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SOME EXISTENCE RESULTS FOR VECTOR QUASI-VARIATIONAL-LIKE INEQUALITIES INVOLVING MULTIFUNCTIONS

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Abstract. In this paper, we introduce and study a class of vector quasi-variational-like inequalities with non-monotonicity and non-compactness assumptions and establish some existence results by using maximal element theorem. The results of this paper generalize and unify the corresponding results of several authors.

1. Introduction

Quasi-variational inequalities involving multifunctions provide us with a unified, natural, innovative and general approach to study a wide class of problems arising in different branches of mathematics, physics and engineering sciences. Very recently Khaliq [13], Khaliq and Rashid [15,16], Khaliq, Siddiqi and Krishan [17] have studied quasi-variational inequalities involving multifunctions in the setting of topological vector spaces and Banach spaces by using one person game theorems, KKM Fan theorem and fixed point theorems. Quasi equilibria constitute an extension of Nash equilibria which are of fundamental importance in the theory of non-cooperative games. Cubiotti

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[7], Ding [8,9] and Noor and Otteli [21] have studied quasi equilibrium problems and recently generalized to vector valued functions by Ansari and Yao [5], Khaliq and Krishan [14], Khaliq [13], Ansari and Flores-Bazan [4] and references therein.

In this paper we introduce a new class of vector quasi-variational inequality problem involving multifunctions which contains equilibrium problem as a special case, so it is interesting to compare these two ways of problem settings. We established some existence results for solution of this type of variational inequality problem by using maximal element theorem. The results of this paper generalize and unify the corresponding results of several authors and can be considered as a significant extension of the previously known results.

Let X be a real locally convex Hausdorff topological vector space and Y be a real Hausdorff topological vector space. We denote by $L(X,Y)$, the space of all continuous linear mappings from X into Y and $\langle u, x\rangle$ the evaluation of $u \in L(X,Y)$ at $x \in X$. Let σ be the family of all bounded subsets of X whose union is total in X, i.e., the linear hull of $\cup\{S : S \in \sigma\}$ is dense in X. Let B be a neighbourhood base of 0 in Y. When S runs through σ, V through B, the family

$$M(S,V) = \{l \in L(X,Y) : \cup_{x\in S}\langle l, x\rangle \subset V\}$$

is a neighbourhood base of 0 in $L(X,Y)$ for a unique translation-invariant topology, called the topology of uniform convergence on the set $S \in \sigma$, or briefly the σ-topology. Also $L(X,Y)$ becomes a locally convex topological vector space under σ-topology , where Y assumed a locally convex topological space, see Schaeffer [23, pp. 80]. Let K be a nonempty convex subset of X and $T : K \to 2^{L(X,Y)}$ and $A : K \to 2^X$ be multifunctions. For given vector valued bifunctions $\theta : K \times L(X,Y) \to L(X,Y)$, $f : K \times K \to Y$ and $\eta : K \times K \to X$, and a mapping $g : K \to K$, we consider the following *vector quasi-variational-like inequality problem*:

Find $x^* \in K$ such that $x^* \in A(x^*)$ and for all $y \in A(x^*)$

$$\exists\ s^* \in T(x^*) : \langle\theta(x^*, s^*), \eta(y, x^*)\rangle + f(g(x^*), y) \notin -int_Y C(x^*). \tag{1.1}$$

It is easy to see that x^* is a solution of the problem (1.1) is equivalent to finding $x^* \in K$ satisfying $x^* \in A(x^*)$ and for all $y \in A(x^*)$

$$\langle\theta(x^*, T(x^*)), \eta(y, x^*)\rangle + f(g(x^*), y) \nsubseteq -int_Y C(x^*), \tag{1.2}$$

where $\langle\theta(x^*, T(x^*)), \eta(y, x^*)\rangle = \bigcup_{s^*\in T(x^*)} \langle\theta(x^*, s^*)), \eta(y, x^*)\rangle$.

If we take T as single valued mapping then as corollary, we consider the problem of finding $x^* \in K$ such that $x^* \in A(x^*)$ and for all $y \in A(x^*)$

$$\langle\theta(x^*, T(x^*)), \eta(y, x^*)\rangle + f(g(x^*), y) \notin -int_Y C(x^*). \tag{1.3}$$

If $\eta(x,y) = x - g(y)$, for all $x, y \in K$, then as corollary, we consider the problem of finding $x^* \in K$ such that $x^* \in A(x^*)$ and for all $y \in A(x^*)$

$$\langle \theta(x^*, T(x^*)), y - g(x^*) \rangle + f(g(x^*), y) \notin -int_Y C(x^*). \tag{1.4}$$

Problems (1.3) and (1.4) also appears to be new.

