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ON THE EXISTENCE OF POSITIVE SOLUTIONS FOR BVPS OF NONLINEAR SINGULAR FOURTH-ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper, by means of the fixed point theorems of the cone expansion and compression of norm type, an explicit interval for λ is derived such that for any λ in this interval, the existence of at least one positive solution to singular boundary value problem in question. Our results extend and improve many known results in the field including both singular and non-singular cases.

1. Introduction

In this paper, we consider the existence of positive solutions for the following boundary value problem(BVP):

$$\begin{cases} u^{(4)}(t) - \lambda a(t) f(t, u, u'') = 0, \quad 0 < t < 1, \\ \alpha_1 u(0) - \beta_1 u'(0) = 0, \\ \gamma_1 u(1) + \delta_1 u'(1) = 0, \\ \alpha_2 u''(0) - \beta_2 u'''(0) = 0, \\ \gamma_2 u''(1) + \delta_2 u'''(1) = 0, \end{cases} \tag{1.1}$$

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where λ is a positive constant, $\alpha_i, \beta_i, \gamma_i, \delta_i \geq 0$, $\Delta_i = \beta_i\gamma_i + \alpha_i\gamma_i + \alpha_i\delta_i > 0$ $(i = 1, 2)$. $a(t)$ can be singular at $t = 0$ and/or 1, $f : [0, 1] \times [0, +\infty) \times (-\infty, 0] \to [0, +\infty)$ is continuous and $f(t, u, v) > 0$ for any $u > 0, v < 0$.

The above boundary value problems for ordinary differential equations play a very important role in both theory and application. It is used to describe a large number of physical, biological and chemical phenomena. For example, when $\lambda = 1$, BVP (1.1) subject to Lidstone boundary value conditions $u(0) = u(1) = u''(0) = u''(1) = 0$ is used to model phenomena such as the deflection of an elastic beam simply supported at the endpoints, and the u'' in f is the bending moment term which represents the bending effect. Owing to its importance in physics, the existence of solutions to this problem has been studied by many authors [1, 2, 3, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Especially, in recent years the existence of positive solutions has caught considerable attention [2, 3, 6, 7, 10, 13, 14, 15]. In addition, it is worth mentioning Ma [7] showed the existence of positive solution for the following BVP:

$$\begin{cases} u^{(4)}(t) - f(t, u, u'') = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.2}$$

where $f \in C([0, 1] \times [0, +\infty) \times (-\infty, 0], [0, +\infty))$ is superlinear or sublinear and Liu [6] improved the results of Ma [7]. Moreover, Ma [8, 9] used bifurcation techniques to obtain nodal solutions to the problems which contain a special bending moment term. We notice that, in all the above mentioned papers, f is required satisfy some growth conditions or assumptions of monotonicity which are essential for the technique used.

For the special case where f does not contain the bending moment term, BVP (1.2) reduces to the following simple fourth-order BVP

$$\begin{cases} u^{(4)}(t) - f(t, u) = 0, \quad 0 < t < 1, \\ u(0) = u'(1) = u''(0) = u''(1) = 0. \end{cases} \tag{1.3}$$

Ma and Wang [10] have shown the existence of positive solutions of the BVP (1.3) under the condition that $f(t, u)$ is either superlinear or sublinear on u by employing Krasnoselskii's fixed point theorem in cones. And then, Bai and Wang [3] improved the results in [10], in which they obtained the following result:

Theorem 1.1 ([3], Theorem 3.5)**.** *Assume that* $f : [0, 1] \times [0, +\infty) \to [0, +\infty)$ *is continuous such that one of the following conditions is satisfied:*

(i) $f^0 < \pi^4 < f_\infty$;
(ii) $f^\infty < \pi^4 < f_0$.

Then BVP (1.3) *has at least one positive solution.*

Motivated by the above results, in this paper, we shall discuss the more general form of BVP (1.1) which contains effect of the bending term and possess the general Sturm-Liouville boundary condition.

The rest of the paper is organized as follows. In Section 2, we firstly present some properties of Green's functions that are to be used to define a positive operator. Then in Section 3 the existence of positive solution for the BVP (1.1) will be established by using the fixed point theory in cones.

