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ON A N-ORDER M-POINT BOUNDARY VALUE PROBLEM AT RESONANCE

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Abstract. By means of the coincidence degree theory of Mawhin, the existence of solutions for higher order multiple point boundary value problem at resonance is investigated. The interesting thing is that the nonlinear term may be noncontinuous.

1. Introduction

The existence of solutions for boundary value problems at resonance has been studied by many authors, we refer the reader to [1-14] and references cited therein. Lu [1], Liu [2] and Du [3] proved the existence of solutions for the following problem:

$$x^{(n)}(t) = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) + e(t),\ t \in (0, 1),$$

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under the following boundary conditions, respectively:

$$x'(0) = x''(0) = \cdots = x^{(n-1)}(0) = 0,\ x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i);$$
$$x'(0) = x''(0) = \cdots = x^{(n-1)}(0) = 0,\ x(1) = \int_0^1 x(s)dy(s);$$
$$x'(0) = x''(0) = \cdots = x^{(n-1)}(0) = 0,\ x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i),\ x(1) = x(\eta),$$

where f is continuous.

Motivated by their results, we study the existence of solutions for the following boundary value problem at resonance:

$$x^{(n)}(t) = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)),\ t \in (0,1), \tag{1.1}$$

$$x(0) = x(\eta), x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0, x^{(n-1)}(1) = \sum_{i=1}^{m-2} \alpha_i x^{(n-1)}(\xi_i), \tag{1.2}$$

where $n \geq 2,\ m \geq 3,\ f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) \in L^1[0,1]$ for any $x(t) \in C^{n-1}[0,1],\ 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1,\ 0 < \eta < 1,\ \alpha_i \in R$. Using the coincidence degree theory of Mawhin, we will prove the existence of solution for the problem (1.1)-(1.2). The interesting thing is that the nonlinear term f may be noncontinuous.

2. Some preliminaries

In order to obtain our results, we introduce some notations and an abstract existence theorem by Mawhin, which can be seen in [15].

Let X and Y be Banach spaces, $L : domL \subset X \to Y$ be a Fredholm operator of index zero, $P : X \to X,\ Q : Y \to Y$ be projectors such that

$$ImP = KerL,\ KerQ = ImL,\ X = KerL \oplus KerP,\ Y = ImL \oplus ImQ.$$

It follows from

$$L\mid_{domL \bigcap KerP}:\ domL \bigcap KerP \to ImL$$

that L is invertible. We denote the inverse of L by K_P.

Suppose Ω is an open bounded subset of X and $domL \bigcap \overline{\Omega} \neq \emptyset$. The map $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.1. *[15] Assume that X, Y are two Banach spaces, L is a Fredholm operator with index zero and N is L-compact on $\overline{\Omega}$. Moreover assume that*

(A_1) $Lx \neq \lambda Nx,\ \forall \lambda \in (0,1)$ *and* $x \in (domL \setminus KerL) \bigcap \partial\Omega$.
(A_2) $Nx \notin ImL,\ \forall x \in KerL \bigcap \partial\Omega$.
(A_3) $deg\{QN \mid_{KerL},\ \Omega \bigcap KerL,\ 0\} \neq 0$.
Then the equation $Lx = Nx$ *has at least one solution in* $domL \bigcap \overline{\Omega}$.

3. Main results

Let $X = C^{n-1}[0,1]$ with norm $\|x\| = max\{\|x\|_\infty, \|x'\|_\infty, \ldots, \|x^{(n-1)}\|_\infty\}$, where $\|x\|_\infty = \max\limits_{t\in[0,1]} |x(t)|$ and $Y = L^1[0,1]$ with norm $\|\cdot\|_1$. $L : domL \subset X \to Y$ is defined as $Lx = x^{(n)}(t)$, where

$$\begin{aligned} domL &= \{x | x^{(n-1)}(t) \text{ is absolutely continuous on } [0,1],\ x(0) = x(\eta), \\ &\quad x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0,\ x^{(n-1)}(1) = \sum_{i=1}^{m-2} \alpha_i x^{(n-1)}(\xi_i)\}. \end{aligned}$$

Let $N : X \to Y$ be defined as

$$Nx(t) = f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)).$$

Then $x(t)$ is a solution of the problem (1.1)-(1.2) if and only if it satisfies $x \in domL$ and $Lx = Nx$.