If $T = 0$, then (1.1) reduces to the problem of finding $x^* \in K$ such that $x^* \in A(x^*)$ and for all $y \in A(x^*)$

$$f(g(x^*), y) \notin -int_Y C(x^*). \tag{1.5}$$

This problem was introduced and studied by Khaliq [12] and is known as *implicit vector quasi-equilibrium problem.* If $g = I$, the identity mapping, then (1.5) was considered by Khaliq and Krishan [14].

If $f = 0$, then (1.1) reduces to the problem of finding $x^* \in K$ such that $x^* \in A(x^*)$ and for all $y \in A(x^*)$

$$\exists\ s^* \in T(x^*) : \langle \theta(x^*, s^*), \eta(y, x^*) \rangle \notin -int_Y C(x^*). \tag{1.6}$$

This problem was considered and studied by Khaliq and Rashid [15].

If $\theta(x^*, s^*) = s^*$, and $g = I$, the identity mapping, problem (1.1) reduces to the problem of finding $x^* \in K$ such that $x^* \in A(x^*)$ and for all $y \in A(x^*)$

$$\exists\ s^* \in T(x^*) : \langle s^*, \eta(y, x^*) \rangle + f(x^*, y) \notin -int_Y C(x^*). \tag{1.7}$$

It is called *generalized vector quasi-variational-like inequality problem* considered and studied by Khaliq and Rashid [16].

If $f = 0$, then the problem (1.7) reduces to the vector quasi-variational-like inequality problem studied with certain monotonocity assumptions by Ding [10], which also contains as special cases the generalized vector variational like inequalities in [1,2,3,24]. For suitable choice of the mappings one can obtain a number of known variational and equilibrium problems as special cases from the above problems, see for example [11,19,20,26].

2. Preliminaries

We need the following.

Definition 2.1. Let X and Y be two real topological vector spaces, K be a nonempty and convex subset of X, $C : K \to 2^Y$ be a multifunction such that $C(x)$ is closed, convex and pointed cone with apex at 0 for each $x \in K$. Let $\theta : K \times L(X,Y) \to L(X,Y)$ and $\eta : K \times K \to X$ be bifunctions. $T : K \to 2^{L(X,Y)}$ is said to satisfy the *generalized $\theta - L - \eta$ condition* iff for any finite set $\{y_1, y_2 ... y_n\}$ in K, $x_o = \sum_{i=1}^{n} \lambda_i y_i$ with $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, there exists

$s \in T(x_o)$ such that

$$\langle \theta(x_o, s), \sum_{i=1}^{n} \lambda_i \eta(y_i, x_o) \rangle \notin -int_Y C(x_o).$$

If $\theta(x_o, s) = s$, then T is said to satisfy the generalized $L - \eta$condition

$$\langle s, \sum_{i=1}^{n} \lambda_i \eta(y_i, x_o) \rangle \notin -int_Y C(x_o).$$

Remark 2.1. If η is affine in first argument and for all $x \in K$, there exists $s \in T(x)$ such that

$$\langle \theta(x, s), \eta(x, x) \rangle \notin -int_Y C(x)$$

and

$$\langle s, \eta(x, x) \rangle \notin -int_Y C(x),$$

then T satisfies the generalized $\theta - L - \eta$ and $L - \eta$conditions, respectively.

Definition 2.2. Let $C : K \to 2^Y$ be a multifunction. The vector valued bifunction $f : K \times K \to Y$ is said to be $0 - C(x)$ *diagonally convex* with respect to the second argument if for any finite subset $\{y_1, y_2...y_n\}$ in K and any $x \in K$ with $x = \sum_{i=1}^{n} \lambda_i y_i$, $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i = 1$, we have

$$\sum_{i=1}^{n} \lambda_i f(x, y_i) \in C(x).$$

The bifunction f is said to be $0 - C(x)$ *diagonally concave* with respect to the second argument if $-f$ is $0 - C(x)$ *diagonally convex* with respect to the second argument.

Remark 2.2. If $C(x) = \{a \in R : a \geq o\}$ and $Y = R \cup \{\pm \infty\}$ then the $0 - C(x)$ diagonally convexity of f reduces to the $0 - C(x)$ diagonally convexity of f, see Zhou and Chen [27].