Throughout this paper, we adopt the following assumptions:

($\mathbf{H}_1$): $f(t,u,v) \in C([0,1]\times[0,+\infty)\times(-\infty,0],[0,+\infty))$.

($\mathbf{H}_2$): $a \in C((0,1),[0,+\infty)), 0 < \int_0^1 \left[\int_0^1 G_1(s,s)G_2(s,\tau)a(\tau)d\tau\right] ds < +\infty$,

where

$$G_1(t,s) = \frac{1}{\Delta_1}\begin{cases} (\gamma_1+\delta_1-\gamma_1 t)(\beta_1+\alpha_1 s), & 0\le s\le t\le 1,\\ (\gamma_1+\delta_1-\gamma_1 s)(\beta_1+\alpha_1 t), & 0\le t\le s\le 1,\end{cases} \tag{1.4}$$

$$G_2(t,s) = \frac{1}{\Delta_2}\begin{cases} (\gamma_2+\delta_2-\gamma_2 t)(\beta_2+\alpha_2 s), & 0\le s\le t\le 1,\\ (\gamma_2+\delta_2-\gamma_2 s)(\beta_2+\alpha_2 t), & 0\le t\le s\le 1.\end{cases} \tag{1.5}$$

It is well known that BVP (1.1) has a solution $u = u(t)$ if and only if u solves the operator equation

$$u(t) = Tu(t) =: \lambda \int_0^1 \left[\int_0^1 G_1(t,s)G_2(s,\tau)a(\tau)f(\tau,u(\tau),u''(\tau))d\tau\right] ds. \tag{1.6}$$

Let

$$E = \{u(t)\in C[0,1] : u''(t) \text{ is continuous, and } u(t)\ge 0, u''(t)\le 0, 0\le t\le 1\}$$

with the norm $\|u\|_2 = \|u\|+\|u''\|$, where $\|u\| = \max_{0\le t\le 1}|u(t)|$, $\|u''\| = \max_{0\le t\le 1}|u''(t)|$. Then $(E,\|\cdot\|)$ is a Banach space.

Let $\mu\in(0,\frac{1}{2})$ be a fixed constant. We introduce the following symbols for convenience:

$$f^0 = \limsup_{|x|+|y|\to 0}\max_{0\le t\le 1}\frac{f(t,x,y)}{|x|+|y|}, \qquad f_\infty = \liminf_{|x|+|y|\to\infty}\min_{\mu\le t\le 1-\mu}\frac{f(t,x,y)}{|x|+|y|},$$

$$f^\infty = \limsup_{|x|+|y|\to\infty}\max_{0\le t\le 1}\frac{f(t,x,y)}{|x|+|y|}, \qquad f_0 = \liminf_{|x|+|y|\to 0}\min_{\mu\le t\le 1-\mu}\frac{f(t,x,y)}{|x|+|y|},$$

$$A_1 = \max_{0\le t\le 1}\int_0^1\int_0^1 G_1(t,s)G_2(s,\tau)a(\tau)d\tau ds,$$
$$A_2 = \max_{0\le t\le 1}\int_0^1 G_2(t,s)a(s)ds,$$
$$B_1 = \min_{\mu\le t\le 1-\mu}\int_0^1\int_\mu^{1-\mu} G_1(t,s)G_2(s,\tau)a(\tau)d\tau ds,$$
$$B_2 = \min_{\mu\le t\le 1-\mu}\int_\mu^{1-\mu} G_2(t,s)a(s)ds,$$
$$L = A_1 + A_2, \quad l = \omega B, \quad B = \min\{B_1, B_2\}.$$

2. Preliminaries and some lemmas

In this section, we give some lemmas that are important to the proof of our main results.

Lemma 2.1 ([4]). *Suppose that E is a Banach space, K is a cone in E. Let Ω_1 and Ω_2 be two bounded open sets in E such that $\theta \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let operator $T : K \cap (\overline{\Omega}_2\backslash\Omega_1) \to K$ be completely continuous. Suppose that one of the following two conditions is satisfied:*

(i) $\|Tu\| \le \|u\|, \quad u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|, \quad u \in K \cap \partial\Omega_2$.
(ii) $\|Tu\| \ge \|u\|, \quad u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|, \quad u \in K \cap \partial\Omega_2$.