If $x(t),\ y(t) \in Y$, and $x(t) \overset{a.e.}{=} y(t)$, we define $x(t) = y(t)$.

Theorem 3.1. *Suppose* $f : [0,1] \times R^n \to R$ *maps bounded subset into bounded subset, and* $f(t, x(t), x'(t), \ldots, x^{(n-1)}(t)) \in L^1[0,1]$ *for any* $x(t) \in X$. *In addition, suppose the following conditions are satisfied:*

(H_1) $a = 1 - \sum_{i=1}^{m-2} \alpha_i \xi_i - \frac{\eta}{n}(1 - \sum_{i=1}^{m-2} \alpha_i) \neq 0$;

(H_2) There is a constant $M > 0$ *such that for any* $x \in domL \setminus KerL$, *if* $|x(t)| > M$ *for all* $t \in [0,1]$, *then*

$$\int_0^1 Nx(s)ds - \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} Nx(s)ds - (1 - \sum_{i=1}^{m-2} \alpha_i) \int_0^{\eta} (1 - \frac{s}{\eta})^{n-1} Nx(s)ds \neq 0;$$

(H_3) There exist functions $b(t),\ r(t) \in L^1[0,1]$ *and* $g_i(t,x) : [0,1] \times R \to R$ *with* $g_i(t, x(t)) \in L^1[0,1]$ *for any* $x(t) \in C[0,1]$ *such that:*

$$\begin{aligned} |f(t, x_1, x_2, \ldots, x_n)| &\leq \sum_{i=1}^{n} g_i(t, x_i) + b(t)|x_k|^\theta + r(t). \\ &\quad (t, x_1, x_2, \ldots, x_n) \in [0,1] \times R^n,\ 1 \leq k \leq n,\ 0 \leq \theta < 1, \end{aligned}$$

where $g_i(t,x)$ maps bounded subset of $[0,1]\times R$ into bounded subset of R and

$$\limsup_{|x|\to\infty}\frac{g_i(t,x)}{|x|}=r_i\in[0,1),\ i=1,2,\ldots,n \text{ with } \sum_{i=1}^{n}r_i<1;$$

(H_4) There is a constant $M^*>0$ such that for any $|c|>M^*$ either

$$c[\int_0^1 Ncds-\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}Ncds-(1-\sum_{i=1}^{m-2}\alpha_i)\int_0^{\eta}(1-\frac{s}{\eta})^{n-1}Ncds]>0,$$

or

$$c[\int_0^1 Ncds-\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}Ncds-(1-\sum_{i=1}^{m-2}\alpha_i)\int_0^{\eta}(1-\frac{s}{\eta})^{n-1}Ncds]<0.$$

Then, the problem (1.1)-(1.2) has at least one solution $x(t)\in domL$.

In order to prove Theorem 3.1, we first give some lemmas.

Lemma 3.2. *If (H_1) holds, then*

(i) $L: domL\to Y$ is a Fredholm operator of index zero and the projector $Q:Y\to Y$ can be defined as

$$Qy=\frac{1}{a}[\int_0^1 y(s)ds-\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}y(s)ds-(1-\sum_{i=1}^{m-2}\alpha_i)\int_0^{\eta}(1-\frac{s}{\eta})^{n-1}y(s)ds]$$

and the inverse of $L: domL\bigcap KerP\to ImL$ can be written by

$$K_Py=\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}y(s)ds-\frac{t^{n-1}}{(n-1)!}\int_0^{\eta}(1-\frac{s}{\eta})^{n-1}y(s)ds;$$

(ii) N is $L-$compact on $\overline{\Omega}$, if $\Omega\subset X$ is an open bounded subset with $\overline{\Omega}\bigcap domL\neq\emptyset$.

