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APPLICATIONS OF THE KKM PRINCIPLE ON ABSTRACT CONVEX MINIMAL SPACES

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Abstract. We introduce a new concept of abstract convex minimal spaces which is used to establish typical results in the KKM theory. Since any minimal space can be made into a topological space, results on abstract convex minimal spaces can be deduced from the theory on abstract convex spaces. In this way, the KKM type theorems are used to obtain coincidence theorems, the Fan-Browder type fixed point theorems, the Fan intersection theorem, and the Nash equilibrium theorem on abstract convex minimal spaces.

1. Introduction

Many problems in nonlinear analysis can be solved by showing the nonemptyness of the intersection of certain family of subsets of an underlying set. One of the remarkable results on the nonempty intersection is the celebrated Knaster-Kuratowski-Mazurkiewicz theorem (simply, the KKM principle) in 1929 [9], which is concerned with certain types of multimaps called the KKM maps.

The KKM theory, first named by the author [11], is nowadays the study of applications of various equivalent formulations of the KKM principle and their generalizations. In the last fifteen years, the KKM theory is extended to generalized convex (G-convex) spaces in a sequence of papers of the author and his followers; for details, see [12-15,18-21] and references therein.

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In our previous paper [16], we introduced a new concept of abstract convex spaces which is adequate to establish the KKM theory. With this new concept, we generalized and simplified known results of the theory on convex spaces, H-spaces, G-convex spaces, and others. Moreover, the KKM type theorems were used there to obtain coincidence theorems and fixed point theorems.

Apparently motivated by the author's works, recently, Alimohammady et al. [1] introduced the notion of minimal G-convex spaces and obtained the open and closed versions of the KKM principle in this new setting. Their method is just replacing the topological structure in the relevant results by the more general minimal structure as in [2].

Our aim in the present paper is to introduce a new concept of abstract convex minimal spaces which is also useful to establish major results in the KKM theory. With this new concept, we obtain generalizations of the KKM principle. Since any minimal space can be made into a topological space, results on abstract convex minimal spaces can be deduced from the theory on abstract convex spaces. In this way, the KKM type theorems on abstract convex spaces in [16] are used to obtain coincidence theorems, the Fan-Browder type fixed point theorems, the Fan intersection theorem, and the Nash equilibrium theorem on abstract convex minimal spaces.

Section 2 is concerned with preliminaries on abstract convex spaces in [16] and on minimal spaces in [1,2]. In Section 3, we deduce various forms of the KKM principle for abstract convex minimal spaces. Section 4 deals with coincidence theorems and the Fan-Browder type fixed point theorems for abstract convex minimal spaces. Finally, in Section 5, applications to the Fan intersection theorem and the Nash equilibrium theorem are obtained. Our new results are just a few examples of the theory which generalize and unify known results in the literature.

2. Abstract convex spaces

In this section, we recall definitions and some basic results on abstract convex spaces given in [16] and some of their new consequences.

A *multimap* $F : X \multimap Y$ is a function from a set X into the power set $\mathcal{P}(Y)$ of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$ and the *fibers* $F^{-}(y) := \{x \in X \mid y \in F(x)\}$ for $y \in Y$. We use the term *map* instead of multimap. For $A \subset X$, let $F(A) := \bigcup\{F(x) \mid x \in A\}$. For any $B \subset Y$, the (*lower*) *inverse* of B under F is defined by

$$F^{-}(B) := \{x \in X \mid F(x) \cap B \neq \emptyset\}.$$

Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D.

Definitions. An *abstract convex space* $(E, D; \Gamma)$ consists of a nonempty set E, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap E$ with nonempty values. We may denote $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$.

For any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

[co is reserved for the convex hull in vector spaces.]

A subset X of E is called a Γ-*convex subset* of $(E, D; \Gamma)$ relative to D' if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$. Then $(X, D'; \Gamma|_{\langle D' \rangle})$ is called a Γ-*convex subspace* of $(E, D; \Gamma)$.

When $D \subset E$, the space is denoted by $(E \supset D; \Gamma)$. In such case, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X \cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X \cap D$. In case $E = D$, let $(E; \Gamma) := (E, E; \Gamma)$.

