_2, y_3).$$

Since $p_1 \in \partial K \subseteq SK$ there exists a point $x_3 \in K$ with $p_1 = Sx_3$ so that

$$d(Tx_2, Sx_3) + d(Sx_3, y_3) = d(Tx_2, y_3).$$

Thus, repeating the foregoing arguments, we obtain two sequences $\{x_n\}$ and $\{y_n\}$ such that

(v) $y_{2n} \in F_{2n}(x_{2n-1})$ for every $(n \in N)$, $y_{2n+1} \in F_{2n+1}(x_{2n})$ for every $(n \in N_0 = N \cup \{0\})$,

(vi) $y_{2n} \in K \Rightarrow y_{2n} = Tx_{2n}$ or $y_{2n} \notin K \Rightarrow Tx_{2n} \in \partial K$ and $d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n})$,

(vii) $y_{2n+1} \in K \Rightarrow y_{2n+1} = Sx_{2n+1}$ or $y_{2n+1} \notin K \Rightarrow Sx_{2n+1} \in \partial K$ and

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}).$$

We denote

$$P_\circ = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} = y_{2i}\},$$
$$P_1 = \{Tx_{2i} \in \{Tx_{2n}\} : Tx_{2i} \neq y_{2i}\},$$
$$Q_\circ = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} = y_{2i+1}\},$$
$$Q_1 = \{Sx_{2i+1} \in \{Sx_{2n+1}\} : Sx_{2i+1} \neq y_{2i+1}\}.$$

One can note that $(Tx_{2n}, Sx_{2n+1}) \notin P_1 \times Q_1$ and $(Sx_{2n-1}, Tx_{2n}) \notin Q_1 \times P_1$.

Now, we distinguish the following three cases.

Case 1. If $(Tx_{2n}, Sx_{2n+1}) \in P_\circ \times Q_\circ$, then

$$\begin{aligned}
&d(Tx_{2n}, Sx_{2n+1})\\
&\leq \delta(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1}))\\
&\leq a\ max\{d(Tx_{2n}, Sx_{2n-1}), d(Tx_{2n}, F_{2n+1}(x_{2n})), d(Sx_{2n-1}, F_{2n}(x_{2n-1}))\}\\
&\quad + b\ [d(Tx_{2n}, F_{2n}(x_{2n-1})) + d(Sx_{2n-1}, F_{2n+1}(x_{2n}))],\\
&\leq a\ max\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n})\}\\
&\quad + b\ [d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})].
\end{aligned}$$

If we suppose that $d(y_{2n-1}, y_{2n}) \leq d(y_{2n}, y_{2n+1})$, then one obtains

$$d(Tx_{2n}, Sx_{2n+1}) \leq (a + 2b)\ d(y_{2n}, y_{2n+1})$$

which is a contradiction. Otherwise if $d(y_{2n}, y_{2n+1}) \leq d(y_{2n-1}, y_{2n})$, then one obtains

$$d(Tx_{2n}, Sx_{2n+1}) \leq (a + b)\ d(y_{2n}, y_{2n-1}) + bd(y_{2n}, y_{2n+1})$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq (\frac{a+b}{1-b})\ d(Sx_{2n-1}, Tx_{2n}).$$

Similarly, if $(Sx_{2n-1}, Tx_{2n}) \in Q_\circ \times P_\circ$, then

$$d(Sx_{2n-1}, Tx_{2n}) \leq (\frac{a+b}{1-b})\ d(Sx_{2n-1}, Tx_{2n-2}).$$

Case 2. If $(Tx_{2n}, Sx_{2n+1}) \in P_\circ \times Q_1$, then

$$d(Tx_{2n}, Sx_{2n+1}) + d(Sx_{2n+1}, y_{2n+1}) = d(Tx_{2n}, y_{2n+1}),$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \le d(Tx_{2n}, y_{2n+1}) = d(y_{2n}, y_{2n+1}),$$

and hence

$$d(Tx_{2n}, Sx_{2n+1}) \le d(y_{2n}, y_{2n+1}) \le \delta(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})).$$

Now, proceeding as in Case 1, we have

$$d(Tx_{2n}, Sx_{2n+1}) \le \left(\frac{a+b}{1-b}\right) \ d(Sx_{2n-1}, Tx_{2n}).$$

Similarly, if $(Sx_{2n-1}, Tx_{2n}) \in Q_1 \times P_\circ$, then

$$d(Sx_{2n-1}, Tx_{2n}) \le \left(\frac{a+b}{1-b}\right) \ d(Sx_{2n-1}, Tx_{2n-2}).$$

