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NONTRIVIAL SOLUTIONS OF SINGULAR STURM-LIOUVILLE PROBLEM WITH BOUNDARY CONDITIONS INVOLVING RIEMANN-STIELTJES INTEGRALS

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Abstract. This paper studies the following singular Sturm-Liouville boundary value problem with integral boundary conditions

$$\begin{cases} -(Lu)(t) = a(t)f(t,u(t)), \quad 0<t<1, \\ \alpha u(0) - \beta u'(0) = \displaystyle\int_0^1 u(s)d\xi(s), \\ \gamma u(1) + \delta u'(1) = \displaystyle\int_0^1 u(s)d\eta(s), \end{cases}$$

where $a(t)$ is allowed to be singular at $t = 0, 1$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a sign-changing continuous function and may be unbounded from below. By applying the topological degree of a completely continuous field and the first eigenvalue and its corresponding eigenfunction of a special linear operator, some existence results of nontrivial solutions are obtained.

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1. Introduction

In this paper, we consider the existence of nontrivial solutions for the following singular second-order Sturm-Liouville boundary value problem (BVP, for short) with integral boundary conditions

$$\begin{cases} -(Lu)(t) = a(t)f(t,u(t)), \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = \displaystyle\int_0^1 u(s)d\xi(s), \\ \gamma u(1) + \delta u'(1) = \displaystyle\int_0^1 u(s)d\eta(s), \end{cases} \tag{1.1}$$

where $(Lu)(t) = (p(t)u'(t))' + q(t)u(t)$, $\alpha, \beta, \gamma, \delta \in \mathbb{R}^+$ are constants such that $\beta\gamma + \alpha\gamma + \alpha\delta > 0$, $\xi(s), \eta(s)$ are nondecreasing functions of bounded variation, and the integrals in (1.1) are Riemann-Stieltjes integrals, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous sign-changing function and f may be unbounded from below for $t \in [0,1]$, $x \in \mathbb{R}$. Moreover, $a : (0,1) \to \mathbb{R}^+$ is continuous and allowed to be singular at $t = 0, 1$, in which $\mathbb{R} = (-\infty, +\infty), \mathbb{R}^+ = [0, +\infty), \mathbb{R}_0^+ = (0, +\infty)$.

The form of (1.1) for differential equations arise from many fields of applied mathematics and physics, and can describe a great deal of nonlinear problems. If $p \equiv 1, q \equiv 0$, $\int_0^1 u(s)d\xi(s) = \int_0^1 u(s)d\eta(s) = 0$, BVP (1.1) reduces to the two-point BVP

$$\begin{cases} -u''(t) = f(t,u(t)), \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0. \end{cases} \tag{1.2}$$

In the case where f is nonnegative, (1.2) has been intensively studied, see [1, 2, 3, 4, 5]. In [6], Zhao studied the property of the positive solutions for the following Sturm-Liouville singular boundary value problems

$$\begin{cases} -(Lu)(t) = f(t,u(t)), \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \end{cases} \tag{1.3}$$

where $f(t,u) : (0,1) \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$, may be singular at $t = 0$, $t = 1$ and/or $u = 0$. The author obtained a relation between the solutions and Green's function. Recently second-order boundary value problems with nonlocal boundary conditions, including multi-point and integral boundary conditions, have received a great deal of attention, and many excellent results are obtained, we refer the readers to [7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and references therein.

But all above work only considered the case of the nonlinearity taking on nonnegative values. As to the nonlinearity f is sign-changed, we refer to [18, 19, 20, 21, 22]. In [21], Sun and Zhang considered the singular nonlinear

Sturm-Liouville problems

$$\begin{cases} -(Lu)(t) = h(t)f(u(t)), \quad 0 < t < 1, \\ R_1(u) = \alpha_1 u(0) + \beta_1 u'(0) = 0, \quad R_2(u) = \alpha_2 u(1) + \beta_2 u'(1) = 0, \end{cases} \tag{1.4}$$

where h is allowed to be singular at $t = 0$ and $t = 1$. Besides, the main condition they assumed that there exists $b > 0$ such that $f(t,u) \geq -b$, i.e., f is bounded from below and is not necessary to be nonnegative. By means of the topological degree theory, the authors established the existence of nontrivial solutions and positive solutions of the problem (1.4). Han and Wu [22] improved the condition of the nonlinear term f in [21], that is, $f(u) \geq -b - c|u|^\kappa$, here $b > 0, c > 0, \kappa \in (0,1)$. Obviously, f is allowed to be unbounded from below, however, f under control by special function $F(u) = -b - c|u|^\kappa$, and the Green function of the boundary value problem is symmetric.

Motivated by [18, 19, 20, 21, 22], the purpose of this paper is to consider the existence of nontrivial solutions of BVP (1.1) under some weaker conditions. The new features of this paper mainly include the following aspects. Firstly, comparing with [20, 21, 22], the nonlinear term f of BVP (1.1) is allowed to be sign-changing and unbounded from below with respect to $t \in [0,1]$, $x \in \mathbb{R}$. Secondly, comparing with [21, 22], the Green function of BVP (1.1) is not necessarily symmetric. Thirdly, comparing with [20, 21, 22], we discuss the boundary value problem with integral boundary conditions, i.e., BVP (1.1) including second-order two-point, three-point, multi-point and nonlocal boundary value problems as special cases. To our knowledge, there are not many references to studied the existence of nontrivial solutions for second-order differential equation with boundary conditions involving Riemann-Stieltjes integrals except [18]. Finally, without making any monotone-type assumption, we established the existence of one nontrivial solution of the BVP (1.1) by using the topological degree of a completely continuous field, the first eigenvalue and its corresponding eigenfunction of a special linear operator.