Lemma 2.1.[10] *Let X and Y be topological vector spaces and let $L(X, Y)$ be equipped with the uniform topology δ. Then the bilinear form $\langle \cdot, \cdot \rangle : L(X, Y) \times X \to Y$ is continuous on $(L(X, Y), \delta) \times X$.*

Lemma 2.2.[6] *Let X and Y be topological vector spaces. If $T : X \to 2^Y$ is upper semicontinuous multifunction with closed values, then T is closed.*

Lemma 2.3.[26] *Let X and Y be two topological vector spaces. Suppose $T : X \to 2^Y$ and $P : X \to 2^Y$ are multifunctions having open lower sections. Then*

(i) *the multifunction* $F : X \to 2^Y$ *defined by, for each* $x \in X$, $F(x) = co(T(x))$ *has open lower sections.*
(ii) *the multifunction* $G : X \to 2^Y$ *defined by, for each* $x \in X$ $\theta(x) = T(x) \cup P(x)$ *has open lower sections.*

Lemma 2.4.[18] *Let* K *be a nonempty convex subset of a Hausdorff topological vector space* X *and* $S : K \to 2^K$ *be a multifunction such that for each* $x \in K$, $x \notin co(S(x))$ *and for each* $y \in K, S^{-1}(y)$ *is open in* K. *Suppose further that there exists a nonempty compact subset* N *of* K *and a nonempty compact convex subset* B *of* K *such that* $co(S(x)) \cap B \neq \emptyset$ *for all* $x \in K \setminus N$. *Then there exists a point* $x^* \in K$ *such that* $S(x^*) = \emptyset$.

Lemma 2.5.[25] *Let* X *and* Y *be topological spaces and* $T : X \to 2^Y$ *be upper semicontinuous multifunction with compact values. Suppose* $\{x_\lambda\}$ *is a net in* X *such that* $x_\lambda \to x_0$. *If* $y_\lambda \in T(x_{x_\lambda})$ *for each* λ, *then there is a* $y_0 \in T(x_0)$ *and a subnet* $\{y_\beta\}$ *of* $\{y_\lambda\}$ *such that* $y_\beta \to y_0$.

3. Existence results

In this section, we state and prove our main existence results.

Theorem 3.1. *Let* K *be a nonempty convex subset of a locally convex real Hausdorff topological vector space* X *and* Y *be a real Hausdorff topological vector space. Let* $L(X,Y)$ *be equipped with the* σ*-topology. Let* $C : K \to 2^Y$ *and* $A : K \to 2^K$ *be the multifunctions such that* $\forall x \in K$, $A(x)$ *is nonempty convex,* $\forall y \in K$, $A^{-1}(y)$ *is open in* K, $\forall x \in K$, $C(x)$ *is closed, convex and pointed cone in* Y *such that* $int_Y C(x)$ *is nonempty and the set* $M = \{x \in K : x \in A(x)\}$ *is closed in* K. *Suppose that the following assumptions hold:*

(i) *the mapping* $g : K \to K$ *is continuous, the bifunction* $f : K \times K \to Y$ *is continuous with respect to the first argument and* $0 - C(x)$ *diagonally convex with respect to the second argument,*
(ii) *the multifunction* $T : K \to 2^{L(X,Y)}$ *is upper semicontinuous on* K *with compact values and* $\eta : K \times K \to X$ *is continuous in the second argument, such that* T *satisfies the generalized* $\theta - L - \eta$*-condition,*
(iii) *the multifunction* $W : K \to 2^Y$ *defined by* $W(x) = Y \setminus (-int_Y C(x))$ *for all* $x \in K$, *is upper semicontinuous on* K,
(iv) $\theta(x_\lambda, s_\lambda) \to \theta(x, s)$, *whenever* $x_\lambda \to x$ *in* K, *for* $s_\lambda \in T(x_\lambda)$ *and* $s \in T(x)$,
(v) *there exists a nonempty and compact subset* N *of* K *and a nonempty, compact and convex subset* B *of* K *such that* $\forall\ x \in K \backslash N$, $\exists y^* \in B$, *such that* $y^* \in A(x)$ *and*

$$\langle \theta(x,s), \eta(y^*,x) \rangle + f(g(x), y^*) \in -int_Y C(x), \forall s \in T(x).$$

Then there exists a point $x^* \in K$ *such that* $x^* \in A(x^*)$ *and for all* $y \in A(x^*)$,

$$\exists\ s^* \in T(x^*) : \langle \theta(x^*, s^*), \eta(y, x^*)\rangle + f(g(x^*), y) \notin -int_Y C(x^*).$$

