Nonlinear Functional Analysis and Applications
Vol. 13, No. 1 (2008), pp. 135-145

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

ON MICCHELLI COMBINATION OF SZÃSZ MIRAKIAN-DURRMEYER OPERATORS

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Abstract. It is shown that the technique of iterative combinations introduced by Micchelli [9] can be used to improve the rate of convergence by Szãsz-Durrmeyer Operators.

1. Introduction

Mazhar and Totik [8] and independently Kasana et al. [6] proposed the following integral modification of Szãsz-Mirakian operators to approximate Lebesgue integrable functions on $[0,\infty)$

$$M_n(f(u),t) = n\sum_{\nu=0}^{\infty} p_{n,\nu}(t)\int_0^{\infty} p_{n,\nu}(u)f(u)\,du,\ \ f\in L_1[0,\infty) \tag{1.1}$$

where $p_{n,\nu}(t) = \frac{e^{-nt}(nt)^{\nu}}{\nu!}$.

Alternatively, (1.1) may be written as

$$M_n(f(u),t) = \int_0^{\infty} W(n,t,u)f(u)\,du,$$

where $W(n,t,u) = n\sum_{\nu=0}^{\infty} p_{n,\nu}(t)\,p_{n,\nu}(u)$.

In order to improve the rate of convergence $O(n^{-1})$ by these operators, the technique of linear combinations introduced by May [7] and Rathore [10] has

[0] Received December 19, 2006. Revised April, 22. 2008.
[0] 2000 Mathematics Subject Classification: 41A36, 41A25.
[0] Keywords: Szãsz-Mirakian Durrmeyer operators, rate of convergence, iterative combinations.

been used (cf., e.g. [1]-[4]). There is yet another approach for improving the order of approximation, which was offered by Micchelli [9] by considering the iterative combinations $U_{n,k} = [I-(I-B_n)^k]$ of the Bernstein polynomials B_n. He proved some direct and saturation results for these operators $U_{n,k}$ using semi-group method.

In the present paper, we have considered the Micchelli combinations of the Szãsz-Mirakian Durrmeyer operators defined by (1.1) and proved some results concerning the degree of approximation.

We begin with the class Ω of all Lebesgue measurable functions on $[0,\infty)$ such that

$$\Omega[0,\infty) = \left\{ f : \int_0^\infty e^{-nt}|f(t)|\,dt < \infty \, \text{for some positive integer} \, n \right\}.$$

Obviously $\Omega[0,\infty) \supset L_1[0,\infty)$ and hence Szsãz Mirakian-Durrmeyer operators M_n may be utilized for studying the approximation of a bigger class of functions.

Let M_n^r the rth iterate (superposition) of the operator M_n, be a mapping from $\Omega[0,\infty)$ into $C^\infty(-\infty,\infty)$(the class of infinitely differentiable functions on the interval $(-\infty,\infty)$), then we define the operator
$T_{n,k} : \Omega[0,\infty) \to C^\infty(-\infty,\infty)$ as

$$\begin{aligned} T_{n,k}(f(u);t) &= \left(I-(I-M_n)^k\right)(f(u);t) \\ &= \sum_{r=1}^{k} (-1)^{r+1} \binom{k}{r} M_n^r(f(u);t). \end{aligned}$$

2. Preliminaries and auxiliary results

In the sequel, we shall require the following results.