Then T has at least one fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.2 ([5]). *Let E be a Banach space, $T_n : E \to E$ $(n = 1, 2, 3, \cdots)$ be a completely continuous operator. Suppose that $T : E \to E$ satisfies*

$$\lim_{n\to\infty}\max_{\|u\|\le r}\|T_n u - Tu\| = 0, \qquad \textit{for some } \ r > 0.$$

Then T is a completely continuous operator.

Now let

$$K = \left\{u \in E : \min_{\mu\le t\le 1-\mu} u(t) \ge \omega\|u\|, \max_{\mu\le t\le 1-\mu} u''(t) \le -\omega\|u''\|\right\},$$

where

$$0 < \omega = \min\left\{\frac{\mu\gamma_1+\delta_1}{\gamma_1+\delta_1}, \frac{\mu\alpha_1+\beta_1}{\alpha_1+\beta_1}, \frac{\mu\gamma_2+\delta_2}{\gamma_2+\delta_2}, \frac{\mu\alpha_2+\beta_2}{\alpha_2+\beta_2}\right\} < 1$$

is a constant satisfying

$$G_1(t,s) \ge \omega G_1(s,s),\ G_2(t,s) \ge \omega G_2(s,s), \ \text{ for } \mu \le t \le 1-\mu,\ s \in [0,1]. \quad (2.1)$$

Clearly, K is a cone of Banach space E.

For any $0 < r < R < +\infty$, let $K_r = \{x \in K : \|x\| < r\}$, $\partial K_r = \{x \in K : \|x\| = r\}$, and $\overline{K}_{r,R} = \{x \in K : r \le \|x\| \le R\}$.

Lemma 2.3. *Suppose that* (H_1) *and* (H_2) *are satisfied, then* $T : K \to K$ *is a completely continuous operator.*

Proof. For any $u \in K$ and $t \in [0,1]$, by (1.4), (1.5) and (1.6), we have

$$\begin{aligned}(Tu)(t) =&\lambda \int_0^1 \left[\int_0^1 G_1(t,s)G_2(s,\tau)a(\tau)f(\tau,u(\tau),u''(\tau))d\tau\right] ds\\ \le&\lambda \int_0^1 \left[\int_0^1 G_1(s,s)G_2(s,\tau)a(\tau)f(\tau,u(\tau),u''(\tau))d\tau\right] ds.\end{aligned}$$

Consequently,

$$\|Tu\| \le \lambda \int_0^1 \left[\int_0^1 G_1(s,s)G_2(s,\tau)a(\tau)f(\tau,u(\tau),u''(\tau))d\tau\right] ds. \tag{2.2}$$

It follows from (1.6) and (2.1) that

$$\begin{aligned}\min_{\mu\le t\le 1-\mu}(Tu)(t) =& \min_{\mu\le t\le 1-\mu}\lambda \int_0^1 \left[\int_0^1 G_1(t,s)G_2(s,\tau)a(\tau)f(\tau,u(\tau),u''(\tau))d\tau\right] ds\\ \ge&\lambda \int_0^1 \left[\int_0^1 \omega G_1(s,s)G_2(s,\tau)a(\tau)f(\tau,u(\tau),u''(\tau))d\tau\right] ds\\ \ge&\omega\lambda \int_0^1 \left[\int_0^1 G_1(s,s)G_2(s,\tau)a(\tau)f(\tau,u(\tau),u''(\tau))d\tau\right] ds.\end{aligned}$$

