Nonlinear Functional Analysis and Applications
Vol. 13, No. 2 (2008), pp. 193-213

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

PARABOLIC AND HYPERBOLIC SYSTEMS DETERMINED BY COERCIVE OPERATOR VALUED MEASURES

N. U. Ahmed

School of Information Technology and Engineering and Department of Mathematics,
University of Ottawa, Ottawa, Canada

Abstract. In this paper we consider a class of distributed parameter systems governed by parabolic and hyperbolic partial differential equations containing operators which are measures. We prove existence, uniqueness and regularity properties of weak solutions. This covers the important and well established class of systems considered by J.L Lions as special case. Control problems with quadratic cost are treated for parabolic problems only. Following similar technique these results can be extended to hyperbolic systems also.

1. Introduction

In two recent papers [1,2] the author studied systems governed by operator valued measures under the assumption that they are countably additive in the uniform operator topology. This assumption has been also used in the semigroup setting [9] where the principal operator is assumed to be the infinitesimal generator C_0-semigroups and the perturbing operator is a measure. The objective of this paper is to study abstract parabolic and hyperbolic partial differential equations where the principal operator itself is an operator valued measure which is assumed to be countably additive only in the weak operator topology. We use the classical Galerkin technique to establish existence and uniqueness of weak solutions and also study the regularity properties. We consider a control problem with quadratic cost functional and present a result

[0]Received December 12, 2006. Revised February 8, 2007.
[0]AMS(MOS) Subject Classification: 35K10, 35L10, 49K20,49N10,49J25, 34G20, 34K30, 35A05, 93C20.
[0]Keywords: Parabolic and hyperbolic systems, coercive operators, operator valued measures, existence of solutions, optimal control.

on the existence of optimal controls. Further we also present necessary conditions of optimality. These results are expected to be useful also in structural control theory as developed in [1].

2. Preliminaries

Function Spaces: Let H be a real separable Hilbert space with scalar product and norms denoted by (v, w) and $|v| \equiv \sqrt{(}v, v)$ respectively for $v, w \in H$. Let V be a linear subspace of the Hilbert space H carrying the structure of a Hilbert space with the scalar product denoted by $(v, w) \equiv (v, w)_V$ and norm denoted by $\| v \|_V$ with V^* denoting its topological dual. Identifying H with its own dual and assuming that V is dense in H, we have the inclusion

$$V \hookrightarrow H \hookrightarrow V^*$$

where the injections are continuous and dense. The duality pairing between $v \in V$ and $w \in V^*$ is denoted by

$$< v, w > \equiv < v, w >_{V,V^*} .$$

In case $w \in H$, this reduces to the scalar product in H. We assume that there exists a complete system of basis vectors $\{v_i\} \subset V$ which is orthogonal in V and V^* and ortho-normal in H and that they span all the three spaces $\{V, H, V^*\}$ known as the Gelfand triple. For more details on these spaces see [6,3].

Let $I \equiv [0, T]$ be an interval with $T < \infty$ and let $\Sigma \equiv \sigma(I)$ denote the sigma algebra of subsets of the set I. Let $B(I, H)$ denote the vector space of bounded Σ measurable functions on I with values in H. Furnished with the sup norm topology, this is a Banach space. Let μ be any countably additive positive measure on Σ having bounded total variation on I. For any of the spaces $X \equiv \{V, H, V^*\}$ and $1 \leq p \leq \infty$, we let $L_p(\mu, X)$ denote the Lebesgue-Bochner space of measurable functions on I with values in X satisfying

$$\int_I \| f(s) \|_X^p \mu(ds) < \infty.$$

Strictly speaking this is the equivalence class of μ measurable X valued functions whose X-norms are $p-th$ power integrable. Furnished with the standard norm topology this is a Banach space. By $L_p(\mu)$ we denote the Banach space of scalar valued p-th power μ integrable functions defined on the interval I. By $BV(I, X)$ we denote the vector space of functions, defined on I and taking values from the Banach space X, having bounded total variation. Furnished with total variation norm this is a Banach space.

Vector Measures: Let F be a Banach space and $I \equiv [0,T]$ a bounded interval with $\Sigma \equiv \sigma(I)$ the sigma algebra of subsets of the set I. Let $\mathcal{M}_c(\Sigma,F)$ denote the space of bounded countably additive vector measures defined on the sigma algebra Σ with values in the Banach space F. This is furnished with the topology induced by the total variation norm. That is, for each $\nu \in \mathcal{M}_c(\Sigma,F)$, we write

$$|\nu| \equiv |\nu|(I) \equiv \sup_{\pi}\{\sum_{\sigma\in\pi} \| \nu(\sigma) \|_F\}$$

where the supremum is taken over all partitions π of the interval I into a finite number of disjoint members of Σ. With respect to this topology, $\mathcal{M}_c(\Sigma,F)$ is a Banach space. For any $\sigma \in \Sigma$, denote the variation of ν on σ by $|\nu|(\sigma)$. Since ν is countably additive and bounded, this defines a countably additive bounded positive measure on Σ see [7, Proposition 9,p3]. In case $F = R$, the real line, we have the space of real valued signed measures. We denote this simply by $\mathcal{M}_c(\Sigma)$ in place of $\mathcal{M}_c(\Sigma,R)$. Clearly for $\nu \in \mathcal{M}_c(\Sigma)$, $|\nu|(\cdot)$ is also a countably additive bounded positive measure. For detailed study of vector measures see [7].

Operator Valued Measures: Let E and F be any pair of Banach spaces and $\mathcal{L}(E,F)$ the space of bounded linear operators from E to F. Let τ_u,τ_s,τ_w denote the uniform, strong and weak operator topologies respectively on $\mathcal{L}(E,F)$ and let $\mathcal{L}_u(E,F),\mathcal{L}_s(E,F),\mathcal{L}_w(E,F)$ denote the corresponding locally convex linear topological spaces. It is well known that $\mathcal{L}_u(E,F)$ is a Banach space, $\mathcal{L}_s(E,F)$ is a locally convex sequentially complete topological vector space. This later fact is a consequence of Banach-Steinhaus theorem and uniform boundedness principle. The space $\mathcal{L}_w(E,F)$ is also a locally convex topological vector space.

A set function Φ mapping Σ to $\mathcal{L}(E,F)$ is said to be an operator valued measure if for each $\sigma \in \Sigma$, $\Phi(\sigma) \in \mathcal{L}(E,F)$ and $\Phi(\emptyset) = 0$ the zero operator. We denote by $M_c(\Sigma,\mathcal{L}_u(E,F)), M_c(\Sigma,\mathcal{L}_s(E,F))$ and $M_c(\Sigma,\mathcal{L}_w(E,F))$ the space of operator valued measures which are countably additive in the uniform operator topology, strong operator topology, and weak operator topology respectively having bounded total variation, and bounded semivariations respectively.

Now we are prepared to undertake the study of dynamic systems and control.