An abstract convex space with a topology on E is called *an abstract convex topological space.*

Examples. 1. Examples of abstract convex spaces were given in Section 5 of [16].

2. A *generalized convex space* or a G-*convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, Δ_n is the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$.

For details on G-convex spaces, see [12-15,18-21], where basic theory was developed and lots of examples of G-convex spaces were given.

Definitions. [1,2] A family $\mathcal{M} \subset \mathcal{P}(X)$ is called a *minimal structure* on a set X if $\emptyset, X \in \mathcal{M}$. In this case, $(X, \mathcal{M})$ is called a *minimal space.* Any element of $\mathcal{M}$ is called an *m-open set* of X and a complement of an m-open set is called an *m-closed set* of X. For minimal spaces $(X, \mathcal{M})$ and $(Y, \mathcal{N})$, a function $f : X \to Y$ is said to be *continuous* (more precisely, *m-continuous* or $(\mathcal{M}, \mathcal{N})$-*continuous*) if $f^{-1}(V) \in \mathcal{M}$ for each $V \in \mathcal{N}$.

From now on, an abstract convex space $(E, D; \Gamma)$ with a minimal structure on E will be called *an abstract convex minimal space.*

Examples. 1. Any topological space is a minimal space and not conversely. However, any minimal space can be made into a topological space; see Proposition 1 below.

2. Any t.v.s. is a minimal vector space. There is some linear minimal space which is not a t.v.s. [1].

3. A *generalized convex minimal space* or a *G-convex minimal space* $(X, D; \Gamma)$ consists of a minimal space X, a nonempty set D, and a map $\Gamma : \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$. See [1].

4. A G-convex space is a G-convex minimal space, and the converse does not hold; for an example, see [1].

5. A ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ consisting of a topological [resp., minimal] space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ for $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$ and an n-simplex Δ_n, is an abstract convex topological [resp., minimal] space by putting $\Gamma_A := \phi_A(\Delta_n)$; see [17].

It is obvious that basic facts on generalized convex spaces (e.g. in [12]) can be extended to corresponding ones on generalized convex minimal spaces. However, we have the following:

Proposition 1. (i) *A minimal space* $(X, \mathcal{M})$ *can be made into a topological space* $(X, \mathcal{T})$.

(ii) *A continuous map* $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ *between minimal spaces can be regarded as a continuous map between the corresponding topological spaces.*

Proof. (i) Any collection $\mathcal{M}$ of subsets of a set X is a subbasis for a topology $\mathcal{T}$ on X.

(ii) A map f is continuous iff the inverse image $f^{-1}(N)$ of each member N of the subbasis $\mathcal{N}$ for Y is $\mathcal{M}$-open in X. $\square$

Proposition 2. *A* ϕ_A*-space* $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ *with a minimal space* $(X, \mathcal{M})$ *can be regarded as the one with a topological space* $(X, \mathcal{T})$.

Consequently, a G-convex minimal space can be made into a G-convex space.

For abstract convex spaces, we can define KKM maps as in [16]:

Definitions. Let $(E, D; \Gamma)$ be an abstract convex space and Z a set. For a map $F : E \multimap Z$ with nonempty values, if a map $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all} \quad A \in \langle D \rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity function 1_E.

A map $F : E \multimap Z$ is said to have the *KKM property* and called a $\mathfrak{K}$*-map* if, for any KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E,Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{-map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-map is defined for closed-valued maps G, and a $\mathfrak{KO}$-map for open-valued maps G. In this case, we have

$$\mathfrak{K}(E,Z) \subset \mathfrak{KC}(E,Z) \cap \mathfrak{KO}(E,Z).$$

Note that if Z is discrete then three classes $\mathfrak{K}$, $\mathfrak{KC}$, and $\mathfrak{KO}$ are identical. Some authors [5] use the notation $\mathrm{KKM}(E,Z)$ instead of $\mathfrak{KC}(E,Z)$.