The remaining part of this paper is organized as follows. Some preliminaries and a number of lemmas useful to the derivation of the main results are given in Section 2, then the proofs of the theorems are given in Section 3, followed by an example in Section 4 to demonstrate the validity of our main results.

2. Preliminaries and some lemmas

In this section, we present some preliminaries and lemmas that are useful to the proof of our main results.

Let $E = C[0,1]$ be a Banach space with the maximum norm

$$\|u\| = \max_{0 \leq t \leq 1} |u(t)|$$

for $u \in E$, E^* be the dual space of E. Define $P = \{u \in E \mid u(t) \geq 0, t \in [0,1]\}$ and $B_r = \{u \in E \mid \|u\| < r\}$. Then P is a total cone in E, that is, $E = \overline{P - P}$. Let P^* be the dual cone of P, namely, $P^* = \{g \in E^* \mid g(u) \geq 0, \text{ for all } u \in P\}$.

For the sake of convenience, we first give the following assumptions:

($\mathbf{H}_1$) $p(t) \in C^1[0,1], p(t) > 0, q(t) \in C[0,1], q(t) \leq 0,\ \alpha, \beta, \gamma, \delta \in \mathbb{R}^+, \alpha\gamma + \alpha\delta + \beta\gamma > 0$, and the homogeneous equation with respect to (1.3),

$$\begin{cases} -(Lu)(t) = 0, \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \end{cases} \tag{2.1}$$

has only the trivial solution.

($\mathbf{H}_2$) $a : (0,1) \to \mathbb{R}^+$ is continuous, $a(t) \not\equiv 0$ and

$$\int_0^1 a(s)ds < +\infty.$$

Let $K(t,s)$ be the Green's function with respect to (2.1), i.e.,

$$K(t,s) = \frac{1}{\omega} \begin{cases} \varphi_1(t)\varphi_2(s), & 0 \leq t \leq s \leq 1, \\ \varphi_1(s)\varphi_2(t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.2}$$

Lemma 2.1 ([23, 24])**.** *Assume that* (H_1) *is satisfied, then the Green's function* $K(t,s)$ *defined by* (2.2) *possesses the following properties:*

(i) $K(t,s)$ *is continuous and symmetrical over* $[0,1] \times [0,1]$;
(ii) $K(t,s) \geq 0$, *and* $K(t,s) \leq K(s,s), \forall\ t,s \in [0,1]$;
(iii) $\varphi_1(t) \in C^2[0,1]$ *is an increasing function,* $\varphi_1(t) > 0, t \in (0,1]$;
(iv) $\varphi_2(t) \in C^2[0,1]$ *is an decreasing function,* $\varphi_1(t) > 0, t \in [0,1)$;
(v) $(L\varphi_1)(t) \equiv 0, \varphi_1(0) = \beta, \varphi_1'(0) = \alpha$;
(vi) $(L\varphi_2)(t) \equiv 0, \varphi_2(1) = \delta, \varphi_2'(1) = -\gamma$;
(vii) ω *is a positive constant.*

According to (2.1), (2.2), it is easy to verify that BVP (1.1) is equivalent to the perturbed integral equation

$$u(t) = \int_0^1 K(t,s)a(s)f(s,u(s))ds + \phi(t)\int_0^1 u(s)d\xi(s) + \psi(t)\int_0^1 u(s)d\eta(s) \tag{2.3}$$

where $\phi(t) \in C^2([0,1], \mathbb{R}^+)$ and $\psi(t) \in C^2([0,1], \mathbb{R}^+)$ solve the following inhomogeneous boundary value problems:

$$\begin{cases} -(Lu)(t) = 0, \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 1,\ \gamma u(1) + \delta u'(1) = 0, \end{cases}$$

and

$$\begin{cases} -(Lu)(t) = 0, \quad 0 < t < 1, \\ \alpha u(0) - \beta u'(0) = 0, \ \gamma u(1) + \delta u'(1) = 1. \end{cases}$$

Remark 2.2. *It follows from Lemma* 2.1 *we know,* $\varphi_1(t), \varphi_2(t)$ *are two linearly independent solutions of homogeneous differential equation* $-(Lu)(t) = 0$, *then* $\phi(t), \psi(t)$ *can be linear representation by* $\varphi_1(t), \varphi_2(t)$, *respectively. By direct calculation, we have* $\phi(t) = \frac{p(0)}{\omega}\varphi_2(t)$ *and* $\psi(t) = \frac{p(1)}{\omega}\varphi_1(t)$. *So* $\psi(t), \phi(t)$ *have the similar properties of* $\varphi_1(t), \varphi_2(t)$, *respectively, i.e.,* ψ *is nonnegative and increasing on* $[0, 1]$, ϕ *is nonnegative and decreasing on* $[0, 1]$.