3. Linear parabolic systems

First let us recall the classical model of J.L Lions

$$\dot{x} + Ax = f, x(0) = \xi,\ t \in I. \tag{1}$$

Suppose the operator A satisfy the following properties: there exist constants $c > 0$, $\lambda \geq 0$ and $\alpha > 0$ such that

(A1): $< Av, w >_{V^*,V} \leq \; c \parallel v \parallel_V \parallel w \parallel_V \;\; \forall \, v, w \in V$

(A2): $< Av, v > + \, \lambda \, |v|_H^2 \geq \alpha \parallel v \parallel_V^2 \;\; \forall \, v \in V.$

Clearly the assumption (A1) implies that $A \in \mathcal{L}(V, V^*)$. Assumption (A2) implies that A is coercive in the sense that

$$\lim_{\|v\|_V \to \infty} \frac{< Av, v >}{\parallel v \parallel_V} = +\infty.$$

The following result is due to Lions and well known [6,3,4].

Theorem 3.1. *Suppose A satisfy the assumptions (A1) and (A2). Then for every $\xi \in H$ and $f \in L_2(I, V^*)$, system (1) has a unique weak solution $x \in C(I, H)$. Further $\dot{x} \in L_2(I, V^*)$.*

Proof. For detailed proof see [6] and [4]. □

This result is classical and has been extensively used in the study of optimal control of systems governed by partial differential equations [3,4,6] of parabolic and also hyperbolic types. In case $f \in L_2(I, H)$ one can also use semigroup theory (see [5]) to construct a unique mild solution given by

$$x(t) = S(t)\xi + \int_0^t S(t - r)f(r)dr, t \in I,$$

where $S(t), t \geq 0$, is the C_0 semigroup in H generated by A_H, the part of A in H. In case $f \in L_2(I, V^*)$ one can use continuity and density of the embedding $H \hookrightarrow V^*$ to prove that there is a sequence $\{f_n\} \in L_2(I, H)$ that converges to $f \in L_2(I, V^*)$ strongly in the norm topology of $L_2(I, V^*)$ and that the corresponding mild solutions converge weakly to the weak solution of the original Cauchy problem with $f \in L_2(I, V^*)$.

Our objective in this section is to prove a result analogous to that of Theorem 3.1 for systems (with measures) of the form

$$dx + A(dt)x = f(t)\alpha(dt), t \in I, x(0) = \xi \in H, \tag{2}$$

where the operator A is a measure, that is, a set function $A : \Sigma \longrightarrow \mathcal{L}(V, V^*)$, and $\alpha(\cdot)$ is a countably additive positive measure having bounded variation. We introduce the following assumptions:

(M1): $A : \Sigma \longrightarrow \mathcal{L}(V, V^*)$ is a weakly countably additive bounded operator valued measure in the sense that for each $v, w \in V$, the scalar valued measure

$$\sigma \ni \Sigma \longrightarrow < A(\sigma)v, w >$$

is countably additive having bounded variation. In other words $A \in M_c(\Sigma, \mathcal{L}_w(V, V^*))$.

(M2): there exist two countably additive nonnegative measures $\lambda(\cdot)$ and $\beta(\cdot)$ having bounded variation on I so that

$$\begin{aligned} | < A(\sigma)v, w > | &\leq \beta(\sigma) \parallel v \parallel_V \parallel w \parallel_V \quad \forall\, \sigma \in \Sigma, v, w \in V, &(3)\\ < A(\sigma)v, v > + \lambda(\sigma)\ |v|_H^2 &\geq \alpha(\sigma) \parallel v \parallel_V^2 \quad \forall\, \sigma \in \Sigma. &(4)\end{aligned}$$

We wish to prove existence and uniqueness of weak solutions for the system (2). For this purpose we need the following Lemma giving a-priori bounds.

Lemma 3.2. *Suppose the operator valued measure $A(\cdot)$ satisfy the assumptions (M1) and (M2) and $\xi \in H$ and $f \in L_2(\alpha, V^*)$. Then if x is any solution of equation (2) we have $x \in B(I, H) \cap L_\infty(I, H) \cap L_2(\alpha, V)$.*

Proof. If x is any solution of the evolution equation (2), x must satisfy the following identity

$$\int_0^t < dx(s), x(s) > + \int_0^t < A(ds)x(s), x(s) > = \int_0^t < f(s), x(s) > \alpha(ds), \quad (5)$$

for all $t \in I$. Using elementary distribution theory one can easily verify that $< dx(t), x(t) >= (1/2)d(|x(t)|_H^2)$. Hence it follows from equation (5) that

$$\begin{aligned} & |x(t)|_H^2 - |\xi|_H^2 + 2\int_0^t < A(ds)x(s), x(s) > \\ = &\quad 2\int_0^t < f(s), x(s) > \alpha(ds), \quad t \in I. \qquad (6)\end{aligned}$$

Using assumption (M2), it follows from the above expression that

$$\begin{aligned} & |x(t)|_H^2 + 2\int_0^t \parallel x(s) \parallel_V^2 \alpha(ds) \\ \leq &\quad |\xi|_H^2 + 2\int_0^t |x(s)|_H^2 \lambda(ds) + 2\int_0^t \parallel f(s) \parallel_{V^*} \parallel x(s) \parallel_V \alpha(ds), t \in I. \ (7)\end{aligned}$$

For any $\varepsilon > 0$, it follows from Cauchy inequality that

$$2\int_0^t \parallel f(s) \parallel_{V^*} \parallel x(s) \parallel_V \alpha(ds) \leq (1/\varepsilon)\int_0^t \parallel f(s) \parallel_{V^*}^2 \alpha(ds) + \varepsilon\int_0^t \parallel x(s) \parallel_V^2 \alpha(ds).$$

Using this expression in (7) for $\varepsilon = 1$, we obtain

$$\begin{aligned} |x(t)|_H^2 + \int_0^t \parallel x(s) \parallel_V^2 \alpha(ds) &\leq |\xi|_H^2 + 2\int_0^t |x(s)|_H^2\ \lambda(ds) \\ &\quad + \int_0^t \parallel f(s) \parallel_{V^*}^2 \alpha(ds), t \in I. \qquad (8)\end{aligned}$$

Define

$$C \equiv |\xi|_H^2 + \int_0^T \| f(s) \|_{V^*}^2 \alpha(ds).$$

Since $\xi \in H$ and $f \in L_2(\alpha, V^*)$ it is clear that $0 < C < \infty$. In view of this observation we have,

$$|x(t)|_H^2 + \int_0^t \| x(s) \|_V^2 \alpha(ds) \quad \leq \quad C + 2\int_0^t |x(s)|_H^2 \lambda(ds), t \in I. \quad (9)$$

Then it follows from a generalized Gronwall inequality [8] valid for arbitrary positive measures (not just Lebesgue measure) that

$$|x(t)|_H^2 \leq C \exp 2\lambda([0,t]) \leq C \exp\{2\lambda(I)\}. \quad (10)$$

Since by assumption $\lambda(I) < \infty$ it follows from the above inequality that $x \in B(I,H) \cap L_\infty(I,H)$. Denoting the bound by $b \equiv C \exp\{2\lambda(I)\}$ and substituting it in (8) we obtain

$$\begin{aligned} &|x(t)|_H^2 + \int_0^t \| x(s) \|_V^2 \alpha(ds) \\ \leq\ &|\xi|_H^2 + 2b\lambda(I) + \int_0^t \| f(s) \|_{V^*}^2 \alpha(ds), \quad t \in I. \end{aligned} \quad (11)$$

Since $f \in L_2(\alpha, V^*)$ it follows from this estimate that $x \in L_2(\alpha, V)$. Combining all these facts we have

$$x \in B(I,H) \cap L_\infty(I,H) \cap L_2(\alpha, V). \quad (12)$$

This completes the proof of the Lemma. □

Remark 3.3. It is interesting to note that the measure λ appearing in assumption (M2) need not be nonnegative. It suffices if it is a signed measure having bounded total variation on bounded intervals.