Case 3. If $(Tx_{2n}, Sx_{2n+1}) \in P_1 \times Q_\circ$, then $Sx_{2n-1} = y_{2n-1}$. Proceeding as in Case 1, one gets

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) &= d(Tx_{2n}, y_{2n+1}) \\ &\le d(Tx_{2n}, y_{2n}) + d(y_{2n}, y_{2n+1}), \\ &\le d(Tx_{2n}, y_{2n}) + \delta(F_{2n+1}(x_{2n}), F_{2n}(x_{2n-1})), \\ &\le d(Tx_{2n}, y_{2n}) + a \ max\{d(Tx_{2n}, Sx_{2n-1}), d(Tx_{2n}, F_{2n+1}(x_{2n})), \\ &\quad d(Sx_{2n-1}, F_{2n}(x_{2n-1}))\} + b \ [d(Tx_{2n}, F_{2n}(x_{2n-1}) \\ &\quad + d(Sx_{2n-1}, F_{2n+1}(x_{2n}))]. \end{aligned}$$

Since

$$d(Sx_{2n-1}, Tx_{2n}) + d(Tx_{2n}, y_{2n}) = d(Sx_{2n-1}, y_{2n}),$$

therefore

$$\begin{aligned} d(Tx_{2n}, Sx_{2n+1}) &\le d(Sx_{2n-1}, y_{2n}) + a \ max\{d(Sx_{2n-1}, y_{2n}), d(Tx_{2n}, y_{2n+1}), \\ &\quad d(Sx_{2n-1}, y_{2n})\} + b \ [d(Tx_{2n}, y_{2n}) + d(y_{2n-1}, y_{2n+1})], \\ &\le d(Sx_{2n-1}, y_{2n}) + a \ max\{d(Sx_{2n-1}, y_{2n}), d(Tx_{2n}, y_{2n+1}), \\ &\quad d(Sx_{2n-1}, y_{2n})\} + b \ [d(Tx_{2n}, y_{2n}) \ + \ d(y_{2n-1}, Tx_{2n}) \\ &\quad + d(Tx_{2n}, y_{2n+1})] \\ &\le d(Sx_{2n-1}, y_{2n}) + a \ max\{d(Sx_{2n-1}, y_{2n}), d(Tx_{2n}, y_{2n+1}), \\ &\quad d(Sx_{2n-1}, y_{2n})\} + b \ [d(y_{2n}, y_{2n-1}) + d(Tx_{2n}, y_{2n+1})] \end{aligned}$$

which in turn yields

$$d(Tx_{2n}, Sx_{2n+1}) \leq \begin{cases} (\frac{1+b}{1-a-b})\ d(Sx_{2n-1}, y_{2n}), & \text{if } d(Sx_{2n-1}, y_{2n}) \leq d(Tx_{2n}, y_{2n+1}) \\ (\frac{1+a+b}{1-b})\ d(Sx_{2n-1}, y_{2n}), & \text{if } d(Sx_{2n-1}, y_{2n}) \geq d(Tx_{2n}, y_{2n+1}). \end{cases}$$

Now, proceeding as earlier, one also obtains

$$d(Sx_{2n-1}, y_{2n}) \leq (\frac{a+b}{1-b})\ d(Sx_{2n-1}, Tx_{2n-2}).$$

Therefore combining the above inequalities, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq k\ d(Sx_{2n-1}, Tx_{2n-2}),$$

where $k = max\{(\frac{1+b}{1-a-b})(\frac{a+b}{1-b}), (\frac{1+a+b}{1-b})(\frac{a+b}{1-b})\} < 1$, since $2a + 3b < 1$.

Thus in all the cases, we have

$$d(Tx_{2n}, Sx_{2n+1}) \leq k\ max\{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n-2}, Sx_{2n-1})\},$$

whereas

$$d(Sx_{2n+1}, Tx_{2n+2}) \leq k\ max\{d(Sx_{2n-1}, Tx_{2n}), d(Tx_{2n}, Sx_{2n+1})\}.$$

Now, on the lines of Assad and Kirk [4], it can be shown by induction that for $n \geq 1$, we have

$$d(Tx_{2n}, Sx_{2n+1}) < k^n\ \alpha \text{ and } d(Sx_{2n+1}, Tx_{2n+2}) < k^{n+\frac{1}{2}}\ \alpha,$$

whereas

$$\alpha = k^{\frac{-1}{2}} max\{d(Tx_0, Sx_1), d(Sx_1, Tx_2)\}.$$

Thus the sequence $\{Tx_0, Sx_1, Tx_2, Sx_3, ..., Sx_{2n-1}, Tx_{2n}, Sx_{2n+1}, ...\}$ is Cauchy and hence converges to the point z in X. Then as noted in [10] there exists at least one subsequence $\{Tx_{2n_k}\}$ or $\{Sx_{2n_k+1}\}$ which is contained in $P_\circ$ or $Q_\circ$ respectively.