Lemma 1. ([6]) *Let the function $\mu_{n,m}(t), m \in N^0$ (the set of all nonnegative integers) be defined by*

$$\mu_{n,m}(t) = M_n\left((u-t)^m;t\right) = \int_0^\infty W(n,t,u)(u-t)^m\,du.$$

Then, $\mu_{n,0}(t)=1$, $\mu_{n,1}(t)=\frac{1}{n}$, $\mu_{n,2}(t)=\frac{2}{n}(t+\frac{1}{n})$ and the following recurrence relation holds

$$\mu_{n,m+1}(t) = t\mu'_{n,m}(t) + (m+1)\mu_{n,m}(t) + 2mt\mu_{n,m-1}(t),\; m \in N. \tag{2.1}$$

From the recurrence relation (2.1) we have the following consequences:
(i) $\mu_{n,m}(t)$ is a polynomial in t of degree $[m/2]$ and in n^{-1} of degree m, where $[\alpha]$ denotes the integral part of α.
(ii) For every $t \in [0,\infty), \mu_{n,m}(t) = O\left(n^{-[(m+1)/2]}\right)$.

For every $m \in N^0$, the mth order moment $\mu_{n,m}^{[p]}(t)$ for the operator M_n^p is defined as

$$\mu_{n,m}^{[p]}(t) = M_n^p\left((u-t)^m; t\right).$$

From Lemma 1 it follows that $\mu_{n,m}^{[p]}(t)$ is a polynomial in t of degree $[m/2]$. We shall write $\mu_{n,m}(t)$ for $\mu_{n,m}^{[1]}(t)$.

Lemma 2. ([5]) *Let γ and δ be two positive numbers and $[a,b] \subset [0,\infty)$. Then, for any $m > 0$ there exists a constant K_m such that*

$$\left\| \int_{|u-t|\geq\delta} W(n,t,u)e^{\gamma u}\,du \right\|_{C[a,b]} \leq K_m n^{-m}.$$

Lemma 3. *There holds the recurrence relation*

$$\mu_{n,m}^{[p+1]}(t) = \sum_{j=0}^{m} \binom{m}{j} \sum_{i=0}^{[(m-j)/2]} \frac{1}{i!} D^i\left(\mu_{n,m-j}^{[p]}(t)\right)\mu_{n,i+j}(t),$$

where D denotes the operator d/dt, $p \in N$ and $m \in N^0$.

Proof. By definition, we have

$$\begin{aligned}
\mu_{n,m}^{[p+1]}(t) &= M_n(M_n^p\left((u-t)^m; x); t\right) \\
&= M_n(M_n^p\left((u-x+x-t)^m; x); t\right) \\
&= \sum_{j=0}^{m} \binom{m}{j} M_n\left((x-t)^j M_n^p\left((u-x)^{m-j}; x\right); t\right).
\end{aligned}$$

Since $M_n^p\left((u-x)^{m-j}; x\right)$ is a polynomial in x of degree $[(m-j)/2]$, by Taylor's expansion we may write it as

$$\sum_{i=0}^{[(m-j)/2]} \frac{(x-t)^i}{i!} D^i\left(\mu_{n,m-j}^{[r]}(t)\right),$$

which proves the lemma. □

Lemma 4. *For $p \in N, m \in N^0$ and every $t \in [0,\infty)$ we have*

$$\mu_{n,m}^{[p]}(t) = O\left(n^{-[(m+1)/2]}\right).$$

Proof. We shall prove this result by mathematical induction on p. For $p = 1$, it follows from Lemma 1 Therefore, assume it for a certain p. Then $\mu_{n,m-j}^{[p]}(t) = O\left(n^{-[(m-j+1)/2]}\right)$. Since $\mu_{n,m-j}^{[p]}(t)$ is a polynomial in t of degree $[(m-j)/2]$, we have
$D^i\left(\mu_{n,m-j}^{[p]}(t)\right) = O\left(n^{-[(m-j+1)/2]}\right)\ \forall\ i,\ 0 \leq i \leq [(m-j)/2]$.