Proof. (i) It is clear that

$$KerL=\{x\in domL|x=c,\ c\in R\}.$$

We will show that

$$\begin{aligned} ImL &= \{y\in Y|\int_0^1 y(s)ds-\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}y(s)ds \\ &\quad -(1-\sum_{i=1}^{m-2}\alpha_i)\int_0^{\eta}(1-\frac{s}{\eta})^{n-1}y(s)ds=0\}.\end{aligned}$$

In fact, if $x^{(n)}(t) = y(t),\ x(t) \in domL$, we can easily get that $y(t)$ satisfies

$$\int_0^1 y(s)ds - \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} y(s)ds - (1 - \sum_{i=1}^{m-2} \alpha_i) \int_0^{\eta} (1 - \frac{s}{\eta})^{n-1} y(s)ds = 0. \quad (3.1)$$

On the other hand, if $y(t) \in Y$ satisfies (3.1), we take

$$x(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)ds - \frac{t^{n-1}}{(n-1)!} \int_0^{\eta} (1 - \frac{s}{\eta})^{n-1} y(s)ds + c,$$

where $c \in R$. Then, $x(t) \in domL$ and $x^{(n)}(t) = y(t)$.

We will show that ImL is a closed subset of Y. Taking $y_n \in ImL \subset Y,\ y_n \to y \in Y$, we have

$$\begin{aligned} &| \int_0^1 y(s)ds - \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} y(s)ds - (1 - \sum_{i=1}^{m-2} \alpha_i) \int_0^{\eta} (1 - \frac{s}{\eta})^{n-1} y(s)ds| \\ &= | \int_0^1 (y(s) - y_n(s))ds - \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} (y(s) - y_n(s))ds \\ &\quad -(1 - \sum_{i=1}^{m-2} \alpha_i) \int_0^{\eta} (1 - \frac{s}{\eta})^{n-1} (y(s) - y_n(s))ds| \\ &\le 2(1 + \sum_{i=1}^{m-2} \alpha_i)\|y - y_n\|_1 \to 0,\ n \to \infty. \end{aligned}$$

So, $y \in ImL$. i.e. ImL is closed.

Let $Q : Y \to Y$ be defined as

$$Qy = \frac{1}{a}[\int_0^1 y(s)ds - \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} y(s)ds - (1 - \sum_{i=1}^{m-2} \alpha_i) \int_0^{\eta} (1 - \frac{s}{\eta})^{n-1} y(s)ds].$$

We have $Qy \in R$ and $Q(y - Qy) = 0$ for $y \in Y$. i.e. $y - Qy \in ImL$. Considering $ImL \cap R = \{0\}$, we get $Y = ImL \oplus R$. Thus, we have

$$dim\ KerL = dim\ R = co\ dim\ ImL = 1.$$

This implies that L is a Fredholm operator of index zero.

Let $P : X \to X$ be defined as

$$Px = x(0).$$

Then the inverse $K_P : ImL \to domL \bigcap KerP$ of L can be written by

$$K_P y = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)ds - \frac{t^{n-1}}{(n-1)!} \int_0^{\eta} (1 - \frac{s}{\eta})^{n-1} y(s)ds.$$

In fact, for $y \in ImL$, we have

$$(LK_P)y(t) = (K_P y)^{(n)} = y(t).$$

On the other hand, for $x \in domL \bigcap KerP$ we have

$$\begin{aligned}(K_PL)x(t) &= K_Px^{(n)}(t)\\ &= \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}x^{(n)}(s)ds - \frac{t^{n-1}}{(n-1)!}\int_0^\eta (1-\frac{s}{\eta})^{n-1}x^{(n)}(s)ds\\ &= x(t).\end{aligned}$$

So, $K_P = (L|_{domL\cap KerP})^{-1}$.

(ii) Since $\Omega \subset X$ is bounded, we can easily get that $QN(\overline{\Omega})$ is bounded. Using the Ascoli-Arzela theorem, we can prove that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Thus N is $L-$compact on $\overline{\Omega}$.