Set

$$k_1 = 1 - \int_0^1 \phi(t) d\xi(t), \quad k_2 = \int_0^1 \psi(t) d\xi(t),$$

$$k_3 = \int_0^1 \phi(t) d\eta(t), \quad k_4 = 1 - \int_0^1 \psi(t) d\eta(t).$$

We also need the following assumptions concerning f and $k_i (i = 1, 2, 3, 4)$:

($\mathbf{H}_3$) $k_1 > 0, k_4 > 0$, $k = k_1 k_4 - k_2 k_3 > 0$.

($\mathbf{H}_4$) $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous, there exist nonnegative functions $b, c \in C[0, 1]$ with $c(t) \not\equiv 0$ and one nondecreasing continuous function $h : \mathbb{R} \to \mathbb{R}^+$ satisfying $\lim\limits_{x\to+\infty} \frac{h(x)}{x} = 0$, such that

$$f(t, u) \geq -b(t) - c(t)h(u), \quad \forall\ u \in \mathbb{R}.$$

Lemma 2.3. *Assume that* $(H_1) - (H_4)$ *hold. Then the integral equation* (2.3) *is equivalent to*

$$u(t) = \int_0^1 G(t, s) a(s) f(s, u(s)) ds, \tag{2.4}$$

where $G(t, s)$ *is the Green function for* (1.1), *taking the form*

$$\begin{aligned} G(t, s) =& K(t, s) + \frac{k_4\phi(t) + k_3\psi(t)}{k} \int_0^1 K(\tau, s) d\xi(\tau) \\ &+ \frac{k_2\phi(t) + k_1\psi(t)}{k} \int_0^1 K(\tau, s) d\eta(\tau). \end{aligned} \tag{2.5}$$

Proof. The proof is similar to Lemma 2.3 of [18], so we omit it. $\square$

Remark 2.4. *If* (H_1) *and* (H_3) *hold, then for any* $t, s \in [0,1]$ *it is easy to testify that*

$$0 \le G(t,s) \le \overline{M} K(s,s), \tag{2.6}$$

where

$$\overline{M} = 1 + \frac{k_4\phi(0) + k_3\psi(1)}{k} \int_0^1 d\xi(\tau) + \frac{k_2\phi(0) + k_1\psi(1)}{k} \int_0^1 d\eta(\tau).$$

Define operators F, J, T and $A : E \to E$ as follows:

$$(Fu)(t) = f(t, u(t)), \quad t \in [0,1], \ u \in E,$$

$$(Ju)(t) = \int_0^1 G(s,t)a(s)u(s)ds, \quad t \in [0,1], \tag{2.7}$$

$$(Tu)(t) = \int_0^1 G(t,s)a(s)u(s)ds, \quad t \in [0,1], \tag{2.8}$$

$$(Au)(t) = (TFu)(t) = \int_0^1 G(t,s)a(s)f(s,u(s))ds, \quad t \in [0,1]. \tag{2.9}$$

Obviously, if u is a fixed point of A, then u is a solution of BVP (1.1) by Lemma 2.3.

Lemma 2.5. *Assume that* $(H_1)-(H_3)$ *hold. Then linear operators* $J, T : E \to E$*, defined by* (2.7) *and* (2.8) *respectively, are completely continuous positive linear operators.*

Proof. By Lemma 2.1, there exists $L > 0$ such that $K(t,s) \le L, \forall\ (t,s) \in [0,1] \times [0,1]$. From (H_2) we know $\int_0^1 K(s,s)a(s)ds \le L \int_0^1 a(s)ds < +\infty$. Then by (2.5), (2.6) and the monotonicity of ϕ, ψ, for any $t \in [0,1]$ we have

$$|(Ju)(t)| \le \|u\| \Big[\int_0^1 K(s,s)a(s)ds + L(\overline{M} - 1) \int_0^1 a(s)ds \Big] < +\infty, \tag{2.10}$$

$$|(Tu)(t)| \le \int_0^1 \overline{M} K(s,s)a(s)u(s)ds \le \overline{M} \|u\| \int_0^1 K(s,s)a(s)ds < +\infty. \tag{2.11}$$

Therefore $J, T : E \to E$ are well defined. By (2.6), we have $J(P) \subset P, T(P) \subset P$. Thus, J and T are positive linear operators. Next we will show that J and

T are completely continuous. For any natural number n ($n \geq 2$), let

$$a_n(t) = \begin{cases} \inf_{t<s\leq \frac{1}{n}} a(s), & 0 \leq t \leq \dfrac{1}{n}, \\ a(t), & \dfrac{1}{n} \leq t \leq \dfrac{n-1}{n}, \\ \inf_{\frac{n-1}{n}\leq s<t} a(s), & \dfrac{n-1}{n} \leq t \leq 1. \end{cases} \tag{2.12}$$

