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A RESULT ON THE EXISTENCE OF COMMON FIXED POINTS OF SOME NONCOMMUTING MAPPINGS WITH AN APPLICATION

M. Abbas[1] and Safeer Hussain Khan[2]

[1]Centre for Advanced Studies in Mathematics, Lahore University of Management Sciences, 54792-Lahore, Pakistan
e-mail: `mujahid@lums.edu.pk`

[2]Department of Mathematics and Physics, Qatar University, Doha 2713, State of Qatar
e-mail: `safeerhussain5@yahoo.com;safeer@qu.edu.qa`

Abstract. Necessary conditions for existence of common fixed points for uniformly R-subweakly commuting mappings in the context of a metrizable topological vector space are obtained. As an application, related results on best approximation are derived. Our results generalize various known results in the literature.

1. Introduction and preliminaries

For the sake of convenience, we gather some basic definitions and set out our terminology needed in the sequel. Throughout this paper $\mathbb{N}$ will denote the set of all positive integers.

Definition 1.1. Let (E, τ) be a topological vector space (TVS). We assume that the topology τ is generated by an F-norm q which has the properties given below

(i) $q(x) \geq 0$ and $q(x) = 0$ if and only if $x = 0$ $(x \in E)$.
(ii) $q(x + y) \leq q(x) + q(y)$ for all $x \in E$
(iii) $q(\lambda x) \leq q(x)$ for all (real or complex) scalar λ with $|\lambda| \leq 1$.
(iv) If $q(x_n) \longrightarrow 0$, then $q(\lambda x_n) \longrightarrow 0$ for all scalars λ.
(v) If $\lambda_n \longrightarrow 0$, then $q(\lambda_n x) \longrightarrow 0$ for all $x \in E$.

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The relation $d(x,y) = q(x-y)$ defines a metric on E. A family Q of F-norms on a vector space E defines a metrizable topology on E. Conversely, the topology of any metrizable topological vector space is determined by the family of τ-continuous F-norms.
Let $u \in C$, the set C is called u-starshaped or starshaped with respect to u if $tx + (1-t)u \in C$ for each $x \in C$. Note that C is convex if C is starshaped with respect to every $u \in C$. Let C be a u-starshaped subset of X and $f, g : C \to C$. Put,

$$C_u^{fx} = \{y_\lambda : y_\lambda = \lambda fx + (1-\lambda)u,\ \lambda \in [0,1]\}.$$

A point $x \in X$ is called a fixed point of f if $f(x) = x$. We denote the set of fixed points of f by $Fix(f)$.

Definition 1.2. Let (X, τ) be a metrizable topological vector space and C be a u-starshaped subset of X, f and g be self mappings on C and $u \in Fix(g)$, then f is said to be:

(1) a *g-contraction* if there exists $k \in (0,1)$ such that

$$q(fx - fy) \leq kq(gx - gy).$$

(2) an *asymptotically g-nonexpansive* if there exists a sequence $\{k_n\}$, $k_n \geq 1$, with $\lim_{n\to\infty} k_n = 1$ such that $q(f^n x - f^n y) \leq k_n q(gx - gy)$ for each x, y in C and each $n \in \mathbb{N}$. If $k_n = 1$, for all $n \in \mathbb{N}$, then f is known as a g-nonexpansive mapping. If $g = I$ (the identity map), then f is asymptotically nonexpansive mapping.

(3) a *uniformly asymptotically regular* on C if for each $\epsilon > 0$, there exists a positive integer N such that $q(f^n x - f^{n+1} x) < \epsilon$ for all $n \geq N$ and for all $x \in C$.

Definition 1.3. Let (X, τ) be a metrizable topological vector space and C be a u-starshaped subset of X, and $u \in Fix(g)$, then a pair $\{f, g\}$ of self maps on C is said to be:

(1) *commuting on C* if $fgx = gfx$ for all $x \in C$.

(2) R-weakly *commuting* on C if there exists a real number $R > 0$ such that

$$q(fgx - gfx) \leq Rq(fx - gx).$$

for all x in C.