We are now ready to prove one of our main results of this section. First we introduce the following definition.

Definition 3.4. An element $x \in B(I,H) \cap L_\infty(I,H) \cap L_2(\alpha, V)$ is said to be a weak solution of the evolution equation (2) if $x(0) = \xi$ and

$$\begin{aligned} &-\int_I < x(t), \dot{\varphi}(t)v > dt + \int_I < x(t), \varphi(t)A(dt)v > \\ &= \int_I < f(t), \varphi(t)v > \alpha(dt), \end{aligned}$$

for every $v \in V$ and every $\varphi \in C_0^1(0,T)$.

Theorem 3.5. *Consider the system (2) and suppose that the operator valued measure A is symmetric and satisfies the assumptions (M1) and (M2). Then for every $\xi \in H$ and $f \in L_2(\alpha, V^*)$, the system (2) has a unique weak solution $x \in B(I,H) \cap L_\infty(I,H) \cap L_2(\alpha, V)$.*

Proof. The proof is based on the method of projection, that is, Galerkin approach. Define

$$x^n(t) \equiv \sum_{i=1}^{n} z_i^n(t) v_i, t \in I, \tag{13}$$

$$x^n(0) = \sum_{i=1}^{n} z_{i,0}^n v_i \tag{14}$$

where $z_{i,0}^n = (\xi, v_i), i = 1, 2, \cdots, n$ are the Fourier coefficients of ξ with respect to the basis $\{v_i\}$ introduced in section 2 and $\{z_i^n, i = 1, 2, \cdots, n\}$ are scalar valued functions satisfying the following system of equations

$$dz_i^n + \sum_{j=1}^{n} < A(dt)v_i, v_j > z_j^n =< f(t), v_i > \alpha(dt), i = 1, 2, \cdots, n \tag{15}$$

$$z_i^n(0) = z_{i,0}^n, i = 1, 2, \cdots, n. \tag{16}$$

This is a system of finite dimensional (n-dimensional) differential equations which can be written as a system in R^n as follows:

$$dZ^n + M(dt)Z^n = f^n \alpha(dt), Z^n(0) = Z_0^n, t \in I \tag{17}$$

where

$$M(\cdot) \equiv \{m_{i,j}(\cdot) \equiv< A(\cdot)v_i, v_j >, i, j = 1, 2, \cdots, n\}$$

is a matrix valued countably additive measure, $Z^n \equiv \{z_i^n, i = 1, 2 \cdots, n\}$ is the R^n valued function representing the solution of the initial value problem (17) if one exists. We write equation (17) as an integral equation,

$$\begin{aligned} Z^n(t) &= Z_0^n + \int_0^t f^n(s)\alpha(ds) - \int_0^t M(ds)Z^n(s), \\ &\equiv \psi_n(t) + \int_0^t -M(ds)Z^n(s) \\ &\equiv (GZ^n)(t), t \in I. \end{aligned} \tag{18}$$

For convenience we have introduced the operator G to represent the expression on the righthand side of the integral equation. This is a fixed point problem. We show that G has a unique fixed point in $B(I, R^n)$. First note that, since $f \in L_2(\alpha, V^*)$, we have $f^n \in L_2(\alpha, R^n)$ and hence $\psi_n \in B(I, R^n)$. Now recall that $\sigma \longrightarrow A(\sigma)$ is a weakly countably additive operator valued measure having bounded variation. Thus the matrix valued measure $\sigma \ni \Sigma \longrightarrow M(\sigma)$

is countably additive having bounded total variation. Further, it follows from a well known result [7, Diestel and Uhl Jr] that a bounded vector measure is countably additive if and only if its variation is countably additive. Hence we may conclude that the scalar valued measure $\mu(\cdot) \equiv |M|(\cdot)$ induced by the variation of the vector measure $M(\cdot)$ is a nonnegative countably additive measure. From these facts we conclude that $G : B(I, R^n) \longrightarrow B(I, R^n)$. We now show that it has a unique fixed point in $B(I, R^n)$. For $y, z \in B(I, R^n)$ define

$$\rho_t(y, z) \equiv \sup\{|y(s) - z(s)|_{R^n}, 0 \leq s \leq t\},\ \rho(x, y) \equiv \rho_T(y, z).$$

Clearly it follows from the definition of the operator G and the measure μ that

$$|(Gy - Gz)(t)|_{R^n} \leq \int_0^t |y(s) - z(s)||M|(ds) = \int_0^t |y(s) - z(s)|\mu(ds), t \in I$$

and hence

$$\rho_t(Gy, Gz) \leq \int_0^t \rho_s(y, z)\mu(ds), t \in I. \tag{19}$$

Since μ is a nonnegative countably additive measure having bounded variation on I, the function m given by $m(t) \equiv \mu([0, t))$ is a nonnegative nondecreasing function of bounded variation on I. By repeated substitution of the expression (19) into itself and using the function m, after k iterations one arrives at the following expression

$$\rho_t(G^k y, G^k z) \leq \big(m^k(t)/k!\big)\rho_t(y, z), t \in I. \tag{20}$$

Clearly this leads to the following inequality,

$$\rho_t(G^k y, G^k z) \leq \big(m^k(T)/k!\big)\rho(y, z), t \in I. \tag{21}$$

From this expression we conclude that for $k \in N$ sufficiently large, the k-th iterate G^k of G is a contraction in the metric space $B(I, R^n)$ endowed with the metric ρ. Hence by Banach fixed point theorem both G^k as well as G has one and the same fixed point in $B(I, R^n)$. Thus we have proved that the system (18) or equivalently the system (17) has a unique solution $Z^n \in B(I, R^n)$ for each $n \in N$. Thus the sequence $\{x^n\}$ given by the expression (13) is well defined V-valued Borel measurable functions. Using this it is easy to verify that equation (15) is equivalent to the following equation

$$< dx^n, v_i > + < A(dt)v_i, x^n >=< f(t), v_i > \alpha(dt), i = 1, 2, \cdots, n. \tag{22}$$

