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EXISTENCE AND UNIQUENESS FOR MILD SOLUTIONS OF THE MHD EQUATIONS IN UNIFORMLY LOCAL L^p SPACES

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Abstract. This paper studies the local existence and uniqueness of mild solutions to the Cauchy problem of the MHD equations in uniformly local L^p spaces $L^p_{uloc,\,\rho}(\mathbb{R}^n)$ for $n < p \leq \infty$ and $\rho > 0$, by using the key $L^p_{uloc,\,\rho} - L^q_{uloc,\,\rho}$ estimates and Banach fixed point theorem. Furthermore, we obtain the existence time estimates of the mild solutions.

1. Introduction

In this paper, we consider the Cauchy problem for the incompressible magnetohydrodynamic (MHD) equations in $\mathbb{R}^n$ where $n \geq 2$:

$$\begin{cases} u_t - \Delta u + (u \cdot \nabla)u - (b \cdot \nabla)b + \nabla p = 0 \\ b_t - \Delta b + (u \cdot \nabla)b - (b \cdot \nabla)u = 0 \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0 \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x) \end{cases} \tag{1.1}$$

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where $x \in \mathbb{R}^n$, $t \geq 0$, $u = u(x,t) = (u_1(x,t), \ldots, u_n(x,t))$, $b = b(x,t) = (b_1(x,t), \ldots, b_n(x,t))$ and $p = p(x,t)$ are non-dimensional quantities corresponding to the flow velocity , the magnetic field and the pressure at the point (x,t), and $u_0(x)$ and $b_0(x)$ are the given initial velocity and initial magnetic field satisfying $\nabla \cdot u_0 = 0$, $\nabla \cdot b_0 = 0$, respectively. For simplicity, we have included the quantity $\frac{1}{2}|b(x,t)|^2$ into $p(x,t)$ and we set the Reynolds number, the magnetic Reynolds number, and the corresponding coefficients to be equal to 1.

It is well-known that there are many works which have constructed different solutions of the MHD equations (1.1) on various function spaces. In the spatial dimension $n > 3$, for any initial data $(u_0, b_0) \in L^2(\mathbb{R}^n)$, the problem (1.1) have been shown to possess at least one global L^2 weak solutions [4, 1]. In [9], Shang established the global existence and uniqueness of strong solutions for the system (1.1) with initial data $u_0 \in PL^n(\mathbb{R}^n) := \{u_0 \in L^n(\mathbb{R}^n), \nabla \cdot u_0 = 0\}$ and $b_0 \in PL^n(\mathbb{R}^n)$. Recently, some improvements were done by Miao etc. In [5], Miao and Yuan studied the Cauchy problem of the MHD system (1.1) for data in larger space than $L^2(\mathbb{R}^2)$ space, the homogeneous Besov space $\dot{B}^{2/p-1}_{p,r}(\mathbb{R}^2)$ for $2 < p < \infty$ and $2 < r < \infty$ to obtain the global existence. And they also proved that for initial data in Besov space $(u_0, b_0) \in \dot{B}^{n/p-1}_{p,r}(\mathbb{R}^n), 1 \leq r \leq \infty, 1 \leq p < \infty$, the Cauchy problem (1.1) have the local strong solutions or global strong solutions for small enough data $(u_0, b_0) \in \dot{B}^{n/p-1}_{p,r}(\mathbb{R}^n)$. In [6], Miao, Yuan and Zhang obtained the global existence of mild solutions to the MHD system (1.1) in the space $BMO^{-1}(\mathbb{R}^n)$ and the local existence of solutions in $BMO^{-1}(\mathbb{R}^n)$. Consequently, the well-posedness of the MHD equations on more general spaces is showed, and the results about well-posedness are improved. Some other related works, we can refer to [2, 3, 7, 8, 10, 12, 13, 14]

Newly, Yasunori and T. Yutaka [11] constructed mild solutions of the incompressible homogeneous Navier-Stokes equations with initial data in uniformly local L^p spaces $L^p_{uloc,\rho}(\mathbb{R}^n)$ where p is grater than or equal to the space dimension n. Inspired by [8], in this paper we will focus on establishing the local existence and uniqueness of the mild solutions to the Cauchy problem of MHD equations (1.1) in $L^p_{uloc,\rho}(\mathbb{R}^n)$ where p is larger than the space dimension n.