Suppose that there exists a subsequence $\{Tx_{2n_k}\}$ which is contained in $P_\circ$ for each $k \in N$, also converges to z. Using compatibility of (F_j, S), we have

$$\lim_{k\to\infty} d(Sx_{2n_k-1}, F_j(x_{2n_k-1})) = 0 \text{ for any even integer } j \in N,$$

which implies that $\lim_{k\to\infty} d(STx_{2n_k}, F_j(Sx_{2n_k-1})) = 0$.

Using the continuity of S and F_j, one obtains $Sz \in F_j(z)$ for any even integer $j \in N$. Similarly, the continuity of T and F_i implies $Tz \in F_i(z)$ for any odd integer $i \in N$. Now

$$\begin{aligned}
d(Tz,Sz) &\le \delta(F_i(z),F_j(z))\\
&\le a\ max\{d(Tz,Sz),d(Tz,F_i(z)),d(Sz,F_j(z))\}\\
&\quad + b\ [d(Tz,F_j(z))+d(Sz,F_i(z))],\\
&\le a\ max\{d(Tz,Sz),0,0\}+b\ [d(Tz,Sz)+d(Tz,Sz)],\\
&\le (a+2b)\ d(Tz,Sz),
\end{aligned}$$

yielding thereby $Tz = Sz$, which shows that z is a common coincidence point of the maps $\{F_n\}, S$ and T. This completes the proof. □

Remark 3.2. By setting $F_n = F$ $(n \in N)$ and $S = T = I_K$ in Theorem 3.1, one deduces a result due to Dhage [7].

Remark 3.3. By setting $S = T = I_K$ in Theorem 3.1, one deduces a result due to Dhage et al. [8].

In our result, we note that Theorem 3.1 remains true for pointwise R-weakly commuting mappings. Thus we have the following.

Theorem 3.4. *Theorem 3.1 remains true if one replaces 'compatibility' by 'pointwise R-weak commutativity' in (iii) and retaining the rest of the hypotheses. Then (F_i,T) as well as (F_j,S) has a point of coincidence.*

Proof. On the lines of the proof of Theorem 3.1, one can show that the sequence $\{Tx_{2n}\}$ converges to a point $z \in X$. Now, we assume that there exists a subsequence $\{Tx_{2n_k}\}$ of $\{Tx_{2n}\}$, which is contained in $P_\circ$. Further subsequence $\{Tx_{2n_k}\}$ and $\{Sx_{2n_k+1}\}$ both converges to $z \in K$ as K is a closed subset of the complete metric space (X,d). Since $Tx_{2n_k} \in F_j(x_{2n_k-1})$ for any even integer $j \in N$ and $Sx_{2n_k-1} \in K$. Using pointwise R-weak commutativity of (F_j,S), we have

$$d(SF_j(x_{2n_k-1}),F_j(Sx_{2n_k-1})) \le R_1\ d(F_j(x_{2n_k-1}),Sx_{2n_k-1}) \tag{3.2}$$

for any even integer $j \in N$ with some $R_1 > 0$. Also

$$d(SF_j(x_{2n_k-1}),F_j(z)) \le d(SF_j(x_{2n_k-1}),F_j(Sx_{2n_k-1}))+\delta(F_j(Sx_{2n_k-1}),F_j(z)). \tag{3.3}$$

Making $k \to \infty$ in (3.2) and (3.3) and using continuity of F_j as well as S, we get $d(Sz,F_j(z)) \le 0$ yielding thereby $Sz \in F_j(z)$ for any even integer $j \in N$.