This together with (2.2) implies that

$$\min_{\mu\le t\le 1-\mu}(Tu)(t) \ge \omega\|Tu\|.$$

On the other hand, by (1.6), for any $t \in [0,1]$, we have

$$\begin{aligned}(Tu)''(t) =& -\lambda \int_0^1 G_2(t,s)a(s)f(s,u(s),u''(s))ds\\ \ge& -\lambda \int_0^1 G_2(s,s)a(s)f(s,u(s),u''(s))ds,\end{aligned}$$

which shows

$$\|(Tu)''\| \le \lambda \int_0^1 G_2(s,s)a(s)f(s,u(s),u''(s))ds.$$

Hence,

$$\begin{aligned}\max_{\mu\le t\le 1-\mu}(Tu)''(t)=&-\min_{\mu\le t\le 1-\mu}\lambda\int_0^1 G_2(t,s)a(s)f\left(s,u(s),u''(s)\right)ds\\ \le&-\omega\lambda\int_0^1 G_2(s,s)a(s)f\left(s,u(s),u''(s)\right)ds\\ \le&-\omega\|(Tu)''\|.\end{aligned}$$

Therefore, $T:K\to K$.

In the following, we prove that T is a completely continuous operator. For any natural number n $(n\ge 2)$, we set

$$a_n(t)=\begin{cases}\inf_{t<s\le\frac{1}{n}}a(s), & 0\le t\le\frac{1}{n},\\ a(t), & \frac{1}{n}\le t\le\frac{n-1}{n},\\ \inf_{\frac{n-1}{n}\le s\le t}a(s), & \frac{n-1}{n}\le t\le 1.\end{cases}\tag{2.3}$$

Then $a_n:[0,1]\to[0,+\infty)$ is continuous and $a_n(t)\le a(t), t\in(0,1)$. Now, for any nature number n, we define an operator $T_n:E\to E$ by

$$(T_nu)(t)=\lambda\int_0^1\left[\int_0^1 G_1(t,s)G_2(s,\tau)a_n(\tau)f(\tau,u(\tau),u''(\tau))d\tau\right]ds.\tag{2.4}$$

It is obvious that $T_n:E\to E$ is completely continuous for each $n\ge 2$. For $r>0$ and $u\in K_r$, by (1.6), (2.3), (2.4) and the absolute continuity of integral, we have

$$\begin{aligned}&\lim_{n\to\infty}\sup_{u\in K_r}\|T_nu-Tu\|\\ \le&\lim_{n\to\infty}\sup_{u\in K_r}\max_{0\le t\le 1}\lambda\int_0^1\left[\int_0^1 G_1(t,s)G_2(s,\tau)(a(\tau)-a_n(\tau))f(\tau,u(\tau),u''(\tau))d\tau\right]ds\\ \le&M\lim_{n\to\infty}\lambda\int_0^1\left[\int_0^1 G_1(s,s)G_2(s,\tau)(a(\tau)-a_n(\tau))d\tau\right]ds\\ =&M\lim_{n\to\infty}\lambda\int_0^1\left[\int_{e(n)} G_1(s,s)G_2(s,\tau)(a(\tau)-a_n(\tau))d\tau\right]ds\\ \le&M\lim_{n\to\infty}\lambda\int_0^1\left[\int_{e(n)} G_1(s,s)G_2(s,\tau)a(\tau)d\tau\right]ds=0,\end{aligned}$$

where $M=\max\{f(t,x,y):t\in[0,1],\ x\in[0,r],\ y\in[-r,0]\}$, $e(n)=[0,\frac{1}{n}]\cup[\frac{n-1}{n},1]$. Therefore, $T:K\to K$ is a completely continuous operator by Lemma 2.2. □

3. Main results

In this section, we present our main results as follows:

Theorem 3.1. *Suppose* $0 \le f^0 < L^{-1}$ *and* $0 < l^{-1} < f_\infty \le +\infty$. *Then for any*

$$\lambda \in \left(\frac{1}{l f_\infty}, \frac{1}{L f^0}\right), \tag{3.1}$$

BVP (1.1) *has at least one positive solution.*

Proof. Let λ satisfy (3.1), and choose $\varepsilon_1 > 0$ such that $f_\infty - \varepsilon_1 > 0$ and

$$\frac{1}{(f_\infty - \varepsilon_1) l} \le \lambda \le \frac{1}{(f^0 + \varepsilon_1) L}. \tag{3.2}$$