The paper is organized as follows. In Section 2, we introduce the definition of uniformly local L^p spaces $L^p_{uloc,\rho}(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ and $\rho > 0$ and give the major results. In Section 3, by using the $L^p_{uloc,\rho}-L^q_{uloc,\rho}$ estimates and Banach fixed point theorem, we construct mild solutions of the MHD equations (1.1) which are local in time. Moreover we obtain the existence time estimates of the mild solutions.

2. Preliminaries and main results

In this section we first introduce the definition of uniformly local L^p spaces $L^p_{uloc,\rho}(\mathbb{R}^n)$ for $1 \le p \le \infty$ and $\rho > 0$. When p is finite, the function space $L^p_{uloc,\rho}(\mathbb{R}^n)$ is defined as follows.

$$\begin{aligned} L^p_{uloc,\rho}(\mathbb{R}^n) &:= \{f \in L^1_{loc}(\mathbb{R}^n);\ \|f\|_{L^p_{uloc,\rho}} \\ &:= \sup_{x\in\mathbb{R}^n} \Big(\int_{|x-y|<\rho} |f(y)|^p\, dy\Big)^{\frac{1}{p}} < \infty\}. \end{aligned}$$

For simplicity of notations we set $L^\infty_{uloc,\rho}(\mathbb{R}^n) := L^\infty(\mathbb{R}^n)$. When p is finite, the space $L^p_{uloc,\rho}$ naturally contains both the space L^p and the space L^∞. We include the parameter ρ here, since the existence time estimate of the mild solutions can be different if ρ is different. More details on the space $L^p_{uloc,\rho}$ can be found in [11].

Now we would like to state the main results of this paper.

Theorem 2.1. *(Existence and uniqueness) Let $n < p \le \infty$ and $\rho > 0$. Then, for all $(u_0, b_0) \in (L^p_{uloc,\rho}(\mathbb{R}^n))^n$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$, there exists $T > 0$ such that the MHD equations (1.1) possess unique mild solutions (u, b) on $(0,T) \times \mathbb{R}^n$, satisfying*

$$\begin{aligned} &(u, b) \in L^\infty((0,T); (L^p_{uloc,\rho})^n), \\ &t^{\frac{n}{2p}}(u, b) \in L^\infty((0,T); (L^\infty)^n). \end{aligned}$$

Theorem 2.2. *(Existence time estimates) Under the same conditions as the Theorem 2.1, the existence time T can be taken as it satisfies*

$$T^{\frac{1}{2}+\frac{n}{2p}}\rho^{-\frac{2n}{p}} + 2T^{\frac{1}{2}}\rho^{-\frac{n}{p}} + T^{\frac{1}{2}-\frac{n}{2p}} \le \frac{\gamma}{\|(u_0, b_0)\|_{L^p_{uloc,\rho}}},$$

where γ is a positive constant depending only on n, p and q.

3. Proof of main results

To solve the MHD equations (1.1) we convert them to the integral equations of the form.