Then $a_n : [0,1] \to [0,+\infty)$ is continuous and $a_n(t) \leq a(t), t \in (0,1)$. Let

$$(T_n u)(t) = \int_0^1 G(t,s)a_n(s)u(s)ds, \quad t \in [0,1]. \tag{2.13}$$

It is clearly that $T_n : E \to E$ is completely continuous. For any $r > 0$ and $u \in B_r$, by (2.12), (2.13) and the absolute continuity of integral, we have

$$\begin{aligned} \lim_{n\to\infty} \|T_n u - Tu\| = &\lim_{n\to\infty} \max_{t\in[0,1]} \left| \int_0^1 G(t,s)\big(a(s) - a_n(s)\big)u(s)ds \right| \\ \leq &\overline{M}\|u\| \lim_{n\to\infty} \int_0^1 K(s,s)\big(a(s) - a_n(s)\big)ds \\ \leq &r\overline{M} \lim_{n\to\infty} \int_{e(n)} K(s,s)\big(a(s) - a_n(s)\big)ds \\ \leq &r\overline{M} \lim_{n\to\infty} \int_{e(n)} K(s,s)a(s)ds = 0, \end{aligned}$$

where $e(n) = [0, \frac{1}{n}] \cup [\frac{n-1}{n}, 1]$. Then by the approximating theorem of completely continuous operators, $T : E \to E$ is completely continuous. Similarly, we can prove that $J : E \to E$ is completely continuous. □

Lemma 2.6. *Assume that* $(H_1) - (H_3)$ *hold. Then the special radius* $r(T) \neq 0$, $r(J) \neq 0$, T *and* J *have positive eigenfunctions corresponding to their first eigenvalues* $r_1 = (r(T))^{-1}$ *and* $\lambda_1 = (r(J))^{-1}$ *respectively.*

Proof. By (H_2), there is $t_1 \in (0,1)$ such that $G(t_1,t_1)a(t_1) > 0$. Thus there exists $[\alpha,\beta] \subset (0,1)$ such that $t_1 \in (\alpha,\beta)$ and $G(t,s)a(s) > 0$, for $t,s \in [\alpha,\beta]$. Take $\widetilde{u} \in P$ such that $\widetilde{u}(t_1) > 0$ and $\widetilde{u}(t) = 0, t \notin [\alpha,\beta]$. Then for $t \in [\alpha,\beta]$, we have

$$(T\widetilde{u})(t) = \int_0^1 G(t,s)a(s)\widetilde{u}(s)ds \geq \int_\alpha^\beta G(t,s)a(s)\widetilde{u}(s)ds > 0.$$

So there exists a constant $c > 0$ such that $c(T\widetilde{u})(t) \geq \widetilde{u}(t), t \in [0,1]$. According to the Krein-Rutman theorem, we know that the special radius $r(T) \neq 0$. Thus, corresponding to $r_1 = (r(T))^{-1}$, the first eigenvalues of T, T has a positive eigenfunctions $\phi_1(t)$, i.e.,

$$r_1 T\phi_1 = \phi_1. \tag{2.14}$$

Similarly, it is easy to prove that the spectral radius $r(J) > 0$ and there exists $\phi_2 \in P, \phi_2(t) > 0, t \in (0,1)$ such that

$$\lambda_1 J\phi_2 = \phi_2, \tag{2.15}$$

where $\lambda_1 = \frac{1}{r(J)}$ is the first eigenvalues of the operator J. $\square$

Let T^* be the dual operator of T. If there exists $g \in P^* \setminus \{\theta\}$ such that

$$\lambda_1 T^* g = g. \tag{2.16}$$

Choose a real number $\delta_0 > 0$ and let

$$P(g, \delta_0) = \{u \in P \mid g(u) \geq \delta_0 \|u\|\}, \tag{2.17}$$

then it is easy to see that $P(g, \delta_0)$ is a cone in E.

Lemma 2.7. *Suppose that the following conditions are satisfied.*

(C_1) *There exist $\phi_1 \in P \setminus \{\theta\}$, $g \in P^* \setminus \{\theta\}$ and $\delta_0 > 0$ such that* (2.14), (2.16), (2.17) *hold and T maps P into $P(g, \delta_0)$,*

(C_2) *$H : E \to P$ is a continuous operator and satisfies that*

$$\lim_{\|u\| \to +\infty} \frac{\|Hu\|}{\|u\|} = 0,$$

(C_3) *$F : E \to E$ is a bounded continuous operator and there exists $u_0 \in E$ such that $Fu + u_0 + Hu \in P$, $\forall\ u \in E$,*

(C_4) *There exists $v_0 \in E$ and $\sigma > 0$ such that*

$$TFu \geq \lambda_1(1+\sigma)Tu - THu - v_0, \quad \forall\ u \in E. \tag{2.18}$$

Let $A = TF$. Then there exists $R > 0$ such that

$$\deg(I - A, B_R, \theta) = 0,$$

where $B_R = \{u \in E \mid \|u\| < R\}$.

Proof. We shall show that

$$u \neq TFu + \mu\varphi_1, \qquad \forall\ u \in \partial B_R. \tag{2.19}$$

provided that R is sufficiently large.