Proof. Define $G : K \to 2^K$ by

$$\begin{aligned} G(x) &= \{y \in K : \langle \theta(x, T(x)), \eta(y, x)\rangle + f(g(x), y) \subseteq -int_Y C(x)\} \\ &= \{y \in K : \langle \theta(x, s), \eta(y, x)\rangle + f(g(x), y) \in -int_Y C(x)\}, \end{aligned}$$

$\forall s \in T(x),\ \forall x \in K$. We first prove that $x \notin coG(x)$, for all $x \in K$. To see this, suppose, by way of contradiction, that there exists some $x_o \in K$ such that $x_o \in coG(x_o)$. Then there exists finite points $y_1, y_2 ... y_n$ in K, and $\lambda_i \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that $x_o = \sum_{i=1}^{n} \lambda_i y_i$ and $y_i \in G(x_o)$ for all $i = 1, 2, ..., n$. This follows

$$\langle \theta(x_o, s), \eta(y_i, x_o)\rangle + f(g(x_o), y_i) \in -int_Y C(x_o) \quad \text{for all} \quad s \in T(x_o)$$

and $i = 1, ..., n$.

Since $int_Y C(x)\}$ is convex set, we obtain

$$\langle \theta(x_o, s), \sum_{i=1}^{n} \lambda_i \eta(y_i, x_o)\rangle + \sum_{i=1}^{n} \lambda_i f(g(x_o), y_i) \in -int_Y C(x_o) \tag{3.1}$$

for all $s \in T(x_o)$. From the $0 - C(x)$ diagonal convexity with respect to the second argument of f, we have

$$\sum_{i=1}^{n} \lambda_i f(g(x_o), y_i) \in C(x_o). \tag{3.2}$$

By (3.1) and (3.2), we get for all $s \in T(x_o)$,

$$\begin{aligned} \langle \theta(x_o, s), \sum_{i=1}^{n} \lambda_i \eta(y_i, x_o)\rangle &\in - \sum_{i=1}^{n} \lambda_i f(g(x_o), y_i) - int_Y C(x_o) \\ &\subseteq -C(x_o) - int_Y C(x_o) \subseteq -int_Y C(x_o), \end{aligned}$$

which contradicts the fact that T satisfies the generalized $\theta - L - \eta$-condition.

Now we show that $G^{-1}(y)$ is open in K, which is equivalent to show that $[G^{-1}(y)]^c = K \setminus G^{-1}(y)$ is closed. Indeed we have for all $s \in T(x)$.

$$\begin{aligned} G^{-1}(y) &= \{x \in K : y \in G(x)\} \\ &= \{x \in K : \langle \theta(x, T(x)), \eta(y, x)\rangle + f(g(x), y) \subseteq -int_Y C(x)\} \\ &= \{x \in K : \langle \theta(x, s), \eta(y, x)\rangle + f(g(x), y) \in -int_Y C(x)\} \end{aligned}$$

and

$$[G^{-1}(y)]^c = \{x \in K : \langle \theta(x, s), \eta(y, x)\rangle + f(g(x), y) \notin -int_Y C(x)\}.$$

Let $u \in \overline{[G^{-1}(y)]^c}$, the closure of $[G^{-1}(y)]^c$ in K. We claim that $u \in [G^{-1}(y)]^c$. Indeed, let $\{x_\lambda\}_{\lambda\in\Lambda}$ be a net in $[G^{-1}(y)]^c$ such that $x_\lambda \to u$. Since $\{x_\lambda\}_{\lambda\in\Lambda} \in [G^{-1}(y)]^c$, there exists $\{s_\lambda\}_{\lambda\in\Lambda} \in T(x_\lambda)$ such that

$$\langle\theta(x_\lambda, s_\lambda), \eta(y, x_\lambda)\rangle + f(g(x_\lambda), y) \notin -int_Y C(x_\lambda), \forall y \in K.$$

That is

$$\langle\theta(x_\lambda, s_\lambda), \eta(y, x_\lambda)\rangle + f(g(x_\lambda), y) \in W(x_\lambda).$$

