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INEQUALITIES FOR THE sth DERIVATIVE OF A POLYNOMIAL WITH PRESCRIBED ZEROS

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Abstract. Let P(z) be a polynomial of degree n which does not vanish outside the closed disk $|z| < k$, where $k \leq 1$. According to a famous result known as Turans Theorem for k=1,we have

$$\max_{|z|=1} |P^{'}(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|$$

In this paper we shall present several interesting generalizations and a refinement of this result which include some results due to Malik,Govil and others.we extend Turans Theorem for the sth derivatives of a polynomial having t-fold zeros at origin and thereby obtain an another generalization of this beautiful result.

1. Introduction

Let P(z) be a polynomial of degree n,then according to a famous result known as Bernstein's inequality(for refrence,see[6,p-531]or[7]),

$$\max_{|z|=1} |P^{'}(z)| \leq n \max_{|z|=1} |P(z)| \tag{1.1}$$

The result is best possible and equality holds for the polynomial having all its zeros at origin.In the reverse direction it was proved by Turan[8] that if P(z) does not vanish in $|z| > 1$, then

$$\max_{|z|=1} |P^{'}(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{1.2}$$

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Inequality (1) was refined by Aziz and Dawood by showing that under the same hypothesis that (1.2) can be replaced by

$$\max_{|z|=1}|P^{'}(z)| \geq \frac{n}{2}\left\{\max_{|z|=1}|P(z)| + \min_{|z|=1}|P(z)|\right\} \tag{1.3}$$

Both the inequalities (1.2) and (1.3) are sharp and equality holds for $P(z) = \alpha z^n + \beta$
where $|\alpha| = |\beta|$.

As a generalization of inequality (1.2) Malik [5] proved that if all the zeros of P(z) lie in $|z| \leq k,\ k \leq 1$, then

$$\max_{|z|=1}|P^{'}(z)| \geq \frac{n}{1+k}\max_{|z|=1}|P(z)| \tag{1.4}$$

Equality in (1.4) holds for the polynomial $P(z) = (z+k)^n,\ k \leq 1$,

In the literature there exists several extensions and generalizations of inequalities (1.3) and (1.4)(see [2],[4]). recently Aziz and Shah [3] have proved the following generalization of inequality (1.2).

Theorem 1.1. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree n having all its zeros in the disk* $|z| \leq k \leq 1$ *with s-fold zeros at origin,then for* $|z| = 1$,

$$\max_{|z|=1}|P^{'}(z)| \geq \frac{n+ks}{1+k}\max_{|z|=1}|P(z)|. \tag{1.5}$$

The result is sharp and extremal polynomial $P(z) = z^s(z+k)^{n-s},\ 0 < s \leq n.$

In this paper we shall first present the following refinement and a generalization of Theorem 1.1.

Theorem 1.2. *If P(z) is a polynomial of degree* $n \geq 1$ *having all its zeros in* $|z| \leq k,\ k \leq 1$ *with t-fold zeros at the origin then for* $1 \leq s \leq t+1 \leq n$,

$$\max_{|z|=1}|P^{s}(z)| \geq \left(\frac{n+kt}{1+k}\right)\left(\frac{n+kt}{1+k}-1\right)\cdots\left(\frac{n+kt}{1+k}-(s-1)\right)\max_{|z|=1}|P(z)|$$

$$+\mathcal{L}^{s}\left(\frac{n-t}{1+k}\right)\frac{1}{k^t}\min_{|z|=k}|P(z)| \tag{1.6}$$

where

$$\mathcal{L}^s = 1 \;\; for \;\; s = 1$$

$$= n(n-1)\cdots(n-s+2) \;\; for \;\; s \geq 2.$$

The result is best possible and equality holds for the polynomial $P(z) = (z+k)^n, \;\; k \leq 1$.

Remark. For t=0 and s=1,this reduces to the result due to Malik.

For k=1,we get the following result.