Further, when $(Z,\mathcal{M})$ is a minimal space, an $m\mathfrak{KC}$-map is defined for m-closed-valued maps G, and an $m\mathfrak{KO}$-map for m-open-valued maps G. In this case, we have

$$\mathfrak{K}(E,Z) \subset m\mathfrak{KC}(E,Z) \cap m\mathfrak{KO}(E,Z).$$

Examples. 1. Every abstract convex space in our sense has a map $F \in \mathfrak{K}(E,Z)$ for any nonempty set Z and for any class of KKM maps $G : D \multimap Z$ with respect to F. In fact, for each $x \in E$, choose $F(x) := Z$ or let $F(x)$ contain some $z_0 \in Z$.

2. Further examples are given in Section 5 of [16].

The following is known in [12,13,21]:

Lemma. *Let $(E,D;\Gamma)$ be a G-convex space and $F : D \multimap E$ a KKM map with closed [resp., open] values. Then $\{F(z)\}_{z\in D}$ has the finite intersection property.*

For a KKM map on a G-convex minimal space, we have the following:

Proposition 3. [1, Theorems 3.2 and 3.5] *Let $(E,D;\Gamma)$ be a G-convex minimal space and $F : D \multimap E$ a KKM map with m-closed [resp., m-open] values. Then $\{F(z)\}_{z\in D}$ has the finite intersection property.*

This is a direct consequence of Lemma in view of Proposition 1. Essentially, the proof of Proposition 3 in [1] is the one in [12,13,21] with minor modifications.

Usually, a KKM type theorem is a claim $1_E \in \mathfrak{K}(E,E)$ for an abstract convex space $(E,D;\Gamma)$. There are a large number of works on various forms of the KKM type theorems for convex spaces, H-spaces, or G-convex spaces and their applications. See Section 5 of [16] and the references at the end.

Definitions. For an abstract convex minimal space $(E,D;\Gamma)$, the *KKM principle* is the statement $1_E \in m\mathfrak{KC}(E,E) \cap m\mathfrak{KO}(E,E)$.

A *minimal KKM space* (or simply, *mKKM space*) is an abstract convex minimal space satisfying the KKM principle.

Example. In view of Proposition 3, a G-convex minimal space is an $mKKM$-space. The converse does not hold; see [17].

3. The KKM type theorems in abstract convex spaces

We begin with the following in [16]:

Theorem 1. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, and $F : E \multimap Z$ a map. Then $F \in \mathfrak{K}(E, Z)$ iff for any map $G : D \multimap Z$ satisfying*

$$F(\Gamma_N) \subset G(N) \quad \textit{for any} N \in \langle D \rangle, \tag{1.1}$$

we have $F(E) \cap \bigcap\{G(y) \mid y \in N\} \neq \emptyset$ for each $N \in \langle D \rangle$.

Remark. If Z has any minimal structure and if $F \in m\mathfrak{KO}(E, Z)$ [resp., $F \in m\mathfrak{KC}(E, Z)$], then we have to assume G is m-open-valued [resp., m-closed-valued].

In this section, we show that some results for abstract convex minimal spaces can be deduced from the corresponding ones for abstract convex spaces in Section 3 of [16].

For an abstract convex minimal space, from Theorem 1 with $E = Z$ and $F = 1_E$, the following recovers the meaning of $1_E \in m\mathfrak{KC}(E, E)$ or $1_E \in m\mathfrak{KO}(E, E)$:

Corollary 1.1. *Let $(E, D; \Gamma)$ be an abstract convex minimal space. Then the identity map 1_E belongs to $m\mathfrak{KC}(E, E)$ [resp., $1_E \in m\mathfrak{KO}(E, E)$] iff for any map $G : D \multimap E$ satisfying*

(1) *G has m-closed [resp., m-open] values, and*
(2) *G is a KKM map,*

$\{G(y)\}_{y \in D}$ has the finite intersection property.

Definitions. [1] A subset K of a minimal space $(Z, \mathcal{M})$ is said to be m-*compact* if any family $\{A_\alpha\}$ of m-open sets such that $K \subset \bigcup_\alpha A_\alpha$ has a finite subfamily $\{A_{\alpha_i}\}$ such that $K \subset \bigcup_i A_{\alpha_i}$.