In fact, if (2.19) is not true, then there exist $u_1 \in \partial B_R$ and $\mu_1 \geq 0$ satisfying

$$u_1 = TFu_1 + \mu_1\phi_1. \tag{2.20}$$

By (2.16), (2.18) and (2.20), we have

$$\begin{aligned} g(u_1) =& g(TFu_1) + \mu_1 g(\phi_1) \geq g(TFu_1) \\ \geq& \lambda_1(1+\sigma)g(Tu_1) - g(THu_1) - g(v_0) \\ =& \lambda_1(1+\sigma)(T^*g)(u_1) - (T^*g)(Hu_1) - g(v_0) \\ =& (1+\sigma)g(u_1) - \lambda_1^{-1} g(Hu_1) - g(v_0). \end{aligned}$$

Thus

$$g(u_1) \leq (\sigma\lambda_1)^{-1} g(Hu_1) + \sigma^{-1} g(v_0). \tag{2.21}$$

It follows from (C_2), (2.14) and (2.21) that

$$\begin{aligned} g(u_1 + THu_1 + Tu_0) =& g(u_1) + (gT)(Hu_1) + (gT)(u_0) \\ =& g(u_1) + \lambda_1^{-1} g(Hu_1) + \lambda_1^{-1} g(u_0) \\ \leq& (\sigma\lambda_1)^{-1} g(Hu_1) + \lambda_1^{-1} g(Hu_1) + \lambda_1^{-1} g(u_0) + \sigma^{-1} g(v_0) \\ \leq& (1+\sigma^{-1})\lambda_1^{-1} \|g\| \cdot \|Hu_1\| + \lambda_1^{-1} g(u_0) + \sigma^{-1} g(v_0) \\ =& l_1 \|Hu_1\| + l_2, \end{aligned} \tag{2.22}$$

where $l_1 = (1+\sigma^{-1})\lambda_1^{-1}\|g\|$, $l_2 = \lambda_1^{-1} g(u_0) + \sigma^{-1} g(v_0)$ are two constant.

Then (C_1) implies $\mu_1\phi_1 = \mu_1\lambda_1 T\phi_1 \in P(g, \delta_0)$. It follows from (C_3) we know that $Fu_1 + u_0 + Hu_1 \in P$. By (C_1) and (2.20) we have

$$u_1 + THu_1 + Tu_0 = T(Fu_1 + Hu_1 + u_0) + \mu_1\phi_1 \in P(g, \delta_0).$$

By the definition of $P(g, \delta_0)$, we have

$$g(u_1+THu_1+Tu_0) \geq \delta_0\|u_1+THu_1+Tu_0\| \geq \delta_0\|u_1\| - \delta_0\|THu_1\| - \delta_0\|Tu_0\|. \tag{2.23}$$

From (2.22) and (2.23), we have

$$\begin{aligned} R = \|u_1\| \leq& \delta_0^{-1} g(u_1 + THu_1 + Tu_0) + \|THu_1\| + \|Tu_0\| \\ \leq& \delta_0^{-1} l_1 \|Hu_1\| + \delta_0^{-1} l_2 + \|T\| \cdot \|Hu_1\| + \|Tu_0\| \\ =& L_1\|Hu_1\| + L_2, \end{aligned}$$

where $L_1 = \delta_0^{-1} l_1 + \|T\|$, $L_2 = \delta_0^{-1} l_2 + \|Tu_0\|$. So

$$1 \leq L_1 \frac{\|Hu_1\|}{\|u_1\|} + \frac{L_2}{\|u_1\|}. \tag{2.24}$$

It follows (C_2) that (2.24) cannot hold as $R \to +\infty$. Therefore, (2.19) holds provided that R is sufficiently large. By virtue of the property of omitting a direction for Leray-Schauder degree, we have $\deg(I - A, B_R, \theta) = 0$. □

3. Main Results

Theorem 3.1. *Suppose that the conditions* $(H_1)-(H_4)$ *are satisfied. If*

$$\liminf_{u\to+\infty}\min_{t\in[0,1]}\frac{f(t,u)}{u}>\lambda_1, \tag{3.1}$$

$$\limsup_{u\to 0}\max_{t\in[0,1]}\left|\frac{f(t,u)}{u}\right|<\lambda_1, \tag{3.2}$$

where λ_1 *is the first eigenvalue of the operator* J *defined by* (2.7)*. Then BVP* (1.1) *has at least one nontrivial solution.*

Proof. First we give some properties of $\phi_2(t)$ which is the positive eigenfunction of J corresponding to its first eigenvalue λ_1, that is, there exist $\delta_2\geq\delta_1>0$ such that

$$\delta_1 K(s,s)\leq\phi_2(s)\leq\delta_2 K(s,s),\qquad \forall\, s\in[0,1]. \tag{3.3}$$

We will show that (3.3) holds according to the following four different cases of boundary condition, respectively.

Cases (i). If $\beta=\delta=0$, then by Lemma 2.1 and (2.5) we have $G(s,0)=G(s,1), s\in[0,1]$. It follows from (2.7) and (2.15) that $\phi_2(0)=\phi_2(1)=0$, which implies that $\phi_2'(0)>0, \phi_2'(1)<0$. We can define

$$\Phi_1(s)=\begin{cases}\phi_2'(0), & s=0,\\ \dfrac{\phi_2(s)}{K(s,s)}, & 0<s<1,\\ -\phi_2'(1), & s=1.\end{cases}$$