After applying $\mathbb{P}$ the equations (1.1) become

$$\begin{cases} u_t - \Delta u + \mathbb{P}(u \cdot \nabla u) - \mathbb{P}(b \cdot \nabla b) = 0 \\ b_t - \Delta b + \mathbb{P}(u \cdot \nabla b) - \mathbb{P}(b \cdot \nabla u) = 0 \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0 \\ u(x,0) = u_0(x), \quad b(x,0) = b_0(x). \end{cases} \tag{3.1}$$

Then the equations (3.1) is converted into the integral equations by using the theory of semigroups

$$\begin{cases} u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)(s)\,ds \\ \qquad + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(b\otimes b)(s)\,ds, \\ b(t) = e^{t\Delta}b_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes b)(s)\,ds \\ \qquad + \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(b\otimes u)(s)\,ds. \end{cases} \tag{3.2}$$

Here $e^{t\Delta}$ is the heat semigroup, $\mathbb{P}$ is Helmholtz projection, and $f\otimes g := (f_ig_j)_{1\le i,j\le n}$ is a tensor product of $f=(f_1,f_2,\cdots,f_n)$ and $g=(g_1,g_2,\cdots,g_n)$.

Next we define mild solutions of the equations (1.1) as the solutions of the integral equations (3.2) associated with the equations (1.1).

Definition 3.1. *(mild solutions) Let $n<p\le\infty$ and $\rho>0$. Then functions $(u,b)\in(L^\infty((0,T);(L^p_{uloc,\rho})^n)$ are called mild solutions of the MHD equations* (1.1) *on $(0,T)\times\mathbb{R}^n$ if there exist $(u_0,b_0)\in(L^p_{uloc,\rho}(\mathbb{R}^n))^n$ with $\nabla\cdot u_0=0$ and $\nabla\cdot b_0=0$ such that the integral equations* (3.2) *hold.*

Remark 3.2. *By the $L^p_{uloc,\rho}-L^q_{uloc,\rho}$ estimates in the following Lemma* 3.1, *the terms $e^{t\Delta}u_0$, $e^{t\Delta}b_0$, $e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes u)$, $e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(b\otimes b)$, $e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(u\otimes b)$ and $e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(b\otimes u)$ can be defined pointwise if $(u_0,b_0)\in(L^p_{uloc,\rho}(\mathbb{R}^n))^n$ and $(u,b)\in(L^\infty((0,T);(L^p_{uloc,\rho})^n)$, $p\ge1$.*

In order to prove the main results in this paper, we also need the key $L^p_{uloc,\rho}-L^q_{uloc,\rho}$ estimates for the convolution operators, $e^{t\Delta}$, $\nabla e^{t\Delta}$ and $e^{t\Delta}\mathbb{P}\nabla\cdot$ and Banach fixed point theorem, which are the followings.

Lemma 3.3. [11]*($L^p_{uloc,\rho}-L^q_{uloc,\rho}$ estimates) Let $1\le q\le p\le\infty$. Then for any $f\in L^p_{uloc,\rho}$, we have*

$$\|e^{t\Delta}f\|_{L^p_{uloc,\rho}} \le \Big(\frac{C_1}{\rho^{n(\frac{1}{q}-\frac{1}{p})}} + \frac{C_2}{t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}}\Big)\|f\|_{L^q_{uloc,\rho}}, \tag{3.3}$$

$$\|\nabla e^{t\Delta}f\|_{L^p_{uloc,\rho}} \le \Big(\frac{C_3}{t^{\frac{1}{2}}\rho^{n(\frac{1}{q}-\frac{1}{p})}} + \frac{C_4}{t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}}}\Big)\|f\|_{L^q_{uloc,\rho}}. \tag{3.4}$$

For $F\in(L^p_{uloc,\rho})^{n\times n}$,

$$\|e^{t\Delta}\mathbb{P}\nabla\cdot F\|_{L^p_{uloc,\rho}} \le \Big(\frac{C_5}{t^{\frac{1}{2}}\rho^{n(\frac{1}{q}-\frac{1}{p})}} + \frac{C_6}{t^{\frac{n}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}}}\Big)\|F\|_{L^q_{uloc,\rho}} \tag{3.5}$$

holds. Here C_1, C_3, C_5 are positive constants depending only on n, and C_2, C_4, C_6 are positive constants depending only on n, p and q.