Now, applying Lemma 2 we get

$$\begin{aligned}\mu_{n,m}^{[p+1]}(t) &= \sum_{j=0}^{m}\sum_{i=0}^{[(m-j)/2]} O\left(n^{-[(m-j+1)/2]+[(i+j+1)/2]}\right)\\ &= \sum_{j=0}^{m}\sum_{i=0}^{[(m-j)/2]} O\left(n^{-[(m+i+1)/2]}\right)\\ &= O\left(n^{-[(m+1)/2]}\right).\end{aligned}$$

□

Lemma 5. *For $k, l \in N$, and every $t \in [0,\infty)$ there holds*

$$T_{n,k}\left((u-t)^l;t\right) = O(n^{-k}).$$

Proof. For $k = 1$, the result follows from Lemma 1. Now, suppose that it is true for some k then we shall prove it for $k + 1$.

$$\begin{aligned}T_{n,k+1}\left((u-t)^l;t\right) &= \sum_{r=1}^{k+1}(-1)^{r+1}\binom{k+1}{r}\mu_{n,l}^{[r]}(t)\\ &= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\mu_{n,l}^{[r]}(t) + \sum_{r=1}^{k+1}(-1)^{r+1}\binom{k}{r-1}\mu_{n,l}^{[r]}(t)\\ &= I_1 + I_2, \text{say.}\end{aligned}$$

We may write I_1 as

$$I_1 = T_{n,k}\left((u-t)^l;t\right). \tag{2.2}$$

Next, by Lemma 3

$$\begin{aligned}I_2 &= \sum_{r=0}^{k}(-1)^{r+2}\binom{k}{r}\mu_{n,l}^{[r+1]}(t)\\ &= -\sum_{j=1}^{l-1}\binom{l}{j}\sum_{i=0}^{[(l-j)/2]}\frac{1}{i!}\left[D^iT_{n,k}\left((u-t)^{l-j};t\right)\right]\mu_{n,i+j}(t) \qquad (2.3)\\ &- T_{n,k}\left((u-t)^l;t\right) - \sum_{i=1}^{[l/2]}\frac{1}{i!}\left[D^iT_{n,k}\left((u-t)^l;t\right)\right]\mu_{n,i}(t)\end{aligned}$$

Thus, combining (2.2) and (2.3)

$$\begin{aligned} T_{n,k+1}\left((u-t)^l;t\right) &= -\sum_{j=1}^{l-1}\binom{l}{j}\sum_{i=0}^{[(l-j)/2]}\frac{1}{i!}\left[D^iT_{n,k}\left((u-t)^{l-j};t\right)\right]\mu_{n,i+j}(t) \\ &\quad -\sum_{i=1}^{[l/2]}\frac{1}{i!}\left[D^iT_{n,k}\left((u-t)^l;t\right)\right]\mu_{n,i}(t) \\ &= O(n^{-(k+1)}). \end{aligned}$$

This completes the proof. □

Lemma 6. ([6]) *There exist polynomials $q_{i,j,r}(t)$ independent of n and ν such that*

$$t^r\frac{d^r}{dt^r}\left(p_{n,\nu}(t)\right)=\sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i\,(\nu-nt)^j\,q_{i,j,r}(t)\,p_{n,\nu}(t).$$

3. Main results

First, we establish a Voronovskaja type asymptotic formula for the operators $T_{n,k}(.,t)$

Theorem 3.1. *Let $f\in\Omega[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$ and $f(t)=O(e^{\alpha t})$ as $t\to\infty$ for some $\alpha>0$. If $f^{(2k)}$ exists at a point $t\in[0,\infty)$ then*

$$\lim_{n\to\infty} n^k\left[T_{n,k}(f;t)-f(t)\right]=\sum_{\nu=2}^{2k}\frac{f^{(\nu)}(t)}{\nu!}Q(\nu,k,t) \tag{3.1}$$

and

$$\lim_{n\to\infty} n^k\left[T_{n,k+1}(f;t)-f(t)\right]=0, \tag{3.2}$$

where $Q(\nu,k,t)$ are certain polynomials in t of degree at most $[\nu/2]$. Further the limits in (3.1-3.2) hold uniformly in $t\in[0,a]$ if $f^{(2k)}\in C[0,b), 0<a<b$.