Cases (ii). If $\beta=0,\delta>0$, then by Lemma 2.1 and (2.5) we have

$$\int_0^1 G(s,1)a(s)u(s)ds>0,\ G(s,0)=0.$$

It follows from (2.7) that $\phi_2(0)=0, \phi_2(1)>0$, which implies that $\phi_2'(0)>0$. We can define

$$\Phi_2(s)=\begin{cases}\phi_2'(0), & s=0,\\ \dfrac{\phi_2(s)}{K(s,s)}, & 0<s\leq 1.\end{cases}$$

Cases (iii). If $\beta>0,\delta=0$, then by Lemma 2.1 and (2.5) we have $\int_0^1 G(s,0)a(s)u(s)ds>0$, $G(s,1)=0$. It follows from (2.7) that $\phi_2(0)>0, \phi_2(1)=0$, which implies that $\phi_2'(1)<0$. We can define

$$\Phi_3(s)=\begin{cases}\dfrac{\phi_2(s)}{K(s,s)}, & 0\leq s<1,\\ -\phi_2'(1), & s=1.\end{cases}$$

Cases (iv). If $\beta > 0, \delta > 0$, then by Lemma 2.1 and (2.5) we have

$$\int_0^1 G(s,0)a(s)u(s)ds > 0, \int_0^1 G(s,1)a(s)u(s)ds > 0.$$

We can define

$$\Phi_4(s) = \frac{\phi_2(s)}{K(s,s)}, \qquad 0 \le s \le 1.$$

It is easy to see that $\Phi_i (i = 1, 2, 3, 4)$ are continuous on $[0, 1]$ and $\Phi_i(s) > 0$ for all $s \in [0, 1]$. So, there exist $\delta_{i1}, \delta_{i2} > 0$ such that $\delta_{i1} \le \Phi_i(s) \le \delta_{i2}$ for all $s \in [0, 1], i = 1, 2, 3, 4$. Let

$$\delta_1 = \min\Big\{\delta_{11}, \delta_{21}, \delta_{31}, \delta_{41}\Big\}, \qquad \delta_2 = \max\Big\{\delta_{12}, \delta_{22}, \delta_{32}, \delta_{42}\Big\}.$$

So, (3.3) holds. Setting

$$g(u) = \int_0^1 a(t)\phi_2(t)u(t)dt, \quad \forall\ u \in E, \tag{3.4}$$

from (H_2) and (3.3), we have

$$\int_0^1 a(t)\phi_2(t)u(t)dt \le \delta_2\|u\| \int_0^1 K(t,t)a(t)dt < +\infty,$$

which shows that g is well defined on E. In the following we shall show that

$$\lambda_1 T^* g = g. \tag{3.5}$$

In fact, by (3.4), (2.15), for any $t, s \in [0, 1]$, we have

$$\begin{aligned}
\lambda_1^{-1} g(u) &= \int_0^1 a(t)\left(\lambda_1^{-1}\phi_2(t)\right)u(t)dt = \int_0^1 a(t)\big(J\phi_2\big)(t)u(t)dt \\
&= \int_0^1 a(t)u(t)\int_0^1 G(s,t)a(s)\phi_2(s)dsdt \\
&= \int_0^1 a(s)\phi_2(s)\int_0^1 G(s,t)h(t)u(t)dtds \\
&= \int_0^1 a(s)\phi_2(s)\big(Tu\big)(s)ds = g(Tu) = \big(T^*g\big)(u), \quad \forall\ u \in E.
\end{aligned} \tag{3.6}$$

So, (3.5) holds. Take $\delta_0 > 0$ in (2.17) such that $\delta_1 = \delta_0\lambda_1\overline{M}$. Next we prove that $T(P) \subset P(g, \delta_0)$. From (3.3) we can obtain

$$\phi_2(s) \ge \delta_0\lambda_1\overline{M}K(s,s). \tag{3.7}$$

For any $u \in P$, by (3.6), (3.7) and (2.11), we have

$$
\begin{aligned}
g(Tu) &= \lambda_1^{-1} g(u) = \lambda_1^{-1} \int_0^1 a(s)\phi_2(s)u(s)ds \\
&\geq \delta_0 \overline{M} \int_0^1 K(s,s)a(s)u(s)ds \geq \delta_0 (Tu)(t), \quad \forall\, t \in [0,1].
\end{aligned}
\tag{3.8}
$$

Hence, $g(Tu) \geq \delta_0 \|Tu\|$, i.e. $T(P) \subset P(g, \delta_0)$. Therefore, T satisfies condition (C_1) of Lemma 2.7.

Let

$$(Hu)(t) = c_0 h(u(t)),\ \forall\, u \in E,$$

where $c_0 = \max_{t\in[0,1]} c(t)$. It follows from (H_4) we know that $H : E \to P$ is continuous. By the monotonicity of h, we have

$$h(u(t)) \leq h(\|u\|), \quad \|Hu\| \leq c_0 h(\|u\|), \quad \forall\, u \in E.$$

So,

$$\lim_{\|u\|\to+\infty} \frac{\|Hu\|}{\|u\|} \leq \lim_{\|u\|\to+\infty} \frac{c_0 h(\|u\|)}{\|u\|} \leq \lim_{x\to+\infty} \frac{c_0 h(x)}{x} = 0,$$

i.e., $\lim\limits_{\|u\|\to+\infty} \frac{\|Hu\|}{\|u\|} = 0$. Therefore H satisfies condition (C_2) in Lemma 2.7.