□

Lemma 3.3. *If conditions (H$_2$) and (H$_3$) hold, then the set* $\Omega_1 = \{x \in domL \setminus KerL | Lx = \lambda Nx$ *for some* $\lambda \in (0,1)\}$ *is bounded .*

Proof. Taking $x \in \Omega_1,$ from $x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0$, we get

$$x^{(i)}(t) = \int_0^t x^{(i+1)}(s)ds,\ i = 1,2,\ldots,n-2.$$

So, we have

$$\|x^{(i)}\|_\infty \le \|x^{(i+1)}\|_1 \le \|x^{(i+1)}\|_\infty,\ i = 1,2,\ldots,n-2. \tag{3.2}$$

From $x(0) = x(\eta),$ and $x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0$, we can get that there exists a $t_0 \in (0,\eta)$ satisfying $x^{(n-1)}(t_0) = 0$. So, we have

$$x^{(n-1)}(t) = \int_{t_0}^t x^{(n)}(s)ds.$$

This implies

$$\|x^{(n-1)}\|_\infty \le \|x^{(n)}\|_1. \tag{3.3}$$

From $Lx = \lambda Nx$, we get $Nx \in ImL$. Therefore, we have

$$\int_0^1 Nx(s)ds - \sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} Nx(s)ds - (1-\sum_{i=1}^{m-2}\alpha_i)\int_0^\eta (1-\frac{s}{\eta})^{n-1}Nx(s)ds = 0.$$

By (H_2), there is a $t_1 \in [0,1]$ such that $x(t_1) \le M$. From $x(t) = x(t_1) + \int_{t_1}^t x'(s)ds$, we can get

$$\|x\|_\infty \le M + \|x'\|_1 \le M + \|x'\|_\infty. \tag{3.4}$$

From (3.2)-(3.4), we get

$$\|x^{(i)}\|_\infty \le \|x^{(n)}\|_1,\ i = 1,2,\ldots,n-1, \tag{3.5}$$

$$\|x\|_\infty \le M + \|x^{(n)}\|_1. \tag{3.6}$$

By (H_3), we can get that there exist constants $\delta > 0$ and $\varepsilon > 0$ such that $\sum_{i=1}^{n}(r_i+\varepsilon) < 1$ and

$$g_i(t,x) < (r_i+\varepsilon)|x|, \text{ for } t \in [0,1], \ |x| > \delta, \ i = 1,2,\ldots,n.$$

Since $g_i(t,x) : [0,1] \times R \to R, \ i = 1,2,\ldots,n$ map bounded subset into bounded subset, there is a constant $M_0 > 0$ such that

$$g_i(t,x) \leq M_0, \text{ for } (t,x) \in [0,1] \times [-\delta,\delta], \ i = 1,2,\ldots,n.$$

It follows from $Lx = \lambda Nx$ and (H_3) that

$$\begin{aligned}
\|x^{(n)}\|_1 &\leq \int_0^1 |f(s,x(s),x'(s),\ldots,x^{(n-1)}(s))|ds \\
&\leq \sum_{i=1}^{n} \int_0^1 g_i(s,x^{(i-1)}(s))ds + \|b\|_1 \cdot \|x^{(k-1)}\|_\infty^\theta + \|r\|_1 \\
&= \sum_{i=1}^{n} [\int_{\triangle_{i,1}} g_i(s,x^{(i-1)}(s))ds + \int_{\triangle_{i,2}} g_i(s,x^{(i-1)}(s))ds] \\
&\quad + \|b\|_1 \cdot \|x^{(k-1)}\|_\infty^\theta + \|r\|_1 \\
&\leq \sum_{i=1}^{n} (r_i+\varepsilon)\|x^{(i-1)}\|_\infty + nM_0 + \|b\|_1 \cdot \|x^{(k-1)}\|_\infty^\theta + \|r\|_1.
\end{aligned}$$

This, together with (3.5)-(3.6), implies

$$\|x^{(n)}\|_1 \leq \frac{\|b\|_1}{1 - \sum_{i=1}^{n}(r_i+\varepsilon)} \|x^{(n)}\|_1^\theta + \frac{nM_0 + \|r\|_1 + (r_1+\varepsilon)M + c}{1 - \sum_{i=1}^{n}(r_i+\varepsilon)},$$

where

$$c = \begin{cases} 0, & 1 < k \leq n, \\ \|b\|_1 M^\theta, & k = 1. \end{cases}$$

So, Ω_1 is bounded.

