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ON MODIFIED PROJECTION METHOD FOR VARIATIONAL INEQUALITIES

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Abstract. Noor [3,4,6] has suggested and considered a modified projection method for solving the variational inequalities using the technique of updating the solution. In this paper, we prove the convergence of the modified projection method under the relaxed monotonicity, which is weaker condition. Our Proofs are very simple as compared with other methods. The results represent an improvement and refinement of the previous results.

1. Introduction

Variational inequality theory, which was introduced and studied by Stampacchia [11] has emerged as an interesting and fascinating branch of applicable mathematics with wide range of applications in industry, finance, economics, transportation, social, pure and applied sciences. The ideas and techniques of variational inequalities are being used in a variety of diverse fields and proved to be innovative and productive, see [1]-[11]. It is well known that the variational inequalities are equivalent to the fixed point formulation. This alternative formulation has been sued to suggest and analyze projection iterative method for solving variational inequalities and is one of the most used method for solving variational inequalities. The convergence of the projection iterative method requires that the underlying operator must be strongly monotone and Lipschitz continuous, which are very strict conditions.This fact has motivated to modified the projection method for solving variational inequalities using novel and different techniques. Using the technique of updating the solution, Noor [3]-[4] has suggested and analyzed a modified projection method and

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proved the convergence of the modified method under the partially relaxed strongly monotonicity, which is a weaker condition than the cocoercivity. In this paper, we use the relaxed monotonicity to prove the convergence of the modified projection method. Note that the partially relaxed strongly monotonicity implies implies relaxed monotonicity, but the converse is not true. In this respect, our results represent an improvement of the results of Noor [3,4,5].

2. Main Results

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. Let K be a closed convex set in H and $T : H \longrightarrow H$ be a nonlinear operator. We consider the problem of finding $u \in K$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in K, \tag{2.1}$$

which is called the variational inequality introduced and studied by Stampacchia [11]. For the recent applications, numerical results, sensitivity analysis, dynamical systems, physical formulation , see [1]-[11] and the references therein.

We now recall the well known basic concepts and results.

Definition 2.1. *The operator $T : H \longrightarrow H$ is said to be*

(a) monotone, if

$$\langle Tu - Tv, u - v \rangle \quad \geq 0, \quad \forall u, v \in H$$

(b) relaxed monotone, if

$$\langle Tu - Tv, z - v \rangle \geq 0, \quad \forall u, v, z \in H.$$

(c). *partially relaxed strongly monotone, if there exists a constant $\alpha > 0$ such that*

$$\langle Tu - Tv, z - u \rangle \geq \alpha \|z - u\|^2, \quad \forall u, v, z \in H.$$

It is clear that partially relaxed strongly monotonicity implies relaxed monotonicity and relaxed monotonicity implies monotonicity, but the converse is not true. We also note that for $z = u$, partially relaxed strongly monotonicity and relaxed monotonicity reduce to monotonicity.

Lemma 2.2. *Let K be a closed and convex set in H. Then, for a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K,$$

if and only if, $u = P_K z$, *where* P_K *is the projection of* H *onto the closed and convex set* K. *It is well known that the projection* P_K *is nonexpansive.*

Lemma 2.2 plays an important and crucial part in the analysis of variational inequalities. Using Lemma 2.2, one can have the following result.

Lemma 2.3. *The variational inequality (2.1) has a solution* $u \in K$ *if and only if* $u \in K$ *satisfies the relation*

$$u = P_K[u - \rho Tu]. \tag{2.2}$$

Lemma 2.3 implies that variational inequalities (2.1) are equivalent to the fixed-point problem (2.2). This alternative equivalent formulation plays an important part in suggesting and analyzing several iterative methods for solving variational inequalities. This fixed-point formulation has been used to suggest the following iterative method for problem (2.1).

Algorithm 2.4. *For a given* $u_0 \in K$, *compute the approximate solution* u_{n+1} *by the iterative scheme*

$$u_{n+1} = P_K[u_n - \rho Tu_n], \quad n = 0, 1, 2, \ldots$$

It is well known that the convergence of Algorithm 2.4 requires that the operator T must be both strongly monotone and Lipschitz continuous. it is worth mentioning that Noor [7] has studied the convergence criteria of Algorithm 2.4 for the relaxed monotone operators.

Using the technique of the updating the solution, one can rewrite (2.2) in the following form:

$$\begin{aligned} w &= P_K[u - \rho Tu] \\ u &= P_K[w - \rho Tw]. \end{aligned}$$

or equivalent in the form:

$$u = P_K[P_K[u - \rho Tu] - \rho T P_K[u - \rho Tu]].$$

This alternative fixed point formulation is used to suggest the following modified projection method for solving the variational inequalities (2.1).