Proof. By Taylor's expansion of f, we have

$$f(u)=\sum_{\nu=0}^{2k}\frac{f^{(\nu)}(t)}{\nu!}(u-t)^\nu+\epsilon(u,t)(u-t)^{2k},$$

$\epsilon(u,t)\to 0$ as $u\to t$ and $\epsilon(u,t)=O(e^{\alpha u})$ as $u\to\infty$. To prove this, let

$$\epsilon(u,t)=\frac{f(u)-\sum_{\nu=0}^{2k}\frac{f^{(\nu)}(t)}{\nu!}(u-t)^\nu}{(u-t)^{2k}}$$

Then

$$\begin{aligned}
\lim_{u\to t}\epsilon(u,t) &= \lim_{u\to t}\frac{f(u)-\sum_{\nu=0}^{2k}\frac{f^{(\nu)}(t)}{\nu!}(u-t)^{\nu}}{(u-t)^{2k}}, \quad \left(\frac{0}{0}\text{ form}\right)\\
&= \lim_{u\to t}\frac{f^{(2k-1)}(u)-(f^{(2k-1)}(t)+(u-t)f^{(2k)}(t))}{2k!(u-t)}\\
&\quad \text{(applying L'Hospital's rule successively}(2k-1)\text{times)}\\
&= \frac{1}{2k!}\lim_{u\to t}\frac{f^{(2k-1)}(u)-f^{(2k-1)}(t)}{(u-t)}-\frac{f^{(2k)}(t)}{2k!}\\
&= \frac{f^{(2k)}(t)}{2k!}-\frac{f^{(2k)}(t)}{2k!}=0.
\end{aligned}$$

Since $M_n(u,t)=t+n^{-1}$, it follows that $M_n^r(u,t)=t+rn^{-1}$ for every $r\in N$. As $M_n^r(1,t)=1$ and M_n^r is a linear positive operator, we have $M_n^r((u-t),t)=rn^{-1}$. Consequently, $T_{n,k}((u-t),t)=n^{-1}\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}r=0$. By the Taylor's expansion of $f(u)$ about $u=t$, we have

$$\begin{aligned}
n^k\left[T_{n,k}(f;t)-f(t)\right] &= n^k\sum_{\nu=1}^{2k}\frac{f^{(\nu)}(t)}{\nu!}T_{n,k}\left((u-t)^{\nu};t\right)\\
&+ n^k\sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}M_n^r\left(\epsilon(u,t)(u-t)^{2k};t\right)\\
&= I_1+I_2,\text{say},
\end{aligned}$$

where $\epsilon(u,t)\to 0$ as $u\to t$ and $\epsilon(u,t)=O(e^{\alpha u})$ as $u\to\infty$.
Since $T_{n,k}(u;t)=t$ by Lemma 5

$$I_1=\sum_{\nu=2}^{2k}\frac{f^{(\nu)}(t)}{\nu!}Q(\nu,k,t)+o(1),$$

where $Q(\nu,k,t)$ is the coefficient of n^{-k} in $T_{n,k}\left((u-t)^{\nu};t\right)$.
Hence, in order to prove (3.1) it is sufficient to show that $I_2\to 0$ as $n\to\infty$.
For a given $\epsilon>0$ there exists a $\delta>0$ such that $|\epsilon(u,t)|<\epsilon$ whenever $0<|u-t|<\delta$. For $|u-t|\geq\delta$, since f is bounded on every finite subinterval of $[0,\infty)$, we have $|\epsilon(u,t)|\leq Me^{\alpha u}$ for some positive constant M.