For a subset A of a minimal space $(Z, \mathcal{M})$, let $\operatorname{Int} A = m\text{-}\operatorname{Int} A := \bigcup\{U \in \mathcal{M} \mid U \subset A\}$ and $\overline{A} = m\text{-}\operatorname{Cl} A := \bigcap\{V \mid A \subset V,\ V^c \in \mathcal{M}\}$. Note that $\overline{A}$ is m-closed if and only if arbitrary union of m-open sets is m-open [22].

Under an additional requirement, we have the whole intersection property for the map-values of a KKM map:

Corollary 1.2. *Let $(E, D; \Gamma)$ be an abstract convex minimal space with the identity map $1_E \in m\mathfrak{KC}(E, E)$, and $G : D \multimap E$ a map satisfying*

(1) *G has m-closed values, and*
(2) *G is a KKM map.*

Then $\{G(y)\}_{y\in D}$ *has the finite intersection property.*

Further if

(3) $\bigcap_{z\in M} G(z)$ *is m-compact for some* $M \in \langle D\rangle$,

then we have

$$\bigcap_{y\in D} G(y) \neq \emptyset.$$

Proof. Since $1_E \in m\mathfrak{KC}(E,E)$, by definition, $\{G(y)\}_{y\in D}$ has the finite intersection property. Now the whole intersection property follows from the compactness (3). □

Remark. Corollary 1.2 reduces to results in [12,13,21] for G-convex spaces, to [16, Proposition 5] for an abstract convex topological spaces, and to [1, Theorem 3.2] for G-convex minimal spaces.

Corollary 1.3. *Let* $(E,D;\Gamma)$ *be an abstract convex minimal space with the identity map* $1_E \in m\mathfrak{KC}(E,E)$, *and* $G : D \multimap E$ *a map satisfying*

(1) $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$,
(2) $\overline{G}$ *has m-closed values,*
(3) $\overline{G}$ *is a KKM map, and*
(4) $\bigcap_{z\in M} \overline{G(z)}$ *is m-compact for some* $M \in \langle D\rangle$.

Then we have

$$\bigcap_{y\in D} G(y) \neq \emptyset.$$

Remarks. 1. Some authors call G a transfer closed map when $\bigcap_{y\in D} \overline{G(y)} = \bigcap_{y\in D} G(y)$.

2. Corollary 1.3 reduces to results in [12,13,21] for G-convex spaces, to [16, Proposition 5] for abstract convex topological spaces, and to [1, Theorem 3.3] for G-convex minimal spaces.

Corollary 1.4. *Let* $(E,D;\Gamma)$ *be an abstract convex minimal space with the identity map* $1_E \in m\mathfrak{KD}(E,E)$, *and* $G : D \multimap E$ *a map satisfying*

(1) G *has m-open values, and*
(2) G *is a KKM map.*

Then $\{G(y)\}_{y\in D}$ *has the finite intersection property.*

Further if

(3) $\bigcap_{z\in M} \overline{G(z)}$ *is m-compact for some* $M \in \langle D\rangle$, *and*
(4) $\overline{G}$ *has m-closed values,*

then we have

$$\bigcap_{y\in D} \overline{G(y)} \neq \emptyset.$$

Proof. Since $1_E \in m\mathfrak{K}\mathfrak{O}(E,E)$, by definition, $\{G(y)\}_{y\in D}$ has the finite intersection property. Now the whole intersection property follows from the compactness (3). $\square$

Remark. Corollary 1.4 reduces to results in [12,13] for G-convex spaces, to [16, Proposition 5] for abstract convex topological spaces, and to [1, Theorem 3.3] for G-convex minimal spaces.