(3) R-subweakly *commuting on C* if there exists a real number $R > 0$ such that for each $x \in C$

$$q(fgx - gfx) \leq R \inf_{\lambda \in [0,1]} q(gx - y_\lambda)$$

where $y_\lambda \in C_u^{fx}$.

(4) a *Uniformly* R-subweakly *commuting on* $C - \{u\}$ if there exists a real number $R > 0$ such that for each $x \in C - \{u\}$ and $n \in \mathbb{N}$

$$d(f^n gx - gf^n x) \leq R \inf_{\lambda \in [0,1]} q(gx - y_\lambda)$$

where $y_\lambda \in C_u^{f^n x}$.

Definition 1.4. Let X be a metrizable topological vector space, M any closed subset of X and $u \in X$. If there exists a $y_0 \in C$ such that

$$q(u - y_0) = \inf_{y \in M} q(u - y),$$

then y_0 is called a *best approximation to u out of M*. We denote by $P_M(u)$, the set of all best approximation to u out of M.

Sessa [11] coined the term weakly commuting maps. Jungck [6] generalized the notion of weak commutativity by introducing compatible maps and then weakly compatible maps ([8]). There are examples that show each of these generalizations of the commutativity is a proper extension of the previous definition. Also during this time a number of authors established fixed point theorems for pairs of maps (See for example, [1], [3], [4], [5], and references therein). It is well known that uniformly R-subweakly commuting map on $C - \{u\}$ is R-subweakly commuting on $C - \{u\}$. For detailed discussion on above mentioned notions and their implications, we refer to [1], [6], [7], [8] and [10] and references mentioned therein. Recently, Shahzad [10] introduced the class of noncommuting mappings called R-subweakly commuting mappings. This paper deals with the study of common fixed points for R-subweakly and uniformly R-subweakly commuting mapping which is general than R-subweakly commuting mapping, in the setting of a metrizable topological vector space. Applying fixed point theorems, some useful results have been obtained in approximation theory as well.

2. Common fixed point results

In this section, we first establish the existence of common fixed points of R-subweakly commuting and uniformly R-subweakly commuting mappings in the setting of metrizable topological vector space.

Theorem 2.1. *Let (X, τ) be a complete metrizable topological vector space and C be a nonempty closed subset of X, $f, g : C \to C$, $u \in Fix(g)$, and $f(C - \{u\}) \subseteq g(C - \{u\})$. Suppose f is g-contractive and continuous. If a pair $\{f, g\}$ is R-weakly commuting on $C - \{u\}$, then $Fix(f) \cap Fix(g)$ is a nonempty.*

Proof. Let $x_0 \in C - \{u\}$ and let x_1 be such that $gx_1 = fx_0$. In general, define a sequence $\{x_n\}$ in $C - \{u\}$ by $gx_n = fx_{n-1}$ for $n \in \mathbb{N}$. We can do this since $f(C - \{u\} \subseteq g(C - \{u\})$. Then,

$$q(gx_{n+1} - gx_n) = q(fx_n - fx_{n-1}) \leq kq(gx_n - gx_{n-1})$$

for some $0 < k < 1$. This shows that $\{gx_n\}$ and, in turn, $\{fx_n\}$ is a Cauchy sequence in $C - \{u\}$. Thus, $fx_n \to y$ and consequently $gx_n \to y$. Now continuity of f implies $f^2x_n \to fy$ and $fgx_n \to fy$. Since the pair $\{f, g\}$ is R-weakly commuting on $C - \{u\}$, therefore

$$\begin{aligned} q(gfx_n - fy) &\leq q(gfx_n - fgx_n) + q(fgx_n - fy) \\ &\leq Rq(fx_n - gx_n) + q(fgx_n - fy). \end{aligned}$$