Since $0 \le f^0 < L^{-1}$, there exists $r > 0$ such that

$$f(t,x,y) \le (f^0 + \varepsilon_1)(|x| + |y|), \quad \text{for } 0 \le t \le 1,\ 0 < |x| + |y| \le r, x \ge 0, y \le 0.$$

Hence, when $u \in K, \|u\|_2 = r$, we have

$$f(t, u(t), u''(t)) \le (f^0 + \varepsilon_1) r = (f^0 + \varepsilon_1)\|u\|_2, \quad \text{for } \ t \in [0,1]. \tag{3.3}$$

For any $u \in \partial K_r$, i.e. $\|u\|_2 = r$, by (3.3), we obtain

$$\begin{aligned} \|Tu\| &= \max_{0 \le t \le 1} \lambda \int_0^1 \int_0^1 G_1(t,s) G_2(s,\tau) a(\tau) f(\tau, u(\tau), u''(\tau)) d\tau ds \\ &\le \lambda \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau) a(\tau) f(\tau, u(\tau), u''(\tau)) d\tau ds \\ &\le \lambda (f^0 + \varepsilon_1)\|u\|_2 \int_0^1 \int_0^1 G_1(s,s) G_2(s,\tau) a(\tau) d\tau ds \\ &= \lambda (f^0 + \varepsilon_1)\|u\|_2 A_1, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \|(Tu)''\| &= \max_{0 \le t \le 1} \lambda \int_0^1 G_2(t,s) a(s) f(s, u(s), u''(s)) ds \\ &\le \lambda \int_0^1 G_2(s,s) a(s) f(s, u(s), u''(s)) ds \\ &\le \lambda (f^0 + \varepsilon_1)\|u\|_2 \int_0^1 G_2(s,s) a(s) ds \\ &= \lambda (f^0 + \varepsilon_1)\|u\|_2 A_2. \end{aligned} \tag{3.5}$$

For any $u \in \partial K_r$, by (3.4) and (3.5), we know

$$\|Tu\|_2 = \|Tu\| + \|(Tu)''\| \le \lambda (f^0 + \varepsilon_1)\|u\|_2 (A_1 + A_2) \le \|u\|_2. \tag{3.6}$$

On the other hand, by $0 < l^{-1} < f_\infty \le +\infty$, there exists $R_0 > 0$ such that

$$f(t,x,y) \ge (f_\infty - \varepsilon_1)(|x| + |y|), \quad \text{for } |x| + |y| \ge R_0,\ \mu \le t \le 1 - \mu. \tag{3.7}$$

Taking $R > \max\{r, \omega^{-1}R_0\}$, when $u \in K$ and $\|u\|_2 = R$, for any $t \in [\mu, 1-\mu]$, we get $|u(t)| + |u''(t)| \geq \omega\|u\|_2 \geq R_0$. Hence, by (3.7), for any $t \in [\mu, 1-\mu]$, we have

$$f(t, u(t), u''(t)) \geq (f_\infty - \varepsilon_1)(|u(t)| + |u''(t)|) \geq (f_\infty - \varepsilon_1)\omega\|u\|_2. \tag{3.8}$$

Therefore, for any $\mu \leq t \leq 1-\mu$ and $u \in \partial K_R$, by (3.2) and (3.8), we have

$$\begin{aligned}
\left|(Tu)''(1/2)\right| &= \lambda \int_0^1 G_2(1/2, s)a(s)f(s, u(s), u''(s))ds \\
&\geq \lambda \int_\mu^{1-\mu} G_2(1/2, s)a(s)f(s, u(s), u''(s))ds \\
&\geq (f_\infty - \varepsilon_1)\omega\|u\|_2\lambda \int_\mu^{1-\mu} G_2(1/2, s)a(s)ds \\
&\geq (f_\infty - \varepsilon_1)\omega\|u\|_2\lambda \min_{\mu\leq t\leq 1-\mu} \int_\mu^{1-\mu} G_2(t, s)a(s)ds \\
&= (f_\infty - \varepsilon_1)\omega\|u\|_2\lambda B_2 \\
&\geq (f_\infty - \varepsilon_1)\|u\|_2\lambda\, l \\
&\geq \|u\|_2.
\end{aligned}$$