The following is a basic observation:

Theorem 2. *Let $(E,D;\Gamma)$ be an abstract convex minimal space and Z a minimal space. If $1_E \in m\mathfrak{K}\mathfrak{C}(E,E)$, then $f \in m\mathfrak{K}\mathfrak{C}(E,Z)$ for any m-continuous function $f : E \to Z$. This also holds for $m\mathfrak{K}\mathfrak{O}$.*

Proof. Let $G : D \multimap Z$ be a m-closed-valued map satisfying $f(\Gamma_N) \subset G(N)$ or $\Gamma_N \subset f^{-1}G(N)$ for each $N \in \langle D\rangle$. Since $1_E \in m\mathfrak{K}\mathfrak{C}(E,E)$ and $f^{-1}G : D \multimap E$ is m-closed-valued, $\{f^{-1}G(y)\}_{y\in D}$ has the finite intersection property. Hence, so does $\{G(y)\}_{y\in D}$. Therefore, $f \in m\mathfrak{K}\mathfrak{C}(E,Z)$. Similarly, we can show the case for $m\mathfrak{K}\mathfrak{O}$. $\square$

4. Coincidence and fixed point theorems

In the KKM theory, there exist some basic results from which we can deduce several equivalent formulations that can be used to applications; see [12]. In this section, we introduce some of such basic results.

For abstract convex spaces, we have the following coincidence theorem as in [16]:

Theorem 3. *Let $(E,D;\Gamma)$ be an abstract convex space, Z a set, $S : D \multimap Z$, $T : E \multimap Z$ maps, and $F \in \mathfrak{K}(E,Z)$. Suppose that*

(3.1) *for each $z \in F(E)$, $\mathrm{co}_\Gamma S^-(z) \subset T^-(z)$ [that is, $T^-(z)$ is Gamma-convex relative to $S^-(z)$]; and*
(3.2) *$F(E) \subset S(N)$ for some $N \in \langle D\rangle$.*

Then there exists an $\bar{x} \in E$ such that $F(\bar{x}) \cap T(\bar{x}) \neq \emptyset$.

Proof. For each $y \in D$, define $R(y) := F(E) \setminus S(y)$. Then $\bigcap_{y\in N} R(y) = F(E)\setminus\bigcup_{y\in N} S(y) = \emptyset$ by (3.2), that is, the values of the map $R : D \multimap Z$ does not have the finite intersection property. Since $F \in \mathfrak{K}(E,Z)$, $F(\Gamma_M) \not\subset R(M)$ for some $M \in \langle D\rangle$. Hence, there exist $\bar{x} \in \Gamma_M$ and $\bar{z} \in F(\bar{x}) \subset F(E)$ such that $\bar{z} \notin R(M)$. Then, $\bar{z} \in S(y)$ or $y \in S^-(\bar{z})$ for all $y \in M$. This implies $\bar{x} \in \Gamma_M \subset T^-(\bar{z})$ by (3.1). Therefore, $\bar{z} \in F(\bar{x}) \cap T(\bar{x})$. □

Remark. If Z has a minimal structure and S has m-open [resp., m-closed] values, then R has relatively m-closed [resp., m-open] values in $F(E)$. Then we can assume $F \in m\mathfrak{KC}(E,Z)$ [resp., $F \in m\mathfrak{KO}(E,Z)$].

From Theorem 3, we have the following prototype of the Fan-Browder fixed point theorem [4]:

Corollary 3.1. *Let $(E,D;\Gamma)$ be a minimal KKM space, and $G : E \multimap D$, $H : E \multimap E$ maps satisfying*

(1) *for each $x \in E$, $\mathrm{co}_\Gamma G(x) \subset H(x)$; and*
(2) *$E = G^-(N)$ for some $N \in \langle D \rangle$.*
(3) *G^- has m-open [resp., m-closed] values.*

Then H has a fixed point $\bar{x} \in E$, that is, $\bar{x} \in F(\bar{x})$.

Proof. In Theorem 3, let $E = Z$, $S := G^-$, $T := H^-$, and $F := 1_E$. □

Remark. Corollary 3.1 is originated from [4] and one of the most useful results in the KKM theory.