Proof of Theorem 2.1. First of all, we define the work space X_T. Let X_T be a Banach space defined as

$$X_T = \{f \in L^\infty((0,T);(L^p_{uloc,\,\rho})^n) \mid t^{\frac{n}{2p}} f(t,\cdot) \in L^\infty((0,T);(L^\infty)^n)\}$$

with norm $\|f\|_{X_T} := \sup_{0<t<T} \|f(t,\cdot)\|_{L^p_{uloc,\,\rho}} + \sup_{0<t<T} t^{\frac{n}{2p}} \|f(t,\cdot)\|_{L^\infty}$.

The proof can be divided into three steps.

Step 1. Estimating for $B(f,g)$. Let us estimate the bilinear form $B(f,g) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(f\otimes g)(s)\,ds$, $0 < t \le T$, where f and g belong to X_T. By the estimate (3.5) in Lemma 3.1 and Hölder's inequality $\|fg\|_{L^{\frac{p}{2}}_{uloc,\,\rho}} \le \|f\|_{L^p_{uloc,\,\rho}} \|g\|_{L^p_{uloc,\,\rho}}$, we have

$$\begin{aligned}
\|B(f,g)\|_{L^p_{uloc,\,\rho}}(t) &\le \int_0^t \|e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(f\otimes g)\|_{L^p_{uloc,\,\rho}}\,ds \\
&\le \int_0^t \Big(\frac{C_5}{(t-s)^{\frac12}\rho^{\frac{n}{p}}} + \frac{C_6}{(t-s)^{\frac12+\frac{n}{2p}}}\Big)\|f\otimes g\|_{L^{\frac{p}{2}}_{uloc,\,\rho}}\,ds \\
&\le \int_0^t \Big(\frac{C_5}{(t-s)^{\frac12}\rho^{\frac{n}{p}}} + \frac{C_6}{(t-s)^{\frac12+\frac{n}{2p}}}\Big)\,ds\,\|f\|_{X_T}\|g\|_{X_T} \\
&= C(t^{\frac12}\rho^{-\frac{n}{p}} + t^{\frac12-\frac{n}{2p}})\|f\|_{X_T}\|g\|_{X_T}
\end{aligned}$$

And we have

$$\begin{aligned}
\|B(f,g)\|_{L_\infty}(t) &\le \int_0^t \|e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(f\otimes g)\|_{L_\infty}\,ds \\
&\le \int_0^t \Big(\frac{C_5}{(t-s)^{\frac12}\rho^{\frac{n}{p}}} + \frac{C_6}{(t-s)^{\frac12+\frac{n}{2p}}}\Big)\|f\otimes g\|_{L^p_{uloc,\,\rho}}\,ds \\
&\le \int_0^t \Big(\frac{C_5}{(t-s)^{\frac12}\rho^{\frac{n}{p}}} + \frac{C_6}{(t-s)^{\frac12+\frac{n}{2p}}}\Big)\frac{1}{s^{\frac{n}{2p}}} s^{\frac{n}{2p}}\|f\|_{L_\infty}\|g\|_{L^p_{uloc,\,\rho}}\,ds \\
&\le \int_0^t \Big(\frac{C_5}{(t-s)^{\frac12}\rho^{\frac{n}{p}}} + \frac{C_6}{(t-s)^{\frac12+\frac{n}{2p}}}\Big)\frac{1}{s^{\frac{n}{2p}}}\,ds\,\|f\|_{X_T}\|g\|_{X_T} \\
&\le C(t^{\frac12-\frac{n}{2p}}\rho^{-\frac{n}{p}} + t^{\frac12-\frac{n}{p}})\|f\|_{X_T}\|g\|_{X_T}
\end{aligned}$$

Thus

$$\sup_{0<t<T} t^{\frac{n}{2p}}\|B(f,g)\|_{L^\infty}(t) \le C(T^{\frac12}\rho^{-\frac{n}{p}} + T^{\frac12-\frac{n}{2p}})\|f\|_{X_T}\|g\|_{X_T}$$