Take $u_0 \equiv b(t)$, then it follows from (H_4) that

$$Fu + u_0 + Hu \in P, \quad \forall\, u \in E,$$

namely condition (C_3) in Lemma 2.7 holds. From (3.1), there exists $\varepsilon_0 > 0$ such that

$$Fu = f(t,u) \geq \lambda_1 (1+\varepsilon_0) u, \quad \forall\, t \in [0,1], \tag{3.9}$$

as $u > 0$ sufficiently large. Combining $(H_4)$and (3.9), there exists $b_0 \geq 0$ such that

$$Fu \geq \lambda_1 (1+\varepsilon_0) u - b_0 - Hu, \quad \forall\, u \in E. \tag{3.10}$$

Since T is a positive linear operator, by (3.10) we have

$$\big(TFu\big)(t) \geq \lambda_1 (1+\varepsilon_0) \big(Tu\big)(t) - Tb_0 - \big(THu\big)(t), \quad \forall\, t \in [0,1]. \tag{3.11}$$

So condition (C_4) in Lemma 2.7 is satisfied. According to Lemma 2.7, there exists a sufficiently large number $R > 0$ such that

$$\deg(I - A, B_R, \theta) = 0. \tag{3.12}$$

It follows from (3.2) that there exist $0 < \varepsilon_1 < 1$ and $0 < r < R$ such that for any $u \in E$ with $\|u\| \leq r$, we have

$$\big|f(t,u(t))\big| \leq (1-\varepsilon_1)\lambda_1 |u(t)|, \qquad \forall\, t \in [0,1]. \tag{3.13}$$

Next we will prove that

$$u \neq \mu A u, \quad \text{ for all } u \in \partial B_r \text{ and } \mu \in [0,1]. \tag{3.14}$$

If there exist $u_1 \in \partial B_r$ and $\mu_1 \in [0,1]$ such that $u_1 = \mu_1 A u_1$, then by (3.6) and (3.13), we have

$$\begin{aligned} g\big(|u_1|\big) =& g\big(|\mu_1 A u_1|\big) = \mu_1 g\big(|Au_1|\big) \le g\big(|TFu_1)|\big) \\ =& g\left(\left| \int_0^1 G(t,s)a(s)f(s,u_1(s))ds \right|\right) \\ \le& (1-\varepsilon_1)\lambda_1 g\left(\int_0^1 G(t,s)a(s)|u_1(s)|ds\right) \\ =& (1-\varepsilon_1)\lambda_1 g\big(T(|u_1(t)|)\big) \\ =& (1-\varepsilon_1)\lambda_1(\lambda_1)^{-1} g\big(|u_1(t)|\big) \\ =& (1-\varepsilon_1) g\big(|u_1(t)|\big). \end{aligned}$$

Therefore, $g\big(|u_1(t)|\big) \le 0$.

On the other hand, $\phi_2(t) > 0$ for all $t \in (0,1)$ by the maximum principle and $u_1(t)$ attains zero on isolated points by the Sturm theorem. Hence

$$g\big(|u_1(t)|\big) = \int_0^1 h(t)\phi_2(t)\,|u_1(t)|\,dt > 0, \quad t \in [0,1].$$

This is a contradiction. Thus (3.14) holds.

It follows from (3.14) and the homotopy invariance of Leray-Shauder degree that

$$\deg(I - A, B_r, \theta) = 1. \tag{3.15}$$

By (3.12), (3.15) and the additivity of Leray-Shauder degree, we obtain

$$\deg(I - A, B_R \setminus \overline{B}_r, \theta) = \deg(I - A, B_R, \theta) - \deg(I - A, B_r, \theta) = -1.$$

As a result, A has at least one fixed point on $B_R \setminus \overline{B}_r$, namely the singular BVP(1.1) has at least one nontrivial solution. □

Remark 3.2. *The Green function of BVP* (1.1) *is not necessarily symmetrical. In order to overcome the difficulties caused by the non-symmetry, we seek one special linear operator J and use its first eigenvalue and its corresponding eigenfunction to construct a linear continuous functional g of P^*, then establish a cone to solve our problem.*

Corollary 3.3. *Using the following condition* (H_4') *instead of* (H_4)*, the conclusion of Theorem* 3.1 *remains true.*

(H_4') *$f(t,u)$ is continuous on $[0,1]\times\mathbb{R}$ and there exist constants $b > 0, c > 0$ and $\kappa \in (0,1)$ such that*

$$f(t,u) \ge -b - c|u|^\kappa, \quad \forall\ t \in [0,1], u \in \mathbb{R}.$$

Corollary 3.4. *Using the following condition* (H_4'') *instead of* (H_4)*, the conclusion of Theorem* 3.1 *remains true.*