Multiplying on either side of this identity by z_i and summing up to $i = n$ and then integrating, we obtain

$$\int_0^t < dx^n(s), x^n(s) > + \int_0^t < A(ds)x^n(s), x^n(s) >$$
$$= \int_0^t < f(s), x^n(s) > \alpha(ds), n \in N, \tag{23}$$

which is identical to equation (5). Hence, it follows from Lemma 3.2 that the sequence $\{x^n\}$ must satisfy the a-priori estimate (11), that is,

$$|x^n(t)|_H^2 + \int_0^t \parallel x^n(s) \parallel_V^2 \alpha(ds)$$
$$\leq |\xi|_H^2 + 2b\lambda(I) + \int_0^t \parallel f(s) \parallel_{V^*}^2 \alpha(ds), t \in I. \tag{24}$$

From this we conclude that $\{x^n\}$ is a bounded sequence in $B(I,H) \cap L_\infty(I,H) \cap L_2(\alpha,V)$. Since $L_2(\alpha,V)$ is a reflexive Banach space and $L_\infty(I,H)$ is the dual of $L_1(I,H)$, there exists a subsequence of the sequence $\{x^n\}$, relabeled as the original sequence, and an element $x^o \in L_\infty(I,H) \cap L_2(\alpha,V)$ such that

$$x^n \stackrel{w}{\longrightarrow} x^o \text{ in } L_2(\alpha,V) \tag{25}$$
$$x^n \stackrel{w^*}{\longrightarrow} x^o \text{ in } L_\infty(I,H). \tag{26}$$

Since $\lambda(I) < \infty$, $x^o \in B(I,H)$ also. Now take a $\varphi \in C_0^1(0,T)$, the space of C^1 functions with compact support, and multiply on either side of (22) and integrate by parts giving

$$-\int_I < x^n(t), \dot{\varphi}(t)v_i > dt + \int_I < x^n(t), A(dt)\varphi(t)v_i >$$
$$= \int_I < f(t), \varphi(t)v_i > \alpha(dt). \tag{27}$$

Using the convergence properties (25), (26) and letting $n \to \infty$ in the above identity we arrive at the following expression,

$$-\int_I < x^o(t), \dot{\varphi}(t)v_i > dt + \int_I < x^o(t), \varphi(t)A(dt)v_i >$$
$$= \int_I < f(t), \varphi(t)v_i > \alpha(dt), \tag{28}$$

which holds for all $i \in N$. Since $\{v_i\}$ is a basis for V, we conclude from this that

$$-\int_I < x^o(t), \dot{\varphi}(t)v > dt + \int_I < x^o(t), \varphi(t)A(dt)v >$$
$$= \int_I < f(t), \varphi(t)v > \alpha(dt), \tag{29}$$

for all $v \in V$ and all $\varphi \in C_0^1(0,T)$. Thus x^o satisfies the evolution equation

$$dx + A(dt)x = f(t)\alpha(dt), t \in (0,T) \tag{30}$$

in the sense of distribution. In order to prove that x^o is a weak solution of (30) in the sense of definition 3.4 we must verify that $x^o(0) = \xi$. We choose a $\varphi \in C^1$ with $\varphi(0)$ arbitrary and then scalar multiply on either side of the evolution equation (30) by $\varphi(\cdot)v_i \in C^1(I,V)$ and integrate over the interval $[0,t]$ to obtain

$$(x^o(t), \varphi(t)v_i) - (x^o(0), \varphi(0)v_i) - \int_0^t < x^o(s), \varphi(s)A(ds)v_i >$$
$$= \int_0^t < f(s), \varphi(s)v_i > \alpha(ds), t \in I.$$

By considering projection on to the linear subspace spanned by $\{v_j, 1 \leq j \leq n\}$, it is clear that x^n (see equation (22)) must satisfy a similar expression given by

$$(x^n(t), \varphi(t)v_i) - (x^n(0), \varphi(0)v_i) - \int_0^t < x^n(s), \varphi(s)A(ds)v_i >$$
$$= \int_0^t < f(s), \varphi(s)v_i > \alpha(ds), t \in I, 1 \leq i \leq n.$$

Subtracting one from the other we obtain the following identity

$$(x^n(t) - x^o(t), \varphi(t)v_i) + (x^o(0) - x_0^n, \varphi(0)v_i) + \int_0^t < x^o(s) - x^n(s), v_i\dot{\varphi}(s) >$$
$$+ \int_0^t < \varphi(s)A(ds)v_i, x^n(s) - x^o(s) >= 0. \tag{31}$$

Recalling the convergence properties (25)-(26) and letting $n \to \infty$ it follows from the above expression that

$$\lim_{n\to\infty} (x^o(0) - x_0^n, \varphi(0)v_i) = 0 \ \forall \ i \in N$$

and hence for all $v \in V$. Since $\varphi \in C^1$ is arbitrary, and $\{v_i\}$ is a basis for all the members of the triple $\{V, H, V^*\}$, and x_0^n converges strongly to ξ in H, we conclude that

$$x^o(0) = \lim_{n\to\infty} x_0^n = \xi. \tag{32}$$

Thus by definition, x^o is a weak solution of the evolution equation (2). Assuming that there are two such solutions $x^o, y^o \in B(I,H) \cap L_\infty(I,H) \cap L_2(\alpha,V)$ with the same data ξ and f, it is easy to verify that the difference z^o must satisfy the identity

$$-\int_I < z^o(t), \dot{\varphi}(t)v > dt + \int_I < z^o(t), \varphi(t)A(dt)v >= 0,\ \forall\ \varphi \in C_0^1,\ v \in V. \tag{33}$$

Since this holds for all $\varphi \in C_0^1(I)$ and all $v \in V$, and V is dense in H, we have $z^o = 0$. An alternative argument is that the identity (33) implies that z^o is a distribution solution of equation

$$dz^o + A(dt)z^o = 0, z^o(0) = 0, t \geq 0.$$

Hence

$$|z^o(t)|_H^2 + 2\int_0^t \parallel z^o(s) \parallel_V^2 \alpha(ds) \leq 2\int_0^t |z^o(s)|_H^2 \lambda(ds), t \geq 0.$$

Thus the conclusion follows from Gronwall inequality. This ends the proof. □

It is known from the classical results not involving measures that the solution $x \in C(I,H)$ and that $\dot{x} \in L_2(I,V^*)$. This is not true in our case. We have seen in the above results that $x \in B(I,H) \cap L_\infty(I,H) \cap L_2(\alpha,V)$. In the following result we prove an additional regularity of the solution.