Let $\chi_\delta(u)$ be the characteristic function of the interval $(t-\delta, t+\delta)$, then

$$\begin{aligned}
|I_2| &\leq n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r \left(|\epsilon(u,t)|(u-t)^{2k}\chi_\delta(u); t \right) \\
&+ n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r \left(|\epsilon(u,t)|(u-t)^{2k}(1-\chi_\delta(u)); t \right) \\
&= I_3 + I_4, \text{say}.
\end{aligned}$$

Using Lemma 4, we get

$$I_3 \leq \epsilon\, n^k \left[\sum_{r=1}^{k} \binom{k}{r} \right] \max_{1\leq r\leq k} M_n^r \left((u-t)^{2k}; t \right) = \epsilon\ O(1).$$

Next, applying Cauchy Schwarz inequality and Lemma 2, for an arbitrary $s>0$, we have

$$\begin{aligned}
I_4 &\leq n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r \left(M e^{\alpha u}(u-t)^{2k}(1-\chi_\delta(u)); t \right) \\
&\leq K n^{-s}.
\end{aligned}$$

Thus, $I_4 = o(1)$ and therefore in view of the arbitrariness of $\epsilon > 0$. we have $|I_2| = o(1)$, as $n \to \infty$.
The assertion (3.2) follows similarly due to the fact that
$T_{n,k+1}((u-t)^l, t) = O(n^{-(k+1)}), l \in N$.
The uniformity assertion follows due to the uniform continuity of $f^{(2k)}$ on $[0,a]$ (enabling δ to become independent of $t \in [0,a]$ and the uniformness of $o(1)$ term occurring in the estimate of I_1 (because, in fact it is a polynomial in n^{-1} and t). □

In the next result we obtain an estimate of the degree of approximation by $T_{n,k}$ for smooth functions.

Theorem 3.2. *Let $f \in \Omega[0,\infty)$ be bounded on every finite subinterval of $[0,\infty)$ and $f(t) = O(e^{\alpha t})$ as $t \to \infty$ for some $\alpha > 0$. If $f^{(p)}$ exists and is continuous on an interval $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$ then for all n sufficiently large there holds*

$$\|T_{n,k}(f) - f\| \leq max\left\{ C_1 n^{-p/2}\omega\left(f^{(p)}; n^{-1/2}\right), C_2 n^{-k} \right\},$$

where $C_1 = C_1(k,p)$,$C_2 = C_2(k,p,f)$, $\omega\left(f^{(p)};\delta\right)$ denotes the modulus of continuity of $f^{(p)}$ on $(a-\eta, b+\eta)$ and $\|.\|$ denotes the sup-norm on $[a,b]$.

Proof. If $u \in (a-\eta, b+\eta)$ and $t \in [a,b]$, we have

$$f(u) = \sum_{i=0}^{p} \frac{f^{(i)}(t)}{i!}(u-t)^i + \frac{\left(f^{(p)}(\xi) - f^{(p)}(t)\right)}{p!}(u-t)^p,$$

where ξ lies between u and t. Hence we can write

$$f(u) = \sum_{i=0}^{p} \frac{f^{(i)}(t)}{i!}(u-t)^i + \frac{\left(f^{(p)}(\xi) - f^{(p)}(t)\right)}{p!}(u-t)^p \chi(u) + F(u,t)(1-\chi(u)), \tag{3.3}$$

where $\chi(u)$ denotes the characteristic function of $(a-\eta, b+\eta)$ and

$$F(u,t) = f(u) - \sum_{i=0}^{p} \frac{f^{(i)}(t)}{i!}(u-t)^i,$$

for all $u \in [0,\infty)$ and $t \in [a,b]$.
Now, operating by $T_{n,k}(;t)$ on (3.3) and breaking the right hand side into three parts I_1, I_2 and I_3, say, corresponding to the three terms on the right hand side of (3.3), by Lemma 5 we have $I_1 = f(t) + O(n^{-k})$ uniformly in $t \in [a,b]$. Next, applying Schwarz inequality and Lemma 4 we get