(H_4'') $f(t,u)$ *is continuous on* $[0,1]\times\mathbb{R}$ *and there exists a nonnegative continuous function* $b:[0,1]\to\mathbb{R}^+$ *such that*

$$f(t,u)\geq -b(t),\quad \forall\ t\in[0,1], u\in\mathbb{R}.$$

Corollary 3.5. *Suppose that the conditions* $(H_1)-(H_3)$ *are satisfied. If*

(H_4''') $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ *is continuous, there exist nonnegative functions* $b_1, c_1\in C[0,1]$ *with* $c_1(t)\not\equiv 0$ *and one nonincreasing continuous function* $h_1:\mathbb{R}\to\mathbb{R}^+$ *satisfying* $\lim\limits_{x\to-\infty}\frac{h_1(x)}{x}=0$*, such that*

$$f(t,u)\leq b_1(t)+c_1(t)h_1(u),\ t\in[0,1],\ u\in\mathbb{R},$$

$$\liminf_{u\to-\infty}\frac{f(t,u)}{u}>\lambda_1,\quad \textit{uniformly on } t\in[0,1],$$

and (3.2) *hold, where* λ_1 *is the first eigenvalue of the operator* J *defined by* (2.7)*. Then BVP* (1.1) *has at least one nontrivial solution.*

Proof. Denote $f_1(t,u)=-f(t,-u),\ t\in[0,1],\ u\in\mathbb{R}$ and define

$$(A_1u)(t)=\int_0^1 G(t,s)a(s)f_1(s,u(s))ds,\quad t\in[0,1].$$

It is easy to verify that the conditions of Theorem 3.1 are satisfied in which the function f replaced by f_1. By Theorem 3.1 we know that A_1 has at least one nontrivial fixed point $\widehat{u}$, i.e., $A_1\widehat{u}=\widehat{u}$. Since $f_1(s,\widehat{u}(s))=-f(s,-\widehat{u}(s)),\ s\in[0,1]$, thus

$$-\widehat{u}(t)=-(A_1\widehat{u})(t)=\int_0^1 G(t,s)a(s)f(s,-\widehat{u}(s))ds=(A(-\widehat{u}))(t),\ t\in[0,1].$$

So, $-\widehat{u}$ is the nontrivial solution of singular BVP (1.1). □

4. An example

In this section, we construct an example to demonstrate the application of our main result obtained in section 3.

Example 4.1. *Consider the following second-order four-point boundary value problem*

$$\begin{cases} -u'' = a(t)f(t,u(t)), \quad 0<t<1, \\ u(0)-u'(0) = \dfrac{1}{4}u\Big(\dfrac{1}{3}\Big) + \dfrac{1}{9}u\Big(\dfrac{2}{3}\Big), \\ u(1)+u'(1) = \dfrac{3}{8}u\Big(\dfrac{1}{3}\Big) + u\Big(\dfrac{2}{3}\Big), \end{cases} \tag{4.1}$$

with $a(t) = \frac{1}{\sqrt{t(1-t)}}$ *and*

$$f(t,u) = \begin{cases} \displaystyle\sum_{i=1}^{n}(-1)^i a_i - (3+2t)\sqrt{|u|}\ln(|u|+2011) & \\ \qquad + (3+2t)\ln 2012, & u\in(-\infty,-1], \\ \displaystyle\sum_{i=1}^{n} a_i u^i, & u\in[-1,+\infty), \end{cases}$$

where $0<a_1<\lambda_1$ *and* $a_n>0$*. Then the singular BVP* (4.1) *has at least one nontrivial solution.*

Proof. It is obvious that a is singular at $t=0,1$ and $\int_0^1 a(t)dt = \pi < +\infty$, f is a continuous sign-changing function and unbounded from below. BVP (4.1) can be regard as a boundary value problem of the form of (1.1). In this situation, $p(t)\equiv 1, q(t)\equiv 0$, $\alpha=\beta=\gamma=\delta=1$ and

$$\xi(s) = \begin{cases} 0, & s\in\Big[0,\dfrac{1}{3}\Big), \\ \dfrac{1}{4}, & s\in\Big[\dfrac{1}{3},\dfrac{2}{3}\Big), \\ \dfrac{13}{36}, & s\in\Big[\dfrac{2}{3},1\Big], \end{cases} \qquad \eta(s) = \begin{cases} 0, & s\in\Big[0,\dfrac{1}{3}\Big), \\ \dfrac{3}{8}, & s\in\Big[\dfrac{1}{3},\dfrac{2}{3}\Big), \\ \dfrac{11}{8}, & s\in\Big[\dfrac{2}{3},1\Big]. \end{cases}$$

Take $b(t) = \sum_{i=1}^{n} a_i + (3+2t)\ln 2012,\ c(t) = 3+2t,\ h(u) = \sqrt{|u|}\ln(|u|+2011)$. Then $h:\mathbb{R}\to\mathbb{R}^+$ is a continuous function and h is nondecreasing on $\mathbb{R}$ satisfying

$$\lim_{u\to+\infty}\frac{h(u)}{u} = 0.$$

Moreover,

$$f(t,u) \geq -b(t) - c(t)h(u), \quad t\in[0,1],\ u\in\mathbb{R}.$$

By calculations, we get

$$\phi(t) = \frac{2-t}{3}, \qquad \psi(t) = \frac{1+t}{3}, \quad t\in[0,1],$$

$$k_1 = \frac{263}{324},\ k_2 = \frac{14}{81},\ k_3 = \frac{47}{72},\ k_4 = \frac{5}{18},\ k \approx 0.112654321 > 0.$$

It is easy to prove that all the conditions of Theorem 3.1 are satisfied. Consequently, we infer that singular BVP (4.1) has at least one nontrivial solution. □

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