Since $y_{2n_k+1} \in F_i(x_{2n_k})$ and $\{Tx_{2n_k}\} \subset K$, pointwise R-weak commutativity of (F_i,T) implies

$$d(TF_i(x_{2n_k}),F_i(Tx_{2n_k})) \le R_2\ d(F_i(x_{2n_k}),Tx_{2n_k})$$

for any odd integer $i \in N$ with some $R_2 > 0$, besides

$$d(TF_i(x_{2n_k}),F_i(z)) \le d(TF_i(x_{2n_k}),F_i(Tx_{2n_k}))+\delta(F_i(Tx_{2n_k}),F_i(z)).$$

Therefore, as earlier the continuity of F_i as well as T implies $d(Tz, F_i(z)) \leq 0$ giving thereby $Tz \in F_i(z)$ as $k \to \infty$.

If we assume that there exists a subsequence $\{Sx_{2n_k+1}\}$ contained in $Q_\circ$, then analogous arguments establish the earlier conclusions. This concludes the proof. $\square$

In the next theorem, we utilize the closedness of TK and SK to replace the continuity and compatibility (i.e (iii) and (iv)) requirements in Theorem 3.1, we have the following.

Theorem 3.5. (*a*) *Theorem 3.1 remains true if we replace conditions (iii) and (iv) by the closedness of TK and SK and retaining the rest of the hypotheses.* (*b*) *Moreover, if in addition to in* (*a*), *T is quasi-coincidentally commuting and coincidentally idempotent w.r.t F_i, then (F_i, T) has a common fixed point. Similarly (F_j, S) has a common fixed point provided S is quasi-coincidentally commuting and coincidentally idempotent w.r.t F_j.*

Proof. On the lines of the Theorem 3.1, one assumes that there exists a subsequence $\{Tx_{2n_k}\}$ which is contained in $P_\circ$ and TK as well as SK are closed subspaces of X. Since $\{Tx_{2n_k}\}$ is Cauchy in TK, it converges to a point $u \in TK$. Let $v \in T^{-1}u$, then $Tv = u$. Since $\{Sx_{2n_k+1}\}$ is a subsequence of Cauchy sequence, $\{Sx_{2n_k+1}\}$ converges to u as well. Using (3.1), one can write

$$\begin{aligned} d(F_i(v), Tx_{2n_k}) &\leq \delta(F_i(v), F_j(x_{2n_k-1})) \\ &\leq a\ max\{d(Tv,Sx_{2n_k-1}),d(Sx_{2n_k-1},F_j(x_{2n_k-1})),d(Tv,F_i(v))\} \\ &\quad + b\ [d(Tv, F_j(x_{2n_k-1})) + d(Sx_{2n_k-1}, F_i(v))], \end{aligned}$$

which on letting $k \to \infty$, reduces to

$$\begin{aligned} d(F_i(v), u) &\leq a\ max\{0, d(u, F_i(v)), 0\} + b\ [0 + d(F_i(v), u)], \\ &\leq (a + b)\ d(u, F_i(v)), \end{aligned}$$

yielding thereby $u \in F_i(v)$, which implies that $u = Tv \in F_i(v)$ as $F_i(v)$ is closed.

Since Cauchy sequence $\{Tx_{2n}\}$ converges to $u \in K$ and $u \in F_i(v)$, $u \in F_i(K) \cap K \subseteq SK$, there exists $w \in K$ such that $Sw = u$. Again using (3.1), one gets

$$\begin{aligned} d(Sw, F_j(w)) &= d(Tv, F_j(w)) \\ &\leq \delta(F_i(v), F_j(w)) \\ &\leq a\ max\{d(Tv, Sw), d(Tv, F_i(v)), d(Sw, F_j(w))\} \\ &\quad + b\ [d(Tv, F_j(w)) + d(Sw, F_i(v))], \\ &\leq (a + b)\ d(Sw, F_j(w)), \end{aligned}$$

implying thereby $Sw \in F_j(w)$, that is w is a coincidence point of (S, F_j).

If one assumes that there exists a subsequence $\{Sx_{2n_k+1}\}$ contained in $Q_\circ$ with TK as well as SK are closed subspaces of X, then noting that $\{Sx_{2n_k+1}\}$ is Cauchy in SK, the foregoing arguments establish that $Tz \in F_i(z)$ and $Sw \in F_j(w)$.

Since z is a coincidence point of (F_i, T) therefore using quasi-coincidentally commuting property of (F_i, T) and coincidentally idempotent property of T w.r.t F_i, one can have

$$Tv \in F_i(v) \text{ and } u = Tv \Rightarrow Tu = TTv = Tv = u,$$

therefore $u = Tu = TTv \in TF_i(v) \subset F_i(Tv) = F_i(u)$, which shows that u is the common fixed point of (F_i, T). Similarly, using the quasi-coincidentally commuting property of (F_j, S) and coincidentally idempotent property of S w.r.t F_j, one can show that (F_j, S) has a common fixed point as well. □

Remark 3.6. Theorem 3.5 remains true if we substitute closedness of 'TK and SK' with closedness of '$F_i(K)$ and $F_j(K)$'.