Corollary. *If P(z) is a polynomial of degree n having all its zeros in* $|z| \leq 1$, *with t-fold zeros at the origin then for* $1 \leq s \leq t+1 \leq n$,

$$\max_{|z|=1} |P^s(z)| \geq \left(\frac{n+t}{2}\right)\left(\frac{n+t}{2} - 1\right)\cdots\left(\frac{n+t}{2} - (s-1)\right) \max_{|z|=1} |P(z)|$$

$$+\mathcal{L}^s \left(\frac{n-t}{2}\right) \min_{|z|=1} |P(z)|,$$

where

$$\mathcal{L}^s = 1 \;\; for \;\; s = 1$$

$$= n(n-1)\cdots(n-s+2) \;\; for \;\; n \geq 2.$$

Next we prove the following result which extends inequality(1.4) to the sth derivative.

Theorem 1.3. *If P(z) is a polynomial of degree* $|z| \leq k, \;\; k \leq 1$, *having all its zeros in* $|z| \leq k, \;\; k \leq 1$,*then*

$$\max_{|z|=1} |P^s(z)| \geq \frac{n(n-1)\cdots(n-s+2)}{(1+k)^s} \max_{|z|=1} |P(z)|. \tag{1.7}$$

The result is best possible with equality in (1.7) for the polynomial $P(z) = (z+k)^n$.

2. Lemmas

For the proofs of these theorems,we need the following Lemmas.

Lemma 2.1. *If P(z) is a polynomial of degree n having all its zeros in* $|z| \leq k$, *then*

$$\min_{|z|=1} |P^{'}(z)| \geq n \min_{|z|=1} |P(z)| \tag{2.1}$$

The result is best possible with equality for the polynomial $P(z) = me^{i\alpha}z^n$, $m > 0$.

Lemma 2.1 is due to Aziz and Dawood [1].

Lemma 2.2. *If P(z) is a polynomial of degree n having all its zeros in* $|z| \leq k,\ k \geq 1$,*then*

$$\min_{|z|=k} |P^{s}(z)| \geq \frac{n(n-1)\cdots(n-s+1)}{k^s} \min_{|z|=k} |P(z)| \tag{2.2}$$

The result is best possible and equality in (2.2) holds for the polynomial $P(z) = (z+k)^n$.

Proof. Since P(z) has all its zeros in $|z| \leq 1$.Let F(z)=P(kz) then F(z) has all its zeros in $|z| \leq 1$.Applying Lemma 2.1 to the polynomial F(z), we get

$$\min_{|z|=1} |F^{'}(z)| \geq n \min_{|z|=1} |F(z)|$$

Equavalently,

$$\min_{|z|=1} |P^{'}(kz)| \geq \frac{n}{k} \min_{|z|=1} |P(kz)|$$

or

$$\min_{|z|=k} |P^{'}(z)| \geq \frac{n}{k} \min_{|z|=k} |P(kz)| \tag{2.3}$$

$P^{'}(z)$ is a polynomial of degree n-1,therefore by (2.3),we have

$$\min_{|z|=k} |P^{"}(z)| \geq \frac{n(n-1)}{k^2} \min_{|z|=k} |P(z)|$$

Proceeding in a similar way it follows that

$$\min_{|z|=k}|P^{s}(z)| \geq \frac{n(n-1)\cdots(n-s+1)}{k^{s}}\min_{|z|=k}|P(z)|$$

This completes the proof of Lemma 2.2. □

Lemma 2.3. *If P(z) is a polynomial of degree n having all its zeros in $|z| \leq k,\ k \geq 1$, with t-fold zeros at the origin, then*

$$\max_{|z|=1}|P^{'}(z)| \geq \frac{n+kt}{1+k}\max_{|z|=1}|P(z)| + \frac{n-t}{(1+k)k^{t}}\min_{|z|=k}|P(z)| \tag{2.4}$$

The result is sharp and equality holds for the polynomial $P(z) = z^{t}(z+k)^{n-t},\ 0 < t \leq n$.