$$\begin{aligned} |I_2| &\leq \sum_{r=1}^{k} \binom{k}{r} M_n^r \left(\frac{|f^{(p)}(\xi) - f^{(p)}(t)|}{p!} |u-t|^p \chi(u); t \right) \\ &\leq \frac{\omega\left(f^{(p)}; n^{-1/2}\right)}{p!} \sum_{r=1}^{k} \binom{k}{r} M_n^r \left((|u-t|^p + n^{1/2}|u-t|^{p+1}); t \right) \\ &= \omega\left(f^{(p)}; n^{-1/2}\right) O(n^{-p/2}), \end{aligned}$$

uniformly in $t \in [a,b]$.
The function $F(u,t)$ for $t \in [a,b]$ is bounded by $Me^{\alpha u}$ for some constant $M > 0$ hence using Lemma 2 we have $I_3 = o(n^{-s})$ uniformly in $[a,b]$, for any $s > 0$. Choosing $s > k$, we obtain $I_3 = o(n^{-k})$ uniformly in $t \in [a,b]$.
Combining the estimates of I_1, I_2 and I_3, the required result follows. □

Finally in the following theorem we show that the derivative $T_{n,k}^{(p)} f$ is an approximation process for $f^{(p)}, p = 1, 2, 3, ...$

Theorem 3.3. *Let* $f \in \Omega[0,\infty)$ *and be bounded on every finite subinterval of* $[0,\infty)$ *admitting a derivative of order* p *at a fixed point* $t \in (0,\infty)$. *Let* $f(t) = O(e^{\alpha t})$ *as* $t \to \infty$ *for some* $\alpha > 0$, *then we have*

$$\lim_{n\to\infty} T_{n,k}^{(p)}(f;t) = f^{(p)}(t). \tag{3.4}$$

Further, if $f^{(p)}$ *exists and is continuous on* $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$ *then (3.4) holds uniformly in* $t \in [a,b]$.

Proof. To prove the theorem, it suffices to show that for each $r \in N$

$$\lim_{n\to\infty} D^p \left(M_n^r(f;t)\right) = f^{(p)}(t),$$

and that it holds uniformly in the uniformity case. By the hypothesis, we have

$$f(u) = \sum_{i=0}^{p} \frac{f^{(i)}(t)}{i!}(u-t)^i + \epsilon(u,t)(u-t)^p, \tag{3.5}$$

where $\epsilon(u,t) \to 0$ as $u \to t$ and $\epsilon(u,t) = O(e^{\alpha u})$ as $u \to \infty$.
We can write

$$\begin{aligned} M_n^r(f(u);t) &= M_n(M_n^{r-1}(f(u);v);t) \\ &= \int_0^\infty W(n,t,v) M_n^{r-1}(f(u);v)\,dv. \end{aligned}$$

Hence, using (3.5) we get

$$\begin{aligned} \frac{d^p}{dt^p} M_n^r(f(u);t) &= \int_0^\infty W^{(p)}(n,t,v) M_n^{r-1}(f(u);v)\,dv \\ &= \sum_{i=0}^{p} \frac{f^{(i)}(t)}{i!} \int_0^\infty W^{(p)}(n,t,v) M_n^{r-1}((u-t)^i;v)\,dv \\ &\quad + \int_0^\infty W^{(p)}(n,t,v) M_n^{r-1}(\epsilon(u,t)(u-t)^p;v)\,dv \\ &= I_1 + I_2, \text{ say.} \end{aligned}$$

Let us estimate I_1 first.

$$I_1 = \sum_{i=0}^{p} \frac{f^{(i)}(t)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-t)^{i-j} \int_0^\infty W^{(p)}(n,t,v) M_n^{r-1}(u^j;v)\,dv.$$

Since $\displaystyle\int_0^\infty W^{(p)}(n,t,v) M_n^{r-1}(u^j;v)\,dv = \frac{d^p}{dt^p} M_n^r(u^j;t)$, we have

$$I_1 = \sum_{i=0}^{p} \frac{f^{(i)}(t)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-t)^{i-j} \frac{d^p}{dt^p} M_n^r(u^j;t).$$