From the upper semicontinuity and compact values of T and Lemma 2.5, without loss of generality, we may assume that s_λ converges to some $s \in T(x)$. By Lemma 2.1 we have for each $x \in K$ and for all $s \in T(x)$, $x \to \langle s, \eta(y, x)\rangle$ is continuous. Since f is continuous in the first argument and g is continuous. From assumption (iv), we have

$$\langle\theta(x_\lambda, s_\lambda), \eta(y, x_\lambda)\rangle + f(g(x_\lambda), y) \to \theta(u, s), \eta(y, u)\rangle + f(g(u), y).$$

By Lemma 2.2 and upper semicontinuity of W, it follows that

$$\theta(u, s), \eta(y, u)\rangle + f(g(u), y) \in W(u),$$

that is

$$\theta(u, s), \eta(y, u)\rangle + f(g(u), y) \notin -int_Y C(u)$$

and so $[G^{-1}(y)]^c$ is closed. Therefore, G has open lower sections in K. By Lemma 2.3, we know that $coG : K \to 2^K$ also has open lower sections. Also define another multifunction $S : K \to 2^K$ by

$$S(x) = \begin{cases} A(x) \cap coG(x) & \text{if } x \in M \\ A(x) & \text{if } x \notin M. \end{cases}$$

Then it is clear that for all $x \in K$, $S(x)$ is convex and $x \notin S(x) = coS(x)$. Since for all $y \in K$,

$$\begin{aligned} S^{-1}(y) &= \{x \in K : y \in S(x)\} \\ &= \{x \in M : y \in A(x) \cap coG(x)\} \\ &\cup \{x \in K \setminus M : y \in A(x)\} \\ &= \{M \cap A^{-1}(y) \cap coG^{-1}(y)\} \cup \{(K \setminus M) \cap A^{-1}(y)\} \\ &= [\{M \cap A^{-1}(y) \cup coG^{-1}(y)\} \cup (K \setminus M)] \\ &\cap [\{M \cap A^{-1}(y) \cup coG^{-1}(y)\} \cup A^{-1}(y)] \\ &= [K \cap [\{(A^{-1}(y) \cap coG^{-1}(y)) \cup (K \setminus M)\}] \\ &\cap [\{M \cup A^{-1}(y)\} \cap A^{-1}(y))] \\ &= [\{(A^{-1}(y) \cap coG^{-1}(y)) \cup (K \setminus M)\} \cap A^{-1}(y)] \\ &= \{A^{-1}(y) \cap coG^{-1}(y)\} \cup \{(K \setminus M) \cap A^{-1}(y)\} \qquad (3.3) \end{aligned}$$

and $A^{-1}(y)$, $coG^{-1}(y)$ and $K \setminus M$ are open in K, we have $S^{-1}(y)$ is open in K. Condition (v) implies that there exist a nonempty compact subset N of K and a nonempty, compact and convex subset B of K such that $S(x) \cap B = coS(x) \cap B \neq \emptyset$ for all $x \in K \setminus N$. Hence by Lemma 2.4, there exist $x^* \in K$ such that $S(x^*) = \emptyset$. Since for all $x \in K$, $A(x)$ is nonempty, we have $x^* \in M$ and $A(x^*) \cap coG(x^*) = \emptyset$. This implies $x^* \in A(x^*)$ and $A(x^*) \cap G(x^*) = \emptyset$. Consequently, $x^* \in A(x^*)$ and for all $y \in A(x^*)$, there exist $s^* \in T(x^*)$ satisfying $\langle \theta(x^*, s^*), \eta(y, x^*) \rangle + f(g(x^*), y) \notin -int_Y C(x^*)$. □

Corollary 3.1. *If in Theorem 3.1 we take T as single valued mapping then there exists $x^* \in K$ such that $x^* \in A(x^*)$ and for all $y \in A(x^*)$*

$$\langle \theta(x^*, T(x^*)), \eta(y, x^*) \rangle + f(g(x^*), y) \notin -int_Y C(x^*).$$

Corollary 3.2. *If in Theorem 3.1 we take $\eta(x, y) = x - g(y)$, for all $x, y \in K$, then there exists $x^* \in K$ such that $x^* \in A(x^*)$ and for all $y \in A(x^*)$ $\langle \theta(x^*, T(x^*)), y - g(x^*) \rangle + f(g(x^*), y) \notin -int_Y C(x^*)$.*

If T is a zero operator, then we have the following existence result for the implicit vector quasi-equilibrium problem.