Proof. If $m = \min_{|z|=k}|P(z)|$,then $m \leq |P(z)|$ for $|z| = k$,which gives $m|\frac{z}{k}|^{t} \leq |P(z)|$ for $|z| = k$.since all the zeros of P(z) lie in $|z| \leq k \leq 1$, with t-fold zeros at the origin,therefore for every complex number α such that $|\alpha| < 1$, it follows (by Rouches Theorem for $m > 0$) that the polynomial $G(z) = P(z) + \frac{\alpha m}{k^{t}}z^{t}$ has all its zeros in $|z| \leq k,\ k \leq 1$, with t-fold zeros at the origin.So that we can write

$$G(z) = z^{t}H(z) \tag{2.5}$$

Where H(z) is a polynomial of degree n-t having all its zeros in $|z| \leq k,\ k \leq 1$.

From (2.5),we get

$$\frac{zG^{'}(z)}{G(z)} = t + \frac{zH^{'}(z)}{H(z)} \tag{2.6}$$

If $z_1, z_2, \cdots, z_{n-t}$ are the zeros of H(z), then $|z_j| \leq k \leq 1$ and from (2.6),we have

$$Re\left\{\frac{e^{i\theta}G^{'}(e^{i\theta})}{G(e^{i\theta})}\right\} = t + Re\left\{\frac{e^{i\theta}H^{'}(e^{i\theta})}{H(e^{i\theta})}\right\}$$

$$= t + Re\sum_{j=1}^{n-t}\frac{e^{i\theta}}{e^{i\theta}-z_j}$$

$$= t + \sum_{j=1}^{n-t} Re \frac{1}{1 - z_j e^{-i\theta}} \tag{2.7}$$

for points $e^{i\theta}$, $0 \le \theta < 2\pi$, which are not the zeros of H(z).

Now if $|w| \le k \le 1$,then it can be easily verified that

$$Re\left(\frac{1}{1-w}\right) \ge \frac{1}{1+k}$$

.
Using this fact in (2.7),we see that

$$\left|\frac{G^{'}(e^{i\theta})}{G(e^{i\theta})}\right| \ge Re\left(\frac{e^{i\theta}G^{'}(e^{i\theta})}{G(e^{i\theta})}\right)$$

$$= t + \sum_{j=1}^{n-t} Re \frac{1}{1 - z_j e^{-i\theta}}$$

$$\ge t + \frac{n-t}{1+k},$$

which gives,

$$|G^{'}(e^{i\theta})| \ge \frac{n+tk}{1+k}|G(e^{i\theta})| \tag{2.8}$$

for points $e^{i\theta}$, $0 \le \theta < 2\pi$, which are not the zeros of G(z).Since inequality (2.8) is trivally true for points $e^{i\theta}$, $0 \le \theta < 2\pi$, which are the zeros of P(z), it follows that

$$|G^{'}(z)| \ge \frac{n+tk}{1+k}|G(z)| \ for \ |z| = 1 \tag{2.9}$$

Replacing G(z) by $P(z) + \frac{\alpha m}{k^t} z^t$ in (2.9),then we get

$$|P^{'}(z) + \alpha t \frac{m}{k^t} z^{t-1}| \ge \frac{n+tk}{1+k}|P(z) + \frac{\alpha m}{k^t} z^t| \ for \ |z| = 1 \tag{2.10}$$

and for every α with $|\alpha| < 1$. Choosing the argument of α such that

$$|P(z) + \frac{\alpha m}{k^t} z^t| = |P(z)| + |\alpha| \frac{m}{k^t} \ for \ |z| = 1,$$

it follows from (2.10), that

$$|P^{'}(z)| + \frac{t|\alpha|m}{k^t} \geq \frac{n+tk}{1+k} \left\{ |P(z)| + \frac{|\alpha|m}{k^t} \right\} \ for \ |z| = 1,$$