Proposition 3.6. *Suppose the assumptions of Theorem 3.5 hold and that the measure β is absolutely continuous with respect to the measure α having Radon-Nikodym derivative $h \in L_2(\alpha)$ such that $\beta(D) = \int_D h(t)\alpha(dt)$ for every $D \in \Sigma$. Then the weak solution of equation (2) belongs to $BV(I,V^*)$.*

Proof. Let x denote the weak solution of the system (2) and define the functional

$$\ell(\varphi) \equiv \int_I < \varphi(t), dx(t) >_{V,V^*}. \tag{34}$$

We show that this is a bounded linear functional on $C(I,V)$. Clearly it follows from the definition of weak solution that

$$\ell(\varphi) = \int_I - < A(dt)\varphi(t), x(t) > + \int_I < \varphi(t), f(t) >_{V,V^*} \alpha(dt). \tag{35}$$

Thus it follows from the assumption (M2-3) that

$$|\ell(\varphi)| \leq \int_I \parallel x(t) \parallel_V \parallel \varphi(t) \parallel_V \beta(dt) + \int_I \parallel \varphi(t) \parallel_V \parallel f(t) \parallel_{V^*} \alpha(dt). \tag{36}$$

Now using the RND of the measure β with respect to the measure α and noting that it is nonnegative, we have

$$\begin{aligned}
|\ell(\varphi)| &\leq \int_I \| x(t) \|_V \| \varphi(t) \|_V \; h(t)\alpha(dt) + \int_I \| \varphi(t) \|_V \| f(t) \|_{V^*} \; \alpha(dt) \\
&\leq \| \varphi \|_{C(I,V)} \left\{ \| x \|_{L_2(\alpha,V)} \| h \|_{L_2(\alpha)} + \| f \|_{L_2(\alpha,V^*)} \sqrt{\alpha}(I) \right\} \\
&\leq b_x \| \varphi \|_{C(I,V)}, \; \forall \; \varphi \in C(I,V), \qquad (37)
\end{aligned}$$

where b_x, given by the expression within the braces, is a finite positive number. Hence $\ell \in (C(I,V))^* \subseteq M_c(\Sigma, V^*)$. In other words the variation of x induces a measure $\mu_x \in M_c(\Sigma, V^*)$ which implies that $x \in BV(I, V^*)$. This completes the proof. □

4. Linear hyperbolic systems

In this section we wish to consider briefly a class of linear hyperbolic systems described by the following evolution equation

$$\begin{aligned}
& d\dot{x} + Axdt + B(dt)\dot{x} = f(t)\nu(dt), t \in I, \\
& x(0) = x_0, \dot{x}(0) = x_1, \qquad (38)
\end{aligned}$$

in a separable Hilbert space H.

We introduce the following assumptions:

(H1): $A \in \mathcal{L}(V, V^*)$ is symmetric and coercive satisfying

$$< Av, v > \geq \gamma \| v \|_V^2, \; \forall \; v \in V$$

for some $\gamma > 0$ so that the positive square root of the operator A, denoted by $\sqrt{A}$ is well defined and it maps V to H.

(H2): The set function ν is a countably additive bounded nonnegative measure (defined on $\Sigma \equiv \sigma(I)$) having bounded total variation on I.

(H3): The operator valued measure $B : \Sigma \longrightarrow \mathcal{L}(V, V^*)$ is countably additive in the weak operator topology satisfying

$$< B(\sigma)h, h >_{V^*,V} \geq \nu(\sigma)|h|_V^2 \; \forall \; \sigma \in \Sigma \text{ and } \forall h \in V.$$

We prove the following a-priori bound.

Lemma 4.1. *Suppose the hypotheses (H1)-(H3) hold. Then any solution, if one exists, of the system (38) corresponding to any $x_0 \in V$, $x_1 \in H$ and $f \in L_2(\nu, V^*)$, must be bounded satisfying $x \in L_\infty(I, V)$, $\dot{x} \in L_\infty(I, H) \cap L_2(\nu, V)$.*

Proof. Suppose x is a weak solution of equation (38) with distributional derivative denoted by $\dot{x}$. Scalar multiplying equation (38) by $\dot{x}$ and then integrating by parts over the interval $[0, t]$, one can easily verify that

$$\begin{aligned} &\{|\dot{x}(t)|_H^2 + < Ax(t), x(t) >\} + 2\int_0^t < B(ds)\dot{x}(s), \dot{x}(s) >_{V^*,V} \\ = \quad &\{|x_1|_H^2 + < Ax_0, x_0 >\} + 2\int_0^t < f(s), \dot{x}(s) > \nu(ds), t \in I. \end{aligned} \tag{39}$$

Using hypothesis (H3) in the above expression we obtain

$$\begin{aligned} &\{|\dot{x}(t)|_H^2 + < Ax(t), x(t) >\} + 2\int_0^t |\dot{x}(s)|_V^2 \nu(ds) \\ \leq \quad &\{|x_1|_H^2 + < Ax_0, x_0 >\} + 2\int_0^t < f(s), \dot{x}(s) > \nu(ds), t \in I. \end{aligned} \tag{40}$$

By virtue of Cauchy inequality applied to the last integral it follows from this that

$$\begin{aligned} &\{|\dot{x}(t)|_H^2 + < Ax(t), x(t) >\} + \int_0^t |\dot{x}(s)|_V^2 \nu(ds) \\ \leq \quad &\{|x_1|_H^2 + < Ax_0, x_0 >\} + \int_0^t |f(s)|_{V^*}^2 \nu(ds), t \in I. \end{aligned} \tag{41}$$

Thus it follows from hypothesis (H1) and the above a-priori estimate that

$$x \in L_\infty(I, V), \dot{x} \in L_\infty(I, H) \cap L_2(\nu, V).$$

This completes the proof. □

Since $A \in \mathcal{L}(V, V^*)$, and it is assumed to be symmetric and positive, the two norms $|\cdot|_V$ and $|\cdot|_{D(\sqrt{A})}$ are equivalent. Hence the solution $x \in L_\infty(I, D(\sqrt{A}))$ also.

Now we can prove the existence of solutions.

Theorem 4.2. *Suppose the assumptions of Lemma 4.1 hold. Then for every $x_0 \in V, x_1 \in H$ and $f \in L_2(\nu, V^*)$, the hyperbolic system has a unique weak solution x satisfying $x \in L_\infty(I, V)$ and $\dot{x} \in L_\infty(I, H) \cap L_2(\nu, V)$.*

Proof. Again by a weak solution we mean any x from $L_\infty(I, V)$ with $\dot{x} \in L_\infty(I, H) \cap L_2(\nu, V)$, that satisfies the identity

$$\begin{aligned} &-\int_I (\dot{x}, \dot{\varphi})_H dt + \int_I < A\varphi(t), x(t) >_{V^*,V} dt + \int_I < \dot{x}, B^*(dt)\varphi(t) >_{V,V^*} \\ &= \int_I < f(t), \varphi >_{V^*,V} \nu(dt) \ \forall \ \varphi \in C_0^1(I, V) \end{aligned}$$

and the initial conditions $x(0) = x_0 \in V, \dot{x}(0) = x_1 \in H$. The proof is based on the same principle as in the parabolic case. So we give only an outline. By use of the projection onto finite dimensional subspaces spanned by $\{v_i, i = 1, 2, \cdots, n\}, n \in N$, one arrives at a second order (in t) evolution equation on R^n. This is then written in the standard way as a first order evolution equation on R^{2n} which is then formulated as an integral equation. Existence of solution of theses finite dimensional problems is proved by fixed point theorem as in the parabolic case. Then using these solutions one constructs the sequence of solutions $\{x^n, \dot{x}^n\}$ for the finite dimensional projections of the (original) infinite dimensional evolution equation. Using the a-priori bounds established in Lemma 4.1, and the facts that V is reflexive and H is Hilbert, one obtains the following convergence results

$$x^n \xrightarrow{w} x^o \text{ in } L_2(I, V),\; x^n \xrightarrow{w^*} x^o \text{ in } L_\infty(I, V)$$
$$\dot{x}^n \xrightarrow{w} \dot{x}^o \text{ in } L_2(\nu, V),\; \dot{x}^n \xrightarrow{w^*} \dot{x}^o \text{ in } L_\infty(I, H).$$

Then using the definition of weak solution and following similar steps as in Theorem 3.5 one can conclude that the system (38) has a unique weak solution. This completes the proof. □

Remark 4.3. Again as in Proposition 3.6, one can verify that $\dot{x} \in BV(I, V^*)$.