From above estimate we have

$$\begin{aligned}
\|B(f,g)\|_{X_T} &\le C(T^{\frac12}\rho^{-\frac{n}{p}} + T^{\frac12-\frac{n}{2p}})\|f\|_{X_T}\|g\|_{X_T} \\
&= C_T\|f\|_{X_T}\|g\|_{X_T} \qquad (3.6)
\end{aligned}$$

where $C(T^{\frac{1}{2}}\rho^{-\frac{n}{p}}+T^{\frac{1}{2}-\frac{n}{2p}})\triangleq C_T$.
Step 2. Existence of mild solutions. Using the estimate (3.3) in Lemma 3.1 we easily see that

$$\|e^{t\Delta}u_0\|_{X_T}\le C(T^{\frac{n}{2p}}\rho^{-\frac{n}{p}}+1)\|u_0\|_{L^p_{uloc,\,\rho}}$$

$$\|e^{t\Delta}b_0\|_{X_T}\le C(T^{\frac{n}{2p}}\rho^{-\frac{n}{p}}+1)\|b_0\|_{L^p_{uloc,\,\rho}}$$

then

$$\begin{aligned}\|(e^{t\Delta}u_0,e^{t\Delta}b_0)\|_{X_T} &\le C(T^{\frac{n}{2p}}\rho^{-\frac{n}{p}}+1)(\|u_0\|_{L^p_{uloc,\,\rho}}+\|b_0\|_{L^p_{uloc,\,\rho}})\\ &= C(T^{\frac{n}{2p}}\rho^{-\frac{n}{p}}+1)\|(u_0,b_0)\|_{L^p_{uloc,\,\rho}}\end{aligned}$$

Let $C(T^{\frac{n}{2p}}\rho^{-\frac{n}{p}}+1)\|(u_0,b_0)\|_{L^p_{uloc,\,\rho}}\triangleq K_0$, we construct a closed subset E of the space X_T as follows.

$$E=\{f|f\in X_T;\ \|f\|_{X_T}\le 2K_0\}$$

For all $f,g\in E$, we define a metric in E.

$$d(f,g)=\|f-g\|_{X_T}$$

It is easy to deduce that E is a complete metric space.
For simplicity, we write the equations (3.2) in the form

$$\begin{pmatrix} u\\ b\end{pmatrix}=\begin{pmatrix}\Phi_1(u,b)\\ \Phi_2(u,b)\end{pmatrix}\triangleq \mathcal{T}(u,b)$$

If $(u,b)\in E$, by (3.6) we obtain

$$\|B(u,u)\|_{X_T}\le 4C_TK_0,\ \|B(b,b)\|_{X_T}\le 4C_TK_0$$

$$\|B(u,b)\|_{X_T}\le 4C_TK_0,\ \|B(b,u)\|_{X_T}\le 4C_TK_0$$

Taking $K_0<\varepsilon=\frac{1}{8C_T}$, according to the integral equations (3.2), we get

$$\|\mathcal{T}(u,b)\|_{X_T}\le 2K_0$$

This implies that $\mathcal{T}(u,b)\in E$.

On the other hand, for any (u_1,b_1), $(u_2,b_2)\in E$ we have

$$\begin{aligned}&\|\Phi_1(u_1,b_1)-\Phi_1(u_2,b_2)\|_{X_T}\\ \le\ &\|B(u_1,u_1)-B(u_2,u_2)\|_{X_T}+\|B(b_1,b_1)-B(b_2,b_2)\|_{X_T}\\ \le\ &\|B(u_1,u_1-u_2)-B(u_1-u_2,u_2)\|_{X_T}\\ &+\|B(b_1-b_2,b_2)-B(b_1,b_1-b_2)\|_{X_T}\\ \le\ &C_T(\|u_1\|_{X_T}+\|u_2\|_{X_T})\|u_1-u_2\|_{X_T}\\ &+C_T(\|b_1\|_{X_T}+\|b_2\|_{X_T})\|b_1-b_2\|_{X_T}\\ \le\ &4C_TK_0(\|u_1-u_2\|_{X_T}+\|b_1-b_2\|_{X_T}).\end{aligned}\tag{3.7}$$