Letting $|\alpha| \to 1$,we obtain

$$|P^{'}(z)| \geq \frac{n+tk}{1+k}|P(z)| + \left\{ \frac{n+tk}{1+k} - t \right\} \frac{m}{k^t}$$

$$= \frac{n+tk}{1+k}|P(z)| + \frac{n-t}{1+k}(\frac{m}{k^t}) \ for \ |z| = 1.$$

This implies,

$$\max_{|z|=1} |P^{'}(z)| \geq \frac{n+kt}{1+k} \max_{|z|=1} |P(z)| + \frac{n-t}{(1+k)k^t} \min_{|z|=k} |P(z)|$$

Which is the desired result. □

3. Proof of the theorems

Proof of Theorem 1.2. We prove this result with the help of mathematical induction.We use induction on s.For s=1,the result follows by Lemma 2.3.Assume that inequality (1.6) is true for s=r, that is we assume for $1 \leq r \leq t+1$,

$$\max_{|z|=1} |P^r(z)| \geq \left(\frac{n+kt}{1+k}\right)\left(\frac{n+kt}{1+k} - 1\right) \cdots \left(\frac{n+kt}{1+k} - (r-1)\right) \max_{|z|=1} |P(z)|$$

$$+\mathcal{L}^r \frac{(n-t)}{(1+k)k^t} \min_{|z|=k} |P(z)| \qquad (3.1)$$

Where

$$\mathcal{L}^r = 1 \ for \ r = 1$$

$$= n(n-1)\cdots(n-r+2) \ for \ r \geq 2.$$

We show (1.6) holds for s=r+1 also.Since P(z) is a polynomial of degree n having all its zeros in $|z| \leq k, \ k \leq 1$, with t-fold zeros at the origin,therefore by Gauss-Lucas Theorem $P^r(z)$ which is a polynomial of degree n-r has all its zeros in $|z| \leq k, \ k \leq 1$, with t-r fold zeros at the origin.Applying Lemma 2.3 to the polynomial $P^r(z)$,we get,

$$\max_{|z|=1} |P^{r+1}(z)| \geq \frac{(n-r)+(t-r)k}{1+k} \max_{|z|=1} |P^r(z)| + \frac{(n-r)-(t-r)}{(1+k)k^{t-r}} \min_{|z|=k} |P^r(z)| \tag{3.2}$$

Using Lemma 2.2,we get

$$\max_{|z|=1} |P^{r+1}(z)| \geq \left\{ \frac{n+tk}{1+k} - r \right\} \max_{|z|=1} |P^r(z)|$$

$$+ \frac{(n-t)}{(1+k)k^t} n(n-1)\cdots(n-r+1) \min_{|z|=k} |P(z)|$$

This implies with the help of Lemma 2.1 that,

$$\max_{|z|=1} |P^{r+1}(z)|$$

$$\geq \left(\frac{n+kt}{1+k} \right) \left(\frac{n+kt}{1+k} - 1 \right) \cdots \left(\frac{n+kt}{1+k} - (r-1) \right) \left(\frac{n+kt}{1+k} - (r+1-1) \right)$$

$$\max_{|z|=1} |P(z)| + \frac{(n-t)}{(1+k)k^t} n(n-1)\cdots(n-(r+1)+2) \min_{|z|=1} |P(z)|. \tag{3.3}$$

(3.3) shows that the result is true for s=r+1 also.We conclude with the help of mathematical induction that (1.6) holds for all $1 \leq s < n$.This completes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Since P(z) has all its zeros in $|z| \leq k, \ k \leq 1$,therefore by Gauss-Lucas Theorem $P^r(z)$ has all its zeros in $|z| \leq k, \ k \leq 1$,for $1 \leq s < n$. Applying inequality (1.4) to the polynomial $P^{s-1}(z)$ and proceeding similarly as in the above Theorem it follows that

$$\max_{|z|=1} |P^s(z)| \geq \frac{n(n-1)\cdots(n-s+1)}{(1+k)^s} \max_{|z|=1} |P(z)|.$$

This proves Theorem 1.3. □

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