Physical implication of the hypothesis (H3) is that the Operator valued measure $B(\cdot)$ is a damping operator providing dissipation of energy. To allow for negative damping, hypothesis (H3) can be replaced by the following assumption,

$(\hat{H}3)$: The operator valued measure $B : \Sigma \longrightarrow \mathcal{L}(V, V^*)$ is countably additive in the weak operator topology satisfying

$$< B(\sigma)h, h > + \nu(\sigma)|h|_V^2 \geq 0 \;\forall\; \sigma \in \Sigma \text{ and } \forall h \in V.$$

Disregarding questions of stability, for any finite time interval I the following result holds.

Corollary 4.4. *Under the modified assumption, the conclusions of Theorem 4.2 remain valid.*

Remark 4.5. The system model (38) can be extended to cover stiffness operators represented by C as follows,

$$d\dot{x} + Axdt + C(dt)x + B(dt)\dot{x} = f(t)\nu(dt), t \in I,$$
$$x(0) = x_0, \dot{x}(0) = x_1.$$

For a semigroup treatment of this system the reader is referred to the recent paper of the author [2].

5. Optimal control

Here we prove some results on optimal control for the parabolic system of section 3. Let U be a Hilbert space and $L_2(\alpha, U)$ the space of α-measurable U valued functions whose norms are square integrable. Consider the Control system

$$dx + A(dt)x = B(t)u(t)\alpha(dt), x(0) = \xi, t \in I, \tag{42}$$

with the cost functional given by

$$J(u) \equiv (1/2) \int_I < Qx, x >_{V^*,V} \alpha(dt) + (1/2) \int_I < Ru, u >_U \alpha(dt) \tag{43}$$

where $Q \in L_\infty(I, \mathcal{L}_n^+(V, V^*))$ and $R \in L_\infty(I, \mathcal{L}^+(U))$. In other words, for each $t \in I$, $Q(t)$ is a positive nuclear operator from V to V^* and $R(t)$ is a positive self adjoint operator in the Hilbert space U. Let $\mathcal{U}_{ad} \subset L_2(\alpha, U)$ denote the class of admissible controls. The problem is to find a control $u^o \in \mathcal{U}_{ad}$ that minimizes the cost functional (43).

We present the following existence result.

Theorem 5.1. *(Existence) Suppose the operator valued measure $A(\cdot)$ and the scalar measure $\alpha(\cdot)$ satisfy the assumptions of Theorem 3.5, and B is a bounded operator valued function uniformly (uniform operator topology) α measurable on I with values in $\mathcal{L}(U, V^*)$. Let $Q \in L_\infty(I, \mathcal{L}_n^+(V, V^*))$, $R \in L_\infty(I, \mathcal{L}^+(U))$ be symmetric, and suppose $\mathcal{U}_{ad}$ is a closed bounded convex subset of $L_2(\alpha, U)$. Then there exists an optimal control $u^o \in \mathcal{U}_{ad}$ minimizing the cost functional (43).*

Proof. Since $\mathcal{U}_{ad}$ is a closed bounded convex subset of $L_2(\alpha, U)$ it is weakly (sequentially) compact. Thus it suffices to verify that $u \longrightarrow J(u)$ is weakly lower semicontinuous. Let $\{u^n\} \in \mathcal{U}_{ad}$ and $\{x^n\} \in B(I, H) \cap L_\infty(I, H) \cap L_2(\alpha, V)$ be the corresponding weak solutions of the system (42). Suppose $u^n \xrightarrow{w} u^o$ in $L_2(\alpha, U)$. Since the set of admissible controls is bounded, it follows from uniform measurability of B that the set $\{Bu^n\}$ is a bounded set in $L_2(\alpha, V^*)$. Thus the sequence of weak solutions is also contained in a bounded subset of $B(I, H) \cap L_\infty(I, H) \cap L_2(\alpha, V)$. Then there exists a unique $x^o \in B(I, H) \cap L_\infty(I, H) \cap L_2(\alpha, V)$ so that along a subsequence, if necessary,

$$x^n \xrightarrow{w} x^o \text{ in } L_2(\alpha, V) \tag{44}$$

$$x^n \xrightarrow{w^*} x^o \text{ in } L_\infty(I, H). \tag{45}$$

Using the definition of weak solution (see Definition 3.4) it is easy to verify that x^o is the weak solution corresponding to the control u^o. Since the cost functional is given by sum of two quadratic functionals and such functionals are weakly lower semi continuous, it follows from weak convergence of u^n to u^o and (43), (44) and (45) that

$$J(u^o) \leq \liminf_{n\to\infty} J(u^n).$$

Hence J is weakly lower semicontinuous on $\mathcal{U}_{ad}$. Since $\mathcal{U}_{ad}$ is weakly compact and J is weakly lower semicontinuous, J attains its infimum on $\mathcal{U}_{ad}$. This completes the proof. □

The following Corollary shows that if the cost functional is radially unbounded, that is, $\lim_{\|u\|\to\infty} J(u) = +\infty$, then it is not necessary for the set $\mathcal{U}_{ad}$ to be weakly compact or even bounded.