Corollary 3.2. *Let $(E,D;\Gamma)$ be a G-convex minimal space, and $G : E \multimap D$, $H : E \multimap E$ maps satisfying* (1) – (3) *in Corollary 3.1. Then H has a fixed point $\bar{x} \in E$, that is, $\bar{x} \in F(\bar{x})$.*

Proof. A G-convex minimal space is a minimal KKM space by Proposition 3. □

From Corollary 3.1, we deduce some new forms of the Fan-Browder type fixed point theorems:

Corollary 3.3. *Let $(E,D;\Gamma)$ be a minimal KKM space and $S : E \multimap D$, $T : E \multimap E$ maps such that*

(1) *for each $x \in E$, $\mathrm{co}_\Gamma S(x) \subset T(x)$; and*
(2) *there exist $D' := \{z_1, z_2, \ldots, z_n\} \in \langle D \rangle$ and m-open [resp., m-closed] subsets $\{G_i\}_{i=1}^n$ of E such that*

$$E = \bigcup_{i=1}^{n} G_i \quad \text{and} \quad G_i \subset S^-(z_i) \quad \text{for each } i.$$

Then T has a fixed point $x_* \in E$.

Proof. Consider the abstract convex space $(E,D';\Gamma)$ where $\Gamma : \langle D' \rangle \multimap X$ is actually the restriction $\Gamma|_{\langle D' \rangle}$ of the original Γ. Define a map $G : E \multimap D'$ by $G^-(z_i) = G_i$ for each $z_i \in D'$. Note that $G(x) \subset S(x)$ for each $x \in X$. Now Corollary 3.1 with $H := T$ works. □

Remarks. 1. In Corollary 3.3, let $E_T := \{x \in E \mid x \notin T(x)\}$. Then condition $E = \bigcup_{i=1}^n G_i$ in (2) can be replaced by $E_T = \bigcup_{i=1}^n G_i$ without affecting the conclusion of Corollary 3.3. In fact, suppose that T has no fixed point, that is, $E = E_T$. Then by Corollary 3.3, T has a fixed point, a contradiction.

2. For a G-convex space, Corollary 3.3 reduces to [15, Theorem 8], which has a number of variants as shown in [15].

Corollary 3.4. *Let $(E, D; \Gamma)$ be a G-convex minimal space and $S : E \multimap D$, $T : E \multimap E$ maps satisfying (1) – (2) in Corollary 3.3. Then T has a fixed point $x_* \in E$.*

Popular generalizations of the Fan-Browder fixed point theorem have the form of Corollary 3.3 for $E = D$ and $S = T$ as follows:

Corollary 3.5. *Let $(E; \Gamma)$ be an m-compact minimal KKM space and $T : E \multimap E$ a map satisfying*

(1) *for each $x \in E$, $T(x)$ is nonempty Γ-convex; and*
(2) *for each $y \in E$, $T^{-1}(y)$ is m-open.*

Then T has a fixed point $x_* \in E$.

5. The nash equilibrium theorem

In this section, from a Fan-Browder type fixed point result, we deduce the Fan intersection theorem and the Nash equilibrium theorem for abstract convex minimal spaces.

Given a cartesian product $X = \prod_{i=1}^n X_i$ of sets, let $X^i = \prod_{j \neq i} X_j$ and $\pi_i : X \to X_i$, $\pi^i : X \to X^i$ be the projections; we write $\pi_i(x) = x_i$ and $\pi^i(x) = x^i$. Given $x, y \in X$, we let

$$(y_i, x^i) = (x_1, \ldots, x_{i-1}, y_i, x_{i+1}, \ldots, x_n).$$

From Corollary 3.5, we have the following Fan type intersection theorem:

Theorem 4. *Let $X = \prod_{i=1}^n X_i$, (X, Γ) be an m-compact minimal KKM space, and $A_1, A_2, \ldots, A_n$ be n subsets of X such that*

(4.1) *for each $x \in X$ and $i = 1, \ldots, n$, the set $A_i(x) := \{y \in X \mid (y_i, x^i) \in A_i\}$ is Γ-convex and nonempty;*
(4.2) *for each $y \in X$ and $i = 1, \ldots, n$, the set $A_i(y) := \{x \in X \mid (y_i, x^i) \in A_i\}$ is m-open; and*
(4.3) *any finite intersection of m-open sets in X is m-open.*

Then $\bigcap_{i=1}^n A_i \neq \emptyset$.