Corollary 3.3. *Let K be a nonempty convex subset of a locally convex real Hausdorff topological vector space X and Y be a real Hausdorff topological vector space. Let $L(X, Y)$ be equipped with the σ-topology. Let $C : K \to 2^Y$ and $A : K \to 2^K$ be the multifunctions such that $\forall x \in K$, $A(x)$ is nonempty convex, $\forall y \in K$, $A^{-1}(y)$ is open in K, $\forall x \in K$, $C(x)$ is closed, convex and pointed cone in Y such that $int_Y C(x)$ is nonempty and the set $M = \{x \in K : x \in A(x)\}$ is closed in K. Suppose that the following assumptions hold:*

(i) *the mapping $g : K \to K$ is continuous, the bifunction $f : K \times K \to Y$ is continuous with respect to the first argument and $0-C(x)$ diagonally convex with respect to the second argument,*
(ii) *the multifunction $W : K \to 2^Y$ defined by $W(x) = Y \setminus (-int_Y C(x))$ for all $x \in K$, is upper semicontinuous on K,*
(iii) *there exists a nonempty and compact subset N of K and a nonempty, compact and convex subset B of K such that $\forall\ x \in K \setminus N$, $\exists y^* \in B$, such that $y^* \in A(x)$ and $f(g(x), y^*) \in -int_Y C(x)$.*

Then there exists a point $x^* \in K$ such that $x^* \in A(x^*)$ and

$$f(g(x^*), y) \notin -int_Y C(x^*), \forall y \in A(x^*).$$

By Theorem 3.1 and Remark 2.1, we have the following corollary.

Corollary 3.4. *Let K be a nonempty convex subset of a locally convex real Hausdorff topological vector space X and Y be a real Hausdorff topological vector space. Let $L(X,Y)$ be equipped with the σ-topology. Let $C : K \to 2^Y$ and $A : K \to 2^K$ be the multifunctions such that $\forall x \in K,\ A(x)$ is nonempty convex, $\forall y \in K,\ A^{-1}(y)$ is open in K, $\forall x \in K,\ C(x)$ is closed, convex and pointed cone in Y such that $int_Y C(x)$ is nonempty and the set $M = \{x \in K : x \in A(x)\}$ is closed in K. Suppose that the following assumptions hold:*

(i) *the mapping $g : K \to K$ is continuous, the bifunction $f : K \times K \to Y$ is continuous with respect to the first argument and $0 - C(x)$ diagonally convex with respect to the second argument,*

(ii) *the multifunction $T : K \to 2^{L(X,Y)}$ is upper semicontinuous on K with compact values and $\eta : K \times K \to X$ is continuous in the second argument and affine with respect to the first argument such that for all $x \in K$, there exists $s \in T(x)$ satisfying, $\langle \theta(x,s), \eta(x,x)\rangle \notin -int_Y C(x)$,*

(iii) *the multifunction $W : K \to 2^Y$ defined by $W(x) = Y \setminus (-int_Y C(x))$ for all $x \in K$, is upper semicontinuous on K,*

(iv) *$\theta(x_\lambda, s_\lambda) \to \theta(x,s)$, whenever $x_\lambda \to x$ in K, for $s_\lambda \in T(x_\lambda)$ and $s \in T(x)$,*

(v) *there exists a nonempty and compact subset N of K and a nonempty, compact and convex subset B of K such that $\forall\ x \in K \backslash N$, $\exists y^* \in B$, such that*

$$y^* \in A(x) \text{ and } \langle \theta(x,s), \eta(y^*,x)\rangle + f(g(x), y^*) \in -int_Y C(x),$$

$$\forall s \in T(x).$$

Then there exists a point $x^* \in K$ such that $x^* \in A(x^*)$ and for all $y \in A(x^*)$,

$$\exists\ s^* \in T(x^*) : \langle \theta(x^*, s^*), \eta(y, x^*)\rangle + f(g(x^*), y) \notin -int_Y C(x^*).$$

If $f = 0$, then by Theorem 3.1 and Corollary 3.2, we have the following.