Corollary 5.2. *(Existence) Suppose the operator valued measure $A(\cdot)$, the scalar measure $\alpha(\cdot)$ satisfy the assumptions of Theorem 5.1 and B is a bounded operator valued function uniformly (uniform operator topology) α measurable on I with values in $\mathcal{L}(U, V^*)$. Let $Q \in L_\infty(I, \mathcal{L}_n^+(V, V^*))$ symmetric, and suppose there exists an $r > 0$ so that $(R(t)w, w)_U \geq r \parallel w \parallel_U^2$ for all $t \in I$ and $\mathcal{U}_{ad} = L_2(\alpha, U)$. Then there exists an optimal control $u^o \in \mathcal{U}_{ad}$ for the problem (43).*

Proof. Since $J(u) \geq 0$, its radial unboundedness implies that any minimizing sequence must be necessarily bounded. Now any bounded sequence in a reflexive Banach space is relatively weakly sequentially compact. Hence the assertion follows from the fact that J is weakly lower semi continuous. □

Theorem 5.3. *(Optimality Conditions) Suppose the assumptions of Theorem 5.1 hold. Then for $u^o \in \mathcal{U}_{ad}$ to be optimal, it is necessary (and sufficient) that there exists a $\psi \in B(I, H) \cap L_\infty(I, H) \cap L_2(\alpha, V)$ such that*

$$\int_I < B^*(t)\psi(t) + R(t)u^o(t), u(t) - u^o(t) >_U \alpha(dt) \geq 0 \ \forall \ u \in \mathcal{U}_{ad}. \quad (46)$$

The function ψ can be chosen as the weak solution of

$$-d\psi + A^*(dt)\psi = Q(t)x^o(t)\alpha(dt), \psi(T) = 0 \quad (47)$$

where x^o is the weak solution of

$$dx^o + A(dt)x^o = B(t)u^o(t) \ \alpha(dt), x(0) = \xi. \quad (48)$$

Proof. The proof is based on variational principle. Letting $\{u^o, x^o\} \in L_2(\alpha, U) \times B(I, H) \cap L_2(\alpha, V)$ denote the optimal pair, it follows from convexity of the admissible controls and symmetry of Q and R that the Gateaux differential of

J at u^o in the direction $u - u^o$ is given by

$$dJ(u^o, u - u^o) = \int_I < Qx^o, y >_{V^*,V} \alpha(dt) + \int_I < Ru^o, u - u^o >_U \ \alpha(dt) \geq 0, \tag{49}$$

where $y \in B(I, H) \cap L_2(\alpha, V)$ is the weak solution of the variational equation

$$dy + A(dt)y = B(t)(u(t) - u^o(t)) \ \alpha(dt), y(0) = 0. \tag{50}$$

From this equation it is clear that the map $Bu \longrightarrow y$ is continuous from $L_2(\alpha, V^*)$ to $B(I, H) \cap L_2(\alpha, V)$. Since $Q \in L_\infty(I, \mathcal{L}_n^+(V, V^*))$ and $x^o \in L_2(\alpha, V)$ we have $Qx^o \in L_2(\alpha, V^*)$. Hence the map $y \longrightarrow \int_I < Qx^o, y > \ \alpha(dt)$ is a continuous linear functional on $L_2(\alpha, V)$. This implies that the the composition map

$$B(u - u^o) \longrightarrow y \longrightarrow \int_I < Qx^o, y >_{V^*,V} \alpha(dt)$$

is a continuous linear functional on the Banach space $L_2(\alpha, V^*)$. Then by Reisz representation theorem there exists a $\psi \in (L_2(\alpha, V^*))^*$ such that

$$\int_I < Qx^o, y >_{V^*,V} \alpha(dt) = \int_I < \psi, B(u - u^o) >_{V,V^*} \ \alpha(dt). \tag{51}$$

Since V is a reflexive Banach space, $(L_2(\alpha, V^*))^* = L_2(\alpha, V)$ and thus $\psi \in L_2(\alpha, V)$. Substituting (51) into equation (49) we obtain

$$\int_I < B^*\psi + Ru^o, u - u^o >_U \ \alpha(dt) \geq 0, \ \forall \ u \in \mathcal{U}_{ad}. \tag{52}$$

This is precisely the necessary condition (46). Now by virtue of the variational equation (50), the expression (51) can be rewritten as follows

$$\int_I < Qx^o, y >_{V^*,V} \alpha(dt) = \int_I < \psi, B(u - u^o) >_{V,V^*} \ \alpha(dt) = \int_I < \psi, dy + A(dt)y >_{V,V^*} . \tag{53}$$

Since $y(0) = 0$, by integration by parts, it follows from the above expression that

$$\int_I < Qx^o, y >_{V^*,V} \alpha(dt) = < \psi(T), y(T) > + \int_I < -d\psi + A^*(dt)\psi, y >_{V^*,V} . \tag{54}$$

Taking $\psi(T) = 0$, it follows from this that ψ, whose existence was proved above, can be chosen as the weak solution of the following evolution equation,

$$-d\psi + A^*(dt)\psi = Qx^o\alpha(dt), \psi(T) = 0. \tag{55}$$

Thus we have derived the adjoint equation (47). Since the adjoint equation is similar to the system equation, the results of Theorem 3.5 apply and we conclude that equation (55) has a unique weak solution $\psi \in B(I, H) \cap L_2(\alpha, V)$. Equation (48) is the system equation given. This completes the proof of necessary conditions. Proof of sufficient condition is straightforward. Let the control-solution pair $\{u^o, x^o\}$ satisfy the necessary conditions given by the expressions (46-48). Let x denote the solution of the evolution equation (42) corresponding to an arbitrary control $u \in \mathcal{U}_{ad}$. Using these control-solution pairs, it follows from the expression for the cost functional given by (43) that

$$J(u) - J(u^o) \geq \int_I < Qx^o, x - x^o >_{V^*,V} \alpha(dt) + \int_I < Ru^o, u - u^o >_U \alpha(dt).$$

Now using the solution of the adjoint equation (47) in the above inequality we arrive at the following inequality

$$J(u) - J(u^o) \geq \int_I < B^*(t)\psi(t) + R(t)u^o(t), u(t) - u^o(t) >_U \alpha(dt).$$

By virtue of the necessary condition (46) the righthand expression is nonnegative. Hence $J(u) \geq J(u^o)$ for all $u \in \mathcal{U}_{ad}$ proving that u^o is optimal. □

From the above result one can easily derive the following operator Riccati equation in its weak form,

$$\begin{aligned} < dKv, w > &= < Kv, A(dt)w > + < A(dt)v, Kw > \\ &\quad + < R^{-1}B^*Kv, B^*Kw > \alpha(dt) - < Qv, w > \alpha(dt), \\ &\quad K(T) = 0 \end{aligned} \tag{56}$$

for all $v, w \in V$.

Remark 5.4. (Feedback Control) If the assumptions of Corollary 5.2 hold, then it follows from the necessary condition (46) that

$$u^o(t) = -R^{-1}(t)B^*(t)\psi(t), t \in I.$$

Further, if the operator Riccati equation (57) has a weak solution, one can construct the optimal feedback control $u^o(t) = -R^{-1}(t)B^*(t)K(t)x^o(t), t \in I$.

Remark 5.5. We leave the question of existence of solutions of the operator Riccati equation (57) as an open problem.