Similar computation follows that

$$\begin{aligned} & |\Phi_2(u_1,b_1)-\Phi_2(u_2,b_2)\|_{X_T} \\ \leq\ & 2C_T\|b_1\|_{X_T}\|u_1-u_2\|_{X_T}+2C_T\|u_2\|_{X_T}\|b_1-b_2\|_{X_T} \\ \leq\ & 4C_TK_0(\|u_1-u_2\|_{X_T}+\|b_1-b_2\|_{X_T}). \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\|\mathcal{T}(u_1,b_1)-\mathcal{T}(u_2,b_2)\|_{X_T}\leq 8C_TK_0\|(u_1,b_1)-(u_2,b_2)\|_{X_T}.$$

Note that $8C_TK_0<1$, then $\mathcal{T}$ is a contraction mapping in E. So $\mathcal{T}$ have a unique fixed point $(u,b)\in E$, by Banach fixed point theorem. Therefore the integral equations (3.5) have unique solutions $(u,b)\in E$. That is to say ,the MHD equations (1.1) possess unique mild solutions $(u,b)\in E\subseteq X_T$.

Step 3. Uniqueness of mild solutions. Next we show the uniqueness of mild solutions in $L^\infty((0,T);(L^p_{uloc,\rho})^n)$. Let $(u_1,b_1),(u_2,b_2)\in L^\infty((0,T);(L^p_{uloc,\rho})^n)$ be two mild solutions with initial data $(u_0,b_0)\in(L^p_{uloc,\rho}(\mathbb{R}^n))^n$, $p>n$. We set

$$M=max\{\sup_{0<t<T}\|u_1\|_{L^p_{uloc,\rho}}(t),\ \sup_{0<t<T}\|u_2\|_{L^p_{uloc,\rho}}(t)\}.$$

Then

$$\begin{aligned} & \|u_1-u_2\|_{L^p_{uloc,\rho}}(t) \\ \leq\ & \int_0^t\|e^{(t-s)\Delta}\mathbb{P}\nabla\cdot((u_1-u_2)\otimes u_1+u_2\otimes(u_1-u_2))\|_{L^p_{uloc,\rho}}\,ds \\ & +\int_0^t\|e^{(t-s)\Delta}\mathbb{P}\nabla\cdot((b_1-b_2)\otimes b_1+b_2\otimes(b_1-b_2))\|_{L^p_{uloc,\rho}}\,ds \\ \leq\ & \int_0^t(\frac{C_5}{(t-s)^{\frac12}\rho^{\frac np}}+\frac{C_6}{(t-s)^{\frac12+\frac{n}{2p}}})\|((u_1-u_2)\otimes u_1 \\ & +u_2\otimes(u_1-u_2))\|_{L^{\frac p2}_{uloc,\rho}}\,ds \\ & +\int_0^t(\frac{C_5}{(t-s)^{\frac12}\rho^{\frac np}}+\frac{C_6}{(t-s)^{\frac12+\frac{n}{2p}}})\|((b_1-b_2)\otimes b_1 \\ & +u_2\otimes(b_1-b_2))\|_{L^{\frac p2}_{uloc,\rho}}\,ds \end{aligned}$$