So,

$$\|Tu\|_2 = \|Tu\| + \|(Tu)''\| \geq \left|(Tu)''(1/2)\right| \geq \|u\|_2, \quad \text{for} \quad u \in \partial K_R. \tag{3.9}$$

It follows from (3.6), (3.9) and Lemma 2.1 that the operator T has a fixed point $u_0 \in K$ satisfying $r \leq \|u_0\|_2 \leq R$ and $u_0(t) \geq 0, u_0''(t) \leq 0$ for any $t \in [0,1]$. Therefore, u_0 is a positive solution of BVP (1.1) by the concavity of u_0 in $[0,1]$, i.e., u_0 satisfies BVP (1.1) and $u_0(t) > 0, t \in (0,1)$ and finishes the proof. □

Remark 3.2. From $lf_\infty > 1, 0 \leq Lf^0 < 1$, we have $1 \in (\frac{1}{lf_\infty}, \frac{1}{Lf^0})$. Therefore, Theorem 3.1 also holds for $\lambda = 1$.

Remark 3.3. From the proof of Theorem 3.1 we can know that $f(t,u,v)$ need not be sub-linear or sup-linear. In fact, the Theorem 3.1 contains one of the following cases:

(i) If $f_\infty = \infty, f^0 > 0$, $\lambda \in (0, \frac{1}{Lf^0})$;
(ii) If $f_\infty = \infty, f^0 = 0$, $\lambda \in (0, +\infty)$;
(iii) If $f_\infty > l^{-1} > 0, f^0 = 0$, $\lambda \in (\frac{1}{lf_\infty}, +\infty)$.

Theorem 3.4. *Suppose* $0 \leq f^\infty < L^{-1}$ *and* $0 < l^{-1} < f_0 \leq +\infty$. *Then for any*

$$\lambda \in \left(\frac{1}{lf_0}, \frac{1}{Lf^\infty}\right), \tag{3.10}$$

BVP (1.1) *has at least one positive solution.*

Proof. Let λ satisfy (3.10), and choose $\varepsilon_2 > 0$ such that $L^{-1} - \varepsilon_2 > 0$, $f_0 - \varepsilon_2 > 0$ and

$$\frac{1}{l(f_0 - \varepsilon_2)} < \lambda < \frac{1}{f^\infty}\left(\frac{1}{L} - \varepsilon_2\right).$$

Since $0 \le f^\infty < L^{-1}$, there exists $R_0 > 0$ such that

$$f(t,x,y) \le \lambda^{-1}(L^{-1} - \varepsilon_2)(|x| + |y|), \quad \text{for } 0 \le t \le 1,\ |x| + |y| \ge R_0. \tag{3.11}$$

Taking a sufficiently large $R > R_0$ such that

$$\max\{f(t,x,y) : 0 \le t \le 1, |x| + |y| \le R_0\} \le \lambda^{-1}(L^{-1} - \varepsilon_2)R. \tag{3.12}$$

Then, by (3.11) and (3.12), for any $u \in K$, $\|u\|_2 = R$, we have

$$0 \le f(t, u(t), u''(t)) \le \lambda^{-1}(L^{-1} - \varepsilon_2)R = \lambda^{-1}(L^{-1} - \varepsilon_2)\|u\|_2. \tag{3.13}$$

From (3.13), for any $u \in \partial K_R$, we obtain

$$\begin{aligned}\|Tu\| &= \max_{0\le t\le 1} \lambda \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)a(\tau)f(\tau,u(\tau),u''(\tau))d\tau ds \\ &\le \lambda^{-1}(L^{-1} - \varepsilon_2)\|u\|_2 \lambda \max_{0\le t\le 1} \int_0^1 \int_0^1 G_1(t,s)G_2(s,\tau)a(\tau)d\tau ds \\ &= (L^{-1} - \varepsilon_2)\|u\|_2 A_1,\end{aligned} \tag{3.14}$$

$$\begin{aligned}\|(Tu)''\| &= \max_{0\le t\le 1} \lambda \int_0^1 G_2(t,s)a(s)f(s,u(s),u''(s))ds \\ &\le \lambda^{-1}(L^{-1} - \varepsilon_2)\|u\|_2 \lambda \max_{0\le t\le 1} \int_0^1 G_2(t,s)a(s)ds \\ &= (L^{-1} - \varepsilon_2)\|u\|_2 A_2.\end{aligned} \tag{3.15}$$