Proof. Define a map $T : X \multimap X$ by $T(x) := \bigcap_{i=1}^n A_i(x)$ for $x \in X$. Then each $T(x)$ is Γ-convex being an intersection of Γ-convex sets by (4.1). For each $x \in X$ and each i, there exists a $y^{(i)} \in A_i(x)$ by (4.1), or $(y_i^{(i)}, x^i) \in A_i$.

Hence, we have $(y_1^{(1)}, \dots, y_n^{(n)}) \in \bigcap_{i=1}^n A_i(x)$. This shows $T(x) \neq \emptyset$. Moreover, $T^-(y) = \bigcap_{i=1}^n A_i(y)$ is m-open for each $y \in X$ by (4.2) and (4.3). Now, the conclusion follows from Corollary 3.5. □

Remarks. 1. If each X_i is a compact G-convex space, so is X.

2. In view of Theorem 3, condition (4.2) can be replaced by the following:

$$X = \bigcup_{y \in X} \operatorname{Int}\left(\bigcap_{i=1}^n A_i(y)\right). \tag{4.2$'$}$$

Particular Forms. For a compact G-convex space (X, Γ), Theorem 4 reduces to [14, Theorem 4.6], which includes the following:

1. Ky Fan [6, Theorem 2]: X_i are compact convex subsets of topological vector spaces in Theorem 4.

2. Bielawski [3, Proposition (4.12) and Theorem (4.15)]: Theorem 4 for X having a finitely local convexity, which is a particular type of his simplicial convexity.

3. Kirk, Sims, and Yuan [8, Theorem 5.2]: Theorem 4 for hyperconvex metric spaces, which are of extremely particular type of G-convex spaces.

From Theorem 4, we deduce the following Nash equilibrium theorem for abstract convex spaces:

Theorem 5. *Let $X = \prod_{i=1}^n X_i$, (X, Γ) be an m-compact minimal KKM space, and $f_1, \dots, f_n : X \to \mathbb{R}$ continuous real functions such that*

(5.1) *any finite intersection of m-open sets in X is m-open; and*

(5.2) *for each $x \in X$, each $i = 1, \dots, n$, and each $r \in \mathbb{R}$, the set $\{(y_i, x^i) \in X \mid f_i(y_i, x^i) > r\}$ is Γ-convex.*

Then there exists a point $x \in X$ such that

$$f_i(x) = \max_{y_i \in X_i} f_i(y_i, x^i) \quad \text{for} \quad i = 1, \dots, n.$$

Proof. Let $\varepsilon > 0$ and, for each i, let

$$A_i^\varepsilon = \{x \in X \mid f_i(x) > \max_{y_i \in X_i} f_i(y_i, x^i) - \varepsilon\}.$$

Then the sets $A_1^\varepsilon, \dots, A_n^\varepsilon$ satisfy conditions (4.1)-(4.3) of Theorem 4, and hence $\bigcap_{i=1}^n A_i^\varepsilon \neq \emptyset$. Then $H_\varepsilon = \bigcap_{i=1}^n \overline{A_i^\varepsilon}$ is a nonempty m-compact set. Since $H_{\varepsilon_1} \subset H_{\varepsilon_2}$ for $\varepsilon_1 < \varepsilon_2$, we have $\bigcap_{\varepsilon > 0} H_\varepsilon \neq \emptyset$. Then $x \in \bigcap_{\varepsilon > 0} H_\varepsilon$ satisfies the conclusion. □

Particular Forms. For a compact G-convex space (X, Γ), Theorem 5 reduces to [14, Theorem 4.7], which includes the following:

1. Nash [10]: Each X_i is a compact convex subset of a Euclidean space in Theorem 5.

2. Fan [7, Theorem 4]: X_i are compact convex subsets of real Hausdorff topological vector spaces in Theorem 5.

3. Bielawski [3, Theorem (4.16)]: Theorem 5 for X having a finitely local convexity.

4. Kirk, Sims, and Yuan [8, Theorem 5.3]: Theorem 5 for hyperconvex metric spaces.

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