$$\begin{aligned}
&\le \int_0^t \Big(\frac{C_5}{(t-s)^{\frac{1}{2}}\rho^{\frac{n}{p}}}+\frac{C_6}{(t-s)^{\frac{1}{2}+\frac{n}{2p}}}\Big)\|u_1-u_2\|_{L^p_{uloc,\,\rho}}(\|u_1\|_{L^p_{uloc,\,\rho}}\\
&\quad +\|u_2\|_{L^p_{uloc,\,\rho}})\,ds+\int_0^t\Big(\frac{C_5}{(t-s)^{\frac{1}{2}}\rho^{\frac{n}{p}}}\\
&\quad +\frac{C_6}{(t-s)^{\frac{1}{2}+\frac{n}{2p}}}\Big)\|b_1-b_2\|_{L^p_{uloc,\,\rho}}(\|b_1\|_{L^p_{uloc,\,\rho}}+\|b_2\|_{L^p_{uloc,\,\rho}})\,ds\\
&\le 2MC(T'^{\frac{1}{2}}\rho^{-\frac{n}{p}}+T'^{\frac{1}{2}-\frac{n}{2p}})\sup_{0<t<T'}\|(u_1,b_1)-(u_2,b_2)\|_{L^p_{uloc,\,\rho}}(t).
\end{aligned}$$

So we have

$$\begin{aligned}
&\sup_{0<t<T'}\|u_1-u_2\|_{L^p_{uloc,\,\rho}}(t)\le 2MC(T'^{\frac{1}{2}}\rho^{-\frac{n}{p}}\\
&+T'^{\frac{1}{2}-\frac{n}{2p}})\sup_{0<t<T'}\|(u_1,b_1)-(u_2,b_2)\|_{L^p_{uloc,\,\rho}}(t). \qquad (3.9)
\end{aligned}$$

Similarly, one can obtain

$$\begin{aligned}
&\sup_{0<t<T'}\|b_1-b_2\|_{L^p_{uloc,\,\rho}}(t)\le 2MC(T'^{\frac{1}{2}}\rho^{-\frac{n}{p}}\\
&+T'^{\frac{1}{2}-\frac{n}{2p}})\sup_{0<t<T'}\|(u_1,b_1)-(u_2,b_2)\|_{L^p_{uloc,\,\rho}}(t). \qquad (3.10)
\end{aligned}$$

Gathering (3.9) and (3.10), we get

$$\begin{aligned}
&\sup_{0<t<T'}\|(u_1,b_1)-(u_2,b_2)\|_{L^p_{uloc,\,\rho}}(t)\\
&\le 2MC(T'^{\frac{1}{2}}\rho^{-\frac{n}{p}}+T'^{\frac{1}{2}-\frac{n}{2p}})\sup_{0<t<T'}\|(u_1,b_1)-(u_2,b_2)\|_{L^p_{uloc,\,\rho}}(t).
\end{aligned}$$

Thus, for sufficiently small $T'>0$ it follows that $(u_1,b_1)=(u_2,b_2)$ in $0<t<T'$. Repeating this argument, we see that $(u_1,b_1)=(u_2,b_2)$ in $0<t<T$.

Proof of Theorem 2.2. According to the previous proof on the existence of mild solutions, we need that

$$K_0<\frac{1}{8C_T}$$

holds.
that is

$$C(T^{\frac{n}{2p}}\rho^{-\frac{n}{p}}+1)\|(u_0,b_0)\|_{L^p_{uloc,\,\rho}}\le\frac{1}{8C(T^{\frac{1}{2}}\rho^{-\frac{n}{p}}+T^{\frac{1}{2}-\frac{n}{2p}})}.$$

It is easy to see the existence time T can be taken as it satisfies

$$T^{\frac{1}{2}+\frac{n}{2p}}\rho^{-\frac{2n}{p}}+2T^{\frac{1}{2}}\rho^{-\frac{n}{p}}+T^{\frac{1}{2}-\frac{n}{2p}}\le\frac{\gamma}{\|(u_0,b_0)\|_{L^p_{uloc,\,\rho}}},$$

where γ is a positive constant depending only on n, p and q.

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