So far we have considered control problems involving the parabolic system (2). Following the same technique as in the parabolic case, one can prove similar necessary conditions of optimality for the hyperbolic system

$$d\dot{x} + Axdt + C(dt)x + B(dt)\dot{x} = f(t)\nu(dt), t \in I,$$
$$x(0) = x_0, \dot{x}(0) = x_1.$$

6. An example

We present a simple example related to transport of pollutants in a region $\Omega \subset R^3$, for example, an aquatic system or the atmosphere or both. This can be described by a partial differential equation of the form

$$(\partial/\partial t)C - \bigtriangledown \cdot (\delta \bigtriangledown C) + \bigtriangledown \cdot (vC) = S \tag{57}$$

where C denotes the concentration of pollutants as a function of time and space, $C = C(t, \xi)$ $t \geq 0$ and $\xi \in \Omega \subset R^3$, where Ω is the region of concern and S is the source. Here we have used $\bigtriangledown \cdot w \equiv div\ w$ for any vector w. The parameter δ represents the diffusivity of the medium, v is the velocity of the medium (wind or water) carrying the pollutants. In case these are measurable functions of time and space one can introduce the abstract differential operator $A(t)$ as follows. Suppose Ω is an open bounded connected domain having smooth boundary $\partial\Omega$ and assume the Drichlet boundary condition and take $H \equiv L_2(\Omega)$, $V = H_0(\Omega)$ with the corresponding dual $V^* = H^{-1}(\Omega)$. Clearly the injections $V \hookrightarrow H \hookrightarrow V^*$ are continuous and even compact. Define the bilinear functional

$$a(t, \varphi, \psi) \equiv \int_\Omega \{\delta(\bigtriangledown\varphi, \bigtriangledown\psi) + \bigtriangledown \cdot (v\varphi)\ \psi\} d\xi,\ \ \varphi, \psi \in V.$$

Then from this bilinear form one can construct the operator $A(t)$ through the identity

$$a(t, \varphi, \psi) \equiv < A(t)\varphi, \psi >_{V^*.V} .$$

This will be the classical formulation if the coefficients are measurable functions of time and space. Let $I \equiv [0, T]$ denote the time interval of concern and $\Sigma \equiv \sigma(I)$ the sigma algebra of Borel subsets of the set I. In the classical case the parameters $\delta : I \times \Omega \longrightarrow R_+$ and $v : I \times \Omega \longrightarrow R^3$ are defined pointwise, whereas there are physical situations where it is more appropriate to consider them as set functions such as

$$\delta : \Sigma \times \Omega \longrightarrow R_+ \text{ and } v : \Sigma \times \Omega \longrightarrow R^3.$$

This arises naturally if the environment experiences violent activities such as twisters, tornados,cyclones,tsunami etc. In this situation the more appropriate bilinear form a is given by

$$a : \Sigma \times V \times V \longrightarrow R$$

which is a set function in the first argument and linear in the second and third arguments defined on $V \times V$ and taking values $a(\sigma, \varphi, \psi)$. Assuming $\delta : \Sigma \times \Omega \longrightarrow R_+$ and $v : \Sigma \times \Omega \longrightarrow R^3$ bounded, with $v(\sigma, \cdot) \in C^1(\bar{\Omega})$, one can justify the assumption for existence of two countably additive bounded nonnegative measures on Σ denoted by $\alpha(\cdot)$ and $\lambda(\cdot)$ such that

$$a(\sigma, v, v) + \lambda(\sigma)|v|_V^2 \geq \alpha(\sigma) \parallel v \parallel_V^2 \ \ \forall \ \sigma \in \Sigma, v \in V.$$

Let $x(t) \equiv C(t, \cdot)$ denote the H valued function representing the state. Since the source is also a part of the ecology, it is natural to consider it as a vector valued measure $\vartheta : \Sigma \longrightarrow V^*$ and hence the appropriate model for such a system is given by the abstract evolution equation

$$dx + A(dt)x = \vartheta(dt), x(0) \equiv C(0, \cdot). \tag{58}$$

Since V, and hence V^*, has reflexive structure, both have Radon-Nikodym property. The source being a part of the ecology, it is natural for the source measure ϑ to be α continuous. Thus there exists an $f \in L_1(\alpha, V^*)$ such that $d\vartheta = f d\alpha$. In other words, f is the Radon-Nikodym derivative of ϑ with respect to the measure α. Since for the example considered here, V as well as V^* have Hilbertian structure, we may take f to be an element of $L_2(\alpha, V^*)$. Thus the evolution equation (58) is similar to our abstract model

$$dx + A(dt)x = f(t)\alpha(dt), x(0) \equiv C(0, \cdot). \tag{59}$$

Hence we conclude that all the results presented in this paper apply to this ecological system. An interesting control problem for the transport system may be posed as follows: Consider the controlled system

$$dx + A(dt)x = f(t)\alpha(dt) + u(t)\alpha(dt), x(0) \equiv C(0, \cdot), \tag{60}$$

and let $\Gamma \subset H$ be a closed convex set. The objective is to drive the pollution level to the target set Γ with minimum possible cost. In other words find a control that minimizes the objective functional given by

$$J(u) \equiv d(x^u(T), \Gamma) + (1/2) \int_I (Qu, u)_U \alpha(dt), \tag{61}$$

where $d(x, \Gamma)$ denotes the distance of $x \in H$ from the set $\Gamma \subset H$. For this problem, one can prove existence of optimal controls and develop necessary conditions of optimality. We will not pursue this further.

Remark 6.1. The results presented in this paper can be easily extended to semilinear problems with $f = f(t, x)$ mapping H to H and satisfying

$|(f(t,x),x)| \leq a + b|x|_H^2$. The same conclusion holds if $f : I \times V \longrightarrow V^*$ and

$$| < f(t,x), x >_{V^*,V} | \leq K(1 + |x|_V^2)$$

for some $K \in (0,1)$. More general f requires further research.

References

[1] N. U. Ahmed, *Differential Inclusions, Operator Valued Measures and Optimal Control, Special Issue of Dynamic Systems and Applications, Set-Valued Methods in Dynamic Systems*, Guest Editors: M. Michta and J. Motyl **16** (2007), 13-36.

[2] N. U. Ahmed, *Evolution Equations Determined by Operator Valued Measures and Optimal Control*, Nonlinear Analsis:TMA series **67(11)** (2007), 3199-3216, (published on line Dec.1,2007).

[3] N. U. Ahmed and K. L. Teo, *Optimal Control of Distributed Parameter Systems*, North Holland, New York, Oxford, (1981).

[4] N. U. Ahmed, *Optimization and Identification of Systems Governed by Evolution Equations on Banach Spaces*, Pitman Research Notes in Mathematics Series **184** (1988), Longman Scientific and Technical, U. K. and John Wiley, New York.

[5] N. U. Ahmed, *Semigroup Theory with Applications to Systems and Control*, Pitman Research Notes in Mathematics Series **246** (1991), Longman Scientific and Technical, U. K. and John Wiley, New York.

[6] J. L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, New York, Heidelberg, Berlin (1971).

[7] J. Diestel and J. J. Uhl, Jr. *Vector Measures*, Mathematical Surveys, no 15, American Mathematical Society, Providence, Rhode Island, (1977).

[8] N. U. Ahmed, *Some Remarks on the Dynamics of Impulsive Systems in Banach Spaces*, Dynamics of Continuous, Discrete and Impulsive Systems **8** (2001), 261-174.

[9] N. U. Ahmed, *Impulsive Perturbation of C_0-semigroups by Operator Valued Measures*, Nonlinear Functional Analysis & Applications **9(1)** (2004), 127-147.