Corollary 3.5. *Let K be a nonempty convex subset of a locally convex real Hausdorff topological vector space X and Y be a real Hausdorff topological vector space. Let $L(X,Y)$ be equipped with the σ-topology. Let $C : K \to 2^Y$ and $A : K \to 2^K$ be the multifunctions such that $\forall x \in K,\ A(x)$ is nonempty convex, $\forall y \in K,\ A^{-1}(y)$ is open in K, $\forall x \in K,\ C(x)$ is closed, convex and pointed cone in Y such that $int_Y C(x)$ is nonempty and the set $M = \{x \in K : x \in A(x)\}$ is closed in K. Suppose that the following assumptions hold:*

(i) *the mapping $g : K \to K$ is continuous and the multifunction $W : K \to 2^Y$ defined by $W(x) = Y \setminus (-int_Y C(x))\ \forall x \in K$, is upper semicontinuous on K,*

(ii) *$\theta(x_\lambda, s_\lambda) \to \theta(x,s)$, whenever $x_\lambda \to x$ in K, for $s_\lambda \in T(x_\lambda)$ and $s \in T(x)$.*

(iii) *there exists a nonempty and compact subset N of K and a nonempty, compact and convex subset B of K such that $\forall\ x \in K\backslash N$, $\exists y^* \in B$, such that*

$$y^* \in A(x) \text{ and } \langle \theta(x,s), \eta(y^*,x)\rangle \in -int_Y C(x), \forall s \in T(x),$$

(iv) *the multifunction $T : K \to 2^{L(X,Y)}$ is upper semicontinuous on K with compact values and $\eta : K \times K \to X$ is continuous in the second argument. Moreover, one of the following conditions satisfied*

(a) *T satisfies the generalized $\theta - L - \eta$-condition,*

(b) *$\eta : K \times K \to X$ is affine with respect to the first argument such that for all $x \in K$, there exists $s \in T(x)$ satisfying, $\langle \theta(x,s), \eta(x,x)\rangle \notin -int_Y C(x)$.*

Then there exists a point $x^* \in K$ such that $x^* \in A(x^*)$ and

$$\forall y \in A(x^*), \exists\ s^* \in T(x^*) : \langle \theta(x^*,s^*), \eta(y,x^*)\rangle \notin -int_Y C(x^*).$$

Theorem 3.2. *Let K be a nonempty convex subset of a locally convex real Hausdorff topological vector space X and Y be a real Hausdorff topological vector space. Let $L(X,Y)$ be equipped with the σ-topology. Let $C : K \to 2^Y$ and $A : K \to 2^K$ be the multifunctions such that $\forall x \in K,\ A(x)$ is nonempty convex, $\forall y \in K,\ A^{-1}(y)$ is open in K, $\forall x \in K,\ C(x)$ is closed, convex and pointed cone in Y such that $int_Y C(x)$ is nonempty and the set $M = \{x \in K : x \in A(x)\}$ is closed in K. Suppose that the following assumptions hold:*

(i) *the mapping $g : K \to K$ is continuous and the multifunction $W : K \to 2^Y$ defined by $W(x) = Y \setminus (-int_Y C(x))\ \forall x \in K$, is upper semicontinuous on K,*

(ii) *$\theta(x_\lambda, s_\lambda) \to \theta(x,s)$, whenever $x_\lambda \to x$ in K, for $s_\lambda \in T(x_\lambda)$ and $s \in T(x)$,*

(iii) *there exists a nonempty and compact subset N of K and a nonempty, compact and convex subset B of K such that $\forall\ x \in K\backslash N$, $\exists y^* \in B$, such that*

$$y^* \in A(x) \text{ and } \langle \theta(x,s), \eta(y^*,x)\rangle \in -int_Y C(x), \forall s \in T(x),$$

(iv) *the multifunction $T : K \to 2^{L(X,Y)}$ is upper semicontinuous on K with compact values, $\eta : K \times K \to X$ is continuous in the second argument and there exists a mapping $h : K \times K \to Y$, such that*

(a) *for all $x, y \in K$, $\exists s \in T(x)$, such that*

$$h(x,y) - \langle \theta(x,s), \eta(y,x)\rangle \in -int_Y C(x), \forall s \in T(x),$$

(b) *for any finite set* $\{y_1, y_2...y_n\} \subseteq K$ *and* $x^* = \sum_{i=1}^{n} \lambda_i y_i$ *with* $\lambda_i \geq 0$ *and* $\sum_{i=1}^{n} \lambda_i = 1$, *there is* $i \in \{1, 2, 3, ..., n\}$ *such that* $h(x^*, y_i) \notin -int_Y C(x^*)$.