By (3.14) and (3.15), we know

$$\begin{aligned}\|Tu\|_2 &= \|Tu\| + \|(Tu)''\| \\ &\le (L^{-1} - \varepsilon_2)\|u\|_2(A_1 + A_2) \le \|u\|_2, \quad \text{for } u \in \partial K_R.\end{aligned} \tag{3.16}$$

On the other hand, since $0 < l^{-1} < f_0 \le +\infty$, there exists $r_0 > 0$ such that

$$f(t,x,y) \ge (f_0 - \varepsilon_2)(|x| + |y|), \quad \text{for } |x| + |y| \le r_0,\ \mu \le t \le 1 - \mu. \tag{3.17}$$

Let $r < r_0$, when $u \in K, \|u\|_2 = r$, by (3.17), for any $t \in [\mu, 1-\mu]$, we have

$$f(t,u(t),u''(t)) \ge (f_0 - \varepsilon_2)(|u(t)| + |u''(t)|) \ge (f_0 - \varepsilon_2)\omega\|u\|_2. \tag{3.18}$$

Therefore, for any $\mu \le t \le 1-\mu$ and $u \in \partial K_r$, by (1.6) and (3.18), we have

$$\begin{aligned}\big|(Tu)''(1/2)\big| &= \lambda \int_0^1 G_2(1/2,s)a(s)f(s,u(s),u''(s))ds \\ &\ge \lambda \int_\mu^{1-\mu} G_2(1/2,s)a(s)f(s,u(s),u''(s))ds \\ &\ge (f_0-\varepsilon_2)\omega\|u\|_2\lambda \int_\mu^{1-\mu} G_2(1/2,s)a(s)ds \\ &\ge (f_0-\varepsilon_2)\omega\|u\|_2\lambda \min_{\mu\le t\le 1-\mu}\int_\mu^{1-\mu} G_2(t,s)a(s)ds \\ &\ge (f_0-\varepsilon_2)\|u\|_2\lambda\, l \\ &\ge \|u\|_2.\end{aligned}$$

So,

$$\|Tu\|_2 = \|Tu\| + \|(Tu)''\| \ge |(Tu)''(1/2)| \ge \|u\|_2, \quad \text{for } u \in \partial K_r. \tag{3.19}$$

It follows from (3.16), (3.19) and Lemma 2.1 that the operator T has a fixed point $u_0 \in K$ satisfying $r \le \|u_0\|_2 \le R$ and $u_0(t) \ge 0, u_0''(t) \le 0$ for any $t \in [0,1]$, and hence u_0 is a positive solution of BVP (1.1) by the concavity of u_0 in $[0,1]$, i.e., u_0 satisfies BVP (1.1) and $u_0(t) > 0, t \in (0,1)$. The proof of Theorem 3.4 is completed. □

Remark 3.5. From $0 \le Lf^\infty < 1, lf_0 > 1$, we know that $1 \in (\frac{1}{lf_0}, \frac{1}{Lf^\infty})$. Therefore, Theorem 3.4 also holds for $\lambda = 1$.

Remark 3.6. From the proof of Theorem 3.4 we can know that $f(t,u,v)$ need not be sub-linear or sup-linear. In fact, the Theorem 3.4 contains one of the following cases:

(i) If $f^\infty = \infty, f_0 > 0$, $\lambda \in (0, \frac{1}{Lf^0})$;
(ii) If $f^\infty = \infty, f_0 = 0$, $\lambda \in (0, +\infty)$;
(iii) If $f^\infty > l^{-1} > 0, f_0 = 0$, $\lambda \in (\frac{1}{lf_\infty}, +\infty)$.

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