Remark 3.7. By setting $F_n = F$ (for all $n \in N$) and $S = T = I_K$ in Theorem 3.5, one deduces a multi-valued version of a result due to Khan et al. [20].

Remark 3.8. If we choose $F_i = F$ (for any odd integer $i \in N$), $F_j = G$ (for any even integer $j \in N$) and $S = T = I_K$ in Theorem 3.5, one deduces a sharpened form of a result due to Khan [19].

Remark 3.9. By restricting $F_i = F$ (for any odd integer $i \in N$), $F_j = G$ (for any even integer $j \in N$) in Theorem 3.5, one deduces sharpened and modified form of result due to Imdad et al. [14].

Remark 3.10. If we choose $S = T = I_K$ in Theorem 3.5, then one deduces a result due to Huang and Cho [12].

Finally, we prove a theorem when 'closedness of K' is replaced by 'compactness of K'.

Theorem 3.11. *Let (X, d) be a complete metrically convex metric space and K a nonempty compact subset of X. Let $\{F_n\}_{n=1}^{\infty} : K \to CB(X)$ and $T : K \to X$ which satisfy:*

(i) $\partial K \subseteq TK,\ (F_i(K) \cup F_j(K)) \cap K \subseteq TK,$

(ii) $Tx \in \partial K \Rightarrow F_i(x) \cup F_j(x) \subseteq K$ *with*

$$\delta(F_i(x), F_j(y)) < M(x, y)$$

when $M(x, y) > 0$, for all $x, y \in K$ where

$$M(x,y) = a\ max\{d(Tx,Ty), d(Tx,F_i(x)), d(Ty,F_j(y))\} \\ + b\ [d(Tx,F_j(y)) + d(Ty,F_i(x))] \tag{3.4}$$

for all $x, y \in X$ with $x \neq y$, where a, b are non-negative reals such that $2a + 3b \leq 1$.

If T is compatible with $\{F_n\}$ $(n \in N)$ along with $\{F_n\}$ and T are continuous on K. Then $\{F_n\}$ and T have a common point of coincidence.

Proof. We assert that $M(x,y) = 0$ for some $x, y \in K$. Otherwise $M(x,y) \neq 0$, for any $x, y \in K$ implies that

$$f(x,y) = \frac{\delta(F_i(x), F_j(y))}{M(x,y)}$$

is continuous and satisfies $f(x,y) < 1$ for all $(x,y) \in K \times K$. Since $K \times K$ is compact, there exists $(u,v) \in K \times K$ such that $f(x,y) \leq f(u,v) = c < 1$ for $x, y \in K$, which in turn yields $\delta(F_i(x), F_j(y)) \leq c\ M(x,y)$ for $x, y \in K$ and $0 < c < 1$. Therefore using (3.4), one obtains $c(2a + 3b) < 1$. Now by Theorem 3.1 (with restriction S = T), we get $Tz \in F_i(z) \cap F_j(z)$ for some $z \in K$ and one concludes $M(z,z) = 0$, contradicting the facts that $M(x,y) > 0$. Therefore $M(x,y) = 0$ for some $x, y \in K$ which implies $Tx \in F_i(x)$ for any odd integer $i \in N$ and $Tx = Ty \in F_j(y)$ for any even integer $j \in N$. If $M(x,x) = 0$ then $Tx \in F_j(x)$ for any even integer $j \in N$ and if $M(x,x) \neq 0$ then using (3.4), one infers that $d(Tx, F_j(x)) \leq 0$ yielding thereby $Tx \in F_j(x)$ for any even integer $j \in N$. Similarly, in either of the cases $M(y,y) = 0$ or $M(y,y) > 0$, one concludes that $Ty \in F_i(y)$ for any odd integer $i \in N$. Thus we have shown that $\{F_n\}$ and T have a common point of coincidence. □

While proving Theorem 3.11 the following question remains unresolved: Does Theorem 3.11 hold for $\{F_n\}, S$ and T instead of $\{F_n\}$ and T?

Acknowledgment. The authors are grateful to the learned referee for the critical reading of our manuscript and suggesting many improvements. The first author is thankful to University Grants Commission in India for financial assistance (Project No. F.30-246/2004(SR)).

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