Then there exists a point $x^* \in K$ *such that* $x^* \in A(x^*)$ *and*

$$\forall y \in A(x^*), \exists\ s^* \in T(x^*) : \langle \theta(x^*, s^*), \eta(y, x^*) \rangle \notin -int_Y C(x^*).$$

Proof. For all $x \in K$ define two multifunctions $G, P : K \to 2^K$ by

$$\begin{aligned} G(x) &= \{y \in K : \langle \theta(x, T(x)), \eta(y, x) \rangle \subseteq -int_Y C(x)\} \\ &= \{y \in K : \langle \theta(x, s), \eta(y, x) \rangle \in -int_Y C(x), \forall s \in T(x)\} \\ P(x) &= \{y \in K : h(x, y) \in -int_Y C(x)\}. \end{aligned}$$

We first prove that $x \notin coP(x)$, for all $x \in K$. To see this, suppose, by way of contradiction, that there exists some $x_o \in K$ such that $x_o \in coP(x_o)$. Then there exists finite points $y_1, y_2...y_n$ in K, and $\lambda_i \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that $x_o = \sum_{i=1}^{n} \lambda_i y_i$ and $y_i \in P(x_o)$ for all $i = 1, 2, ..., n$. This follows

$$h(x_o, y_i) \in -int_Y C(x_o),\ i = 1, 2, 3, ..., n.$$

This contradicts the condition $(iv)(b)$. Hence $x \notin coP(x)$ for all $x \in K$. The condition $(vi)(a)$ implies that $P(x) \supseteq G(x)$ for all $x \in K$. Hence $x \notin coG(x)$ for all $x \in K$. The remaining proof is similar to that of the proof of Theorem 3.1. □

Corollary 3.6. *Let* K *be a nonempty convex subset of a locally convex real Hausdorff topological vector space* X *and* Y *be a real Hausdorff topological vector space. Let* $L(X, Y)$ *be equipped with the* σ*-topology. Let* $C : K \to 2^Y$ *and* $A : K \to 2^K$ *be the multifunctions such that* $\forall x \in K,\ A(x)$ *is nonempty convex,* $\forall y \in K,\ A^{-1}(y)$ *is open in* K, $\forall x \in K,\ C(x)$ *is closed, convex and pointed cone in* Y *such that* $int_Y C(x)$ *is nonempty and the set* $M = \{x \in K : x \in A(x)\}$ *is closed in* K. *Suppose that the following assumptions hold:*

(i) *the mapping* $g : K \to K$ *is continuous and the multifunction* $W : K \to 2^Y$ *defined by* $W(x) = Y \setminus (-int_Y C(x))\ \forall x \in K$, *is upper semicontinuous on* K,

(ii) $\theta(x_\lambda, s_\lambda) \to \theta(x, s)$, *whenever* $x_\lambda \to x$ *in* K, *for* $s_\lambda \in T(x_\lambda)$ *and* $s \in T(x)$,

(iii) *there exists a nonempty and compact subset* N *of* K *and a nonempty, compact and convex subset* B *of* K *such that* $\forall\ x \in K \setminus N,\ \exists y^* \in B$, *such that*

$$y^* \in A(x) \text{ and } \langle \theta(x,s), \eta(y^*,x)\rangle \in -int_Y C(x), \forall s \in T(x),$$

(iv) *the multifunction* $T : K \to 2^{L(X,Y)}$ *is upper semicontinuous on* K *with compact values,* $\eta : K \times K \to X$ *is continuous in the second argument and there exists a mapping* $h : K \times K \to Y$*, such that*

(a) *for all* $x, y \in K$*,* $\exists s \in T(x)$*, such that*

$$h(x,y) - \langle \theta(x,s), \eta(y,x)\rangle \in -int_Y C(x), \forall s \in T(x),$$

(b) *the set* $\{y \in K : H(x,y) \in -int_Y C(x)\}$ *is convex for all* $x \in K$*,*

(c) $h(x,x) \notin -intC(x)$*, for all* $x \in K$*.*

Then there exists a point $x^* \in K$ *such that* $x^* \in A(x^*)$ *and*

$$\forall y \in A(x^*), \exists\ s^* \in T(x^*) : \langle \theta(x^*,s^*), \eta(y,x^*)\rangle \notin -int_Y C(x^*).$$

Proof. Following the arguments similar to those used in proving [11, Corollary 3], by the condition (iv)(b) and (iv)(c), the condition (iv)(b) of Theorem 3.2 holds. By Theorem 3.4, we know that the conclusion is correct. □

Remark 3.1. Theorem 3.1 generalizes and improves the corresponding results in [1, 2, 11, 15, 17, 19]. Corollary 3.3 and [12, Theorem 3.1] are different. Also Corollary 3.5 and [15, Theorem 2.1] are different. We note that our proof of Corollary 3.3 and 3.5 depends on the maximal element theorem instead of one person game theorem.

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