□

Lemma 3.4. *If (H_4) holds, then $\Omega_2 = \{x | x \in KerL, \ Nx \in ImL\}$ is bounded.*

Proof. By $x \in \Omega_2$, we get $x = c$ and

$$\int_0^1 Ncds - \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} Ncds - (1 - \sum_{i=1}^{m-2} \alpha_i) \int_0^\eta (1 - \frac{s}{\eta})^{n-1} Ncds = 0.$$

From (H_4), we get $|c| \leq M^*$. So, Ω_2 is bounded.

□

Lemma 3.5. *If either the first part of (H_4) holds and $a > 0$, or the second part of (H_4) holds and $a < 0$, then*

$$\Omega_3 = \{x \in KerL | \lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}$$

is bounded;

If either the first part of (H_4) holds and $a < 0$, or the second part of (H_4) holds and $a > 0$, then

$$\Omega_3 = \{x \in KerL | -\lambda Jx + (1-\lambda)QNx = 0,\ \lambda \in [0,1]\}$$

is bounded, where $J : KerL \to ImQ$ is a linear isomorphism given by $J(c) = c,\ \forall c \in R$.

Proof. Suppose the first part of (H_4) holds and $a > 0$. For $x \in KerL$ and $\lambda Jx = (1-\lambda)QNx$ we have

$$\lambda c = -\frac{1-\lambda}{a}[\int_0^1 Ncds - \sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} Ncds - (1-\sum_{i=1}^{m-2}\alpha_i)\int_0^{\eta}(1-\frac{s}{\eta})^{n-1}Ncds]. \tag{3.7}$$

If $\lambda = 0$, by (H_4), we get $|c| \leq M^*$. If $\lambda = 1$, then $c = 0$. For $\lambda \in (0,1)$, if $|c| > M^*$, multiplying two sides of (3.7) by c, we get

$$\begin{aligned}\lambda c^2 = \ & -\frac{1-\lambda}{a}c \cdot [\int_0^1 Ncds - \sum_{i=1}^{m-2}\alpha_i \int_0^{\xi_i} Ncds \\ & -(1-\sum_{i=1}^{m-2}\alpha_i)\int_0^{\eta}(1-\frac{s}{\eta})^{n-1}Ncds] < 0.\end{aligned}$$

A contradiction. So, $|c| \leq M^*$. i.e. Ω_3 is bounded.

Similarly, we can get that Ω_3 is bounded under the other conditions. □

Proof of Theorem 3.1. Let Ω be an open bounded ball centered at zero of X such that $\Omega \supset \bigcup_{i=1}^{3}\overline{\Omega}_i$. By Lemma 3.2, L is a Fredhold operator of index zero and N is $L-$compact on $\overline{\Omega}$. By Lemma 3.3, Lemma 3.4 and $\Omega \supset \overline{\Omega}_1 \bigcup \overline{\Omega}_2$, we have $Lx \neq \lambda Nx$ for $x \in (domL \setminus KerL)\bigcap \partial\Omega,\ \lambda \in (0,1)$ and $Nx \notin ImL$ for $x \in KerL \bigcap \partial\Omega$.

Now, let $H(x,\lambda) = \pm\lambda Jx + (1-\lambda)QNx$. By Lemma 3.5 and $\Omega \supset \overline{\Omega}_3$, we get $H(x,\lambda) \neq 0$ for $(x,\lambda) \in (KerL\bigcap\partial\Omega) \times [0,1]$. Therefore, by homotopy property of degree, we get

$$\begin{aligned} deg(QN|_{KerL},\ \Omega\bigcap KerL,\ 0) &= deg(H(\cdot,0),\ \Omega\bigcap KerL,\ 0) \\ &= deg(H(\cdot,1),\ \Omega\bigcap KerL,\ 0) \\ &= deg(J,\ \Omega\bigcap KerL,\ 0) \neq 0.\end{aligned}$$

It follows from Theorem 2.1 that $Lx = Nx$ has at least one solution in $domL \bigcap \overline{\Omega}$, which is a solution of the problem (1.1)-(1.2).

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