By Lemma 1, it follows that $M_n(u^j;t)$ is a polynomial in t of degree j and the coefficient of t^j is 1. Consequently, $M_n^r(u^j;t), r \in N$ is also a polynomial in t of degree j and the coefficient of t^j is 1. Hence, as long as $0 \le j \le p-1$, $\frac{d^p}{dt^p}M_n^r(u^j;t) = 0$. Thus

$$\begin{aligned} I_1 &= \frac{f^{(p)}(t)}{p!}\frac{d^p}{dt^p}M_n^r(u^p;t) \\ &= \frac{f^{(p)}(t)}{p!}(p!) = f^{(p)}(t). \end{aligned}$$

To estimate I_2, $\epsilon(u,t) \to 0$ as $u \to t$ implies that for a given $\epsilon > 0$ there exists a $\delta > 0$ such that $|\epsilon(u,t)| < \epsilon$ whenever $0 < |u-t| < \delta$ and for $|u-t| \ge \delta$, $|\epsilon(u,t)||u-t|^p < Me^{\alpha u}$ for some $M > 0$.
Hence, using Lemma 6

$$\begin{aligned} I_2 &\le C_1 \sum_{\substack{2i+j\le p\\ i,j\ge 0}} n^{i+1} \sum_{\nu=0}^{\infty} |\nu - nt|^j \left\{ \epsilon \int_0^\infty p_{n,\nu}(v) M_n^{r-1}\left(|u-t|^p \chi_\delta(u); v\right) dv \right. \\ &\quad \left. + \int_0^\infty p_{n,\nu}(v) M_n^{r-1}\left(Me^{\alpha u}(1-\chi_\delta(u)); v\right) dv \right\} \\ &= I_3 + I_4, \text{ say,} \end{aligned}$$

where, $C_1 = \sup\limits_{\substack{2i+j\le p\\ i,j\ge 0}} \frac{|q_{i,j,p}(t)|}{|t|^p}$, M is a constant independent of u and $\chi_\delta(u)$ is the characteristic function of $(t-\delta, t+\delta)$.
Applying Cauchy Schwarz inequality three times, we get

$$\begin{aligned} I_3 &\le \epsilon C_1 \sum_{\substack{2i+j\le p\\ i,j\ge 0}} n^i \left(\sum_{\nu=0}^{\infty} p_{n,\nu}(t)(\nu - nt)^{2j} \right)^{1/2} \times \\ &\quad \times \left[n \sum_{\nu=0}^{\infty} p_{n,\nu}(t) \left(\int_0^\infty p_{n,\nu}(v) M_n^{r-1}\left((u-t)^{2p}; v\right) d\nu \right) \right]^{1/2}, \end{aligned}$$

in view of $\int_0^\infty p_{n,\nu}(u)\,du = n^{-1}$.
It is known [6] that for each $t \in [0,\infty)$ and $m \in N^0$,

$$\sum_{\nu=0}^{\infty} p_{n,\nu}(t)\left(\frac{\nu}{n} - t\right)^m = O\left(n^{-[(m+1)/2]}\right). \tag{3.6}$$

Consequently, using Lemma 4 we obtain

$$I_3 = \epsilon C_1 \sum_{\substack{2i+j\le p\\ i,j\ge 0}} n^i O\left(n^{j/2}\right) O\left(n^{-p/2}\right) = \epsilon O(1).$$

Now, again applying Cauchy Schwarz inequality, Lemma 2 and (3.6) it follows that
$I_4 = O\left(n^{-(p-m)/2}\right)$ for any $m > 0$. Choosing $m > p$, we get $I_4 = o(1)$ and therefore in view of the arbitrariness of $\epsilon > 0$, we have $I_2 = o(1)$.
Combining the estimates of I_1 and I_2, we obtain (3.4). The second assertion follows as in the proof of Theorem 3.1. □

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