Taking limit as $n \to \infty$, we have $gfx_n \to fy$. Also since f is a g-contraction, therefore

$$q(f^2x_n - fx_n) \leq kq(gfx_n - gx_n).$$

Again letting $n \to \infty$, we arrive at the following inequality

$$q(y - fy) \leq kq(y - fy)$$

which implies $y = fy$. As, the pair $\{f, g\}$ is R-weakly commuting on $C - \{u\}$, so $y \in C - \{u\}$. Now there exists a z in $C - \{u\}$ such that $y = gz$.Next, we show that $fz = gz$. Using the hypothesis, we obtain

$$q(fz - fx_n) \leq kq(gz - gx_n).$$

As $n \to \infty$,this yields

$$q(fz - gz) \leq kq(gz - gz)$$

which implies that $fz = gz$. Since the pair $\{f, g\}$ is R-weakly commuting on $C - \{u\}$ and $fz = gz$, therefore the pair $\{f, g\}$ commutes at z and hence $y = fy = fgz = gfz = gy$. The uniqueness follows from contraction condition. □

Corollary 2.2. *Let (X, τ) be a complete metrizable topological vector space and C be a nonempty closed subset of X. If $f : C \to C$ is a contraction map, then f has a fixed point.*

Theorem 2.1, above extends Lemma 2.3 of [2] to case of metrizable topological vector spaces. In following theorem, we obtain common fixed point of class of uniformly R-subweakly commuting mappings which contains R-subweakly commuting maps as a subclass.

Theorem 2.3. *Let C be a non empty closed subset of complete metrizable topological vector space (X, τ), $f, g : C \to C$ are continuous, $g(C) = C$, $u \in Fix(g)$, and $f(C - \{u\}) \subseteq g(C - \{u\})$. Suppose that f^n is a g-contraction with sequence of contractivity factors $\{k_n\}$, g is linear and f is uniformly asymptotically regular. If the pair $\{f, g\}$ is uniformly R-subweakly commuting* *on* $C-\{u\}$, C *is starshaped with respect to* u *and* $\overline{f(C-\{u\}}$ *is compact, then* $Fix(f)\cap Fix(g)$ *is nonempty.*

Proof. For each $n\in\mathbb{N}$, define a mapping f_n on C by $f_nx=\alpha_nf^nx+(1-\alpha_n)u$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ with $\lim_{n\to\infty}\alpha_n=1$. Since C is starshaped with respect to u, $f(C-\{u\})\subseteq g(C-\{u\})$, and $g(C)=C$, therefore f_n is a self mapping on C such that $f_n(C-\{u\})\subseteq g(C-\{u\})$ for each $n\in\mathbb{N}$. Consider,

$$q(f_nx-f_ny)\leq q(f^nx-f^ny)\leq k_nq(x-y).$$

Also,

$$\begin{aligned}d(f_ngx-gf_nx) &= q(\alpha_nf^ngx-\alpha_ngf^nx)\\ &\leq q(f^ngx-gf^nx)\\ &\leq R\inf_{\lambda\in[0,1]}q(gx-y_\lambda)\end{aligned}$$

where $y_\lambda\in C_u^{f^nx}$. This implies

$$d(f_ngx-gf_nx)\leq Rq(gx-f_nx).$$

This further implies that the pair $\{f_n,g\}$ is R-weakly commuting for each n. Theorem 2.1 assures the existence of x_n in C such that x_n is a common fixed point of g and f_n for each $n\geq 1$. Thus for each $n\geq 1$, we have

$$gx_n=x_n=\alpha_nf^nx_n+(1-\alpha_n)u$$

and

$$x_n-f^nx_n=(\alpha_n-1)(f^nx_n-u).$$

Since $f(C-\{u\})$ is bounded which implies $q(x_n-f^nx_n)\to 0$, as $n\to\infty$. As, g is continuous, linear and f is uniformly asymptotically regular, so

$$\begin{aligned}&q(x_n-fx_n)\\ \leq\ & q(x_n-f^nx_n)+q(f^nx_n-f^{n+1}x_n)+q(f^{n+1}x_n-fx_n)\\ \leq\ & q(x_n-f^nx_n)+q(f^nx_n-f^{n+1}x_n)+k_1q(gf^nx_n-gx_n)\\ \leq\ & q(x_n-f^nx_n)+q(f^nx_n-f^{n+1}x_n)+k_1q(g(f^nx_n-x_n)).\end{aligned}$$

Let $n\to\infty$ to obtain $q(x_n-fx_n)\to 0$. Since $\overline{f(C-\{u\}}$ is compact, there exists a subsequence $\{x_{n_j}\}$ such that $x_{n_j}\to x_0\in C$ as $j\to\infty$. By continuity of f, we obtain $x_0\in Fix(f)$. Since $f(C-\{u\})\subset g(C-\{u\})$, it follows that $x_0=fx_0=gy$ for some y in C. Also,

$$q(fx_{n_j}-fy)\leq k_1q(gx_{n_j}-gy)=k_1q(x_{n_j}-x_0).$$

Taking the limit as $j\to\infty$, we obtain $fx_0=fy=gy=x_0$. Also,

$$q(fx_0-gx_0)=q(fgy-gfy)\leq Rq(fy-gy).$$

Hence, we have $fx_0=gx_0$.This completes the proof. □

We have a question whether Theorem 2.3 is extendable to the case when f is asymptotically g-nonexpansive mapping in the set up of metrizable topological vector spaces.

Meinardus [9] was the first to employ a fixed point theorem to prove the existence of an invariant approximation in Banach spaces. Subsequently, several interesting and valuable results appeared in the literature of approximation theory ([1], [10] and [12]). As an application of Theorem 2.3, we obtain the following result on best approximation as a fixed point of uniformly R-subweakly commuting mappings.

Theorem 2.4. *Let M be a nonempty subset of a complete metrizable topological vector space (X,τ), $f,g : X \to X$ continuous mappings, and $u \in Fix(f) \cap Fix(g)$. Suppose that f^n is g-contraction with sequence of contractivity factors $\{k_n\}$, g is linear, f is uniformly asymptotically regular on $P_M(u)$ and $g(P_M(u) - \{p\} = P_M(u) - \{p\}$, $p \in Fix(g)$. Moreover, assume that $P_M(u)$ is closed and starshaped with respect to p and $\overline{f(P_M(u))}$ is compact. If a pair $\{f,g\}$ is uniformly R-subweakly commuting on $P_M(u)$ satisfying $q(x - fa) < q(x - ga)$ for each $a \in P_M(u) - \{p\}$ and $x \in X$. Then $Fix(f) \cap Fix(g) \cap P_M(u)$ is nonempty.*

Proof. Let $a \in P_M(u) - \{p\}$ which implies $g(b) \in P_M(u) - \{p\}$.Now, we prove that $P_M(u) - \{p\}$ is f-invariant. If not, then there exists b in $P_M(u) - \{p\}$ such that $f(b) \notin P_M(u) - \{p\}$. Therefore,

$$\begin{aligned} q(u-b) &= \inf_{c \in M} q(u-c) \leq q(u - fb) \\ &< q(u - gb) = q(u-b), \end{aligned}$$

a contradiction. Thus $P_M(u) - \{p\}$ is f-invariant. Also, $P_M(u)$ being a closed subset of a complete metrizable topological vector space is complete. Now by Theorem 2.3, f and g have a common fixed point in $P_M(u)$. □

Theorem 2.5. *Let M be a nonempty subset of a complete metrizable topological vector space (X,τ), $f,g : X \to X$, and $u \in Fix(f) \cap Fix(g)$. Suppose that f is g-contraction on $P_M(u)$ and continuous. Also, assume that $g(P_M(u) - \{p\}) = P_M(u) - \{p\}$, $p \in Fix(g)$ and $P_M(u)$ is closed. If the pair $\{f,g\}$ is R-weakly commuting on $P_M(u) - \{p\}$ satisfying $q(x - fa) < q(x - ga)$ for each $a \in P_M(u) - \{p\}$ and $x \in X$. Then $Fix(f) \cap Fix(g) \cap P_M(u)$ is nonempty.*

Proof. Using the similar arguments as those in Theorem 2.4 and taking $C = P_M(u)$ in Theorem 2.1, the result follows. □

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