Algorithm 2.5. *For a given* $u_0 \in K$, *compute the approximate solution* u_{n+1} *by the iterative schemes:*

$$w_n = P_K[u_n - \rho Tu_n] \tag{2.3}$$

$$u_{n+1} = P_K[w_n - \rho Tw_n], \quad n = 0, 1, 2, \ldots \tag{2.4}$$

Algorithm 2.6. *For a given* $u_0 \in K$, *compute the approximate solution* u_{n+1} *by the iterative scheme:*

$$u_{n+1} = P_K[P_K[u_n - \rho Tu_n] - \rho T P_K[u_n - \rho Tu_n]], \quad n = 0, 1, 2, \ldots$$

We would like to mention that Algorithm 2.5 and Algorithm 2.6 are remarkably different than the so-called extragradient method of Koperlevich [2], which is as:

Algorithm 2.7. *For a given $u_0 \in K$, compute the approximate solution u_{n+1} by the iterative schemes:*

$$\begin{aligned} w_n &= P_K[u_n - \rho T u_n] \\ u_{n+1} &= P_K[u_n - \rho T w_n], \quad n = 0, 1, 2, \ldots, \end{aligned}$$

which can be written as

$$u_{n+1} = P_K[u_n - \rho T P_K[u_n - \rho T u_n]], \quad n = 0, 1, 2, \ldots$$

Using Lemma 2.3, one can rewrite (2.3) and (2.4) in the following equivalent forms as:

$$\langle \rho T u_n + w_n - u_n, v - w_n \rangle \geq 0, \quad \forall v \in K \tag{2.5}$$

$$\langle \rho T w_n + u_{n+1} - w_n, v - u_{n+1} \rangle \geq 0, \quad \forall v \in K. \tag{2.6}$$

respectively. This equivalent alternative formulation plays a crucial role in the convergence analysis of the modified projection method. It is clear that this equivalent formulation is independent of the projection. This is another advantage of this formulation.

Theorem 2.8. *Let $u \in K$ be a solution of problem (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 2.5. If $T : H \longrightarrow H$ is relaxed monotone, then*

$$\|u - u_{n+1}\|^2 \leq \|u - w_n\|^2 - \|w_n - u_{n+1}\|^2 \tag{2.7}$$

$$\|u - w_n\|^2 \leq \|u - u_n\|^2 - \|u_n - w_n\|^2. \tag{2.8}$$

Proof. Let $\bar{u} \in K$ be a solution of (2.1). Then, taking $v = u_{n+1}$ in (2.1), we have

$$\langle Tu, u_{n+1} - u \rangle \geq 0. \tag{2.9}$$

Take $v = u$ in (2.6) to have

$$\langle \rho T w_n + u_{n+1} - w_n, u - u_{n+1} \rangle \geq 0, \quad \forall v \in K. \tag{2.10}$$

Adding (2.9), (2.10) and using the relaxed monotonicity of T, we have

$$\langle u_{n+1} - w_n, u - u_{n+1} \rangle \geq \rho \langle T w_n - Tu, u_{n+1} - u \rangle \geq 0, \tag{2.11}$$

from which, using

$$2\langle v, u \rangle = \|u + v\|^2 - \|u\|^2 - \|v\|^2, \tag{2.12}$$

we have

$$\|u - u_{n+1}\|^2 \leq \|u - w_n\|^2 - \|w_n - u_{n+1}\|^2,$$

the required (2.7).

Taking $v = u$ in (2.5) and $v = w_n$ in (2.1), we have

$$\langle \rho Tu_n + w_n - u_n, u - w_n\rangle \geq 0. \quad (2.13)$$

$$\langle Tu, w_n - u\rangle \geq 0. \quad (2.14)$$

Adding (2.13) and (2.14), rearranging the terms and using the relaxed monotonicity of T, we have

$$\langle w_n - u_n, u - w_n\rangle \geq \rho\langle Tu_n - Tu, w_n - u\rangle \geq 0,$$

from which, using (2.12), we have

$$\|u - w_n\|^2 \leq \|u - u_n\|^2 - \|u_n - w_n\|^2,$$

the required (2.8). □

Theorem 2.9. *Let $u \in K$ be a solution of (2.1) and u_{n+1} be the approximate solution obtained from Algorithm 2.5. Then, $\lim_{n\longrightarrow\infty} u_{n+1} = u$.*

Proof. Let $\bar{u} \in K$ be a solution of problem (2.1). From (2.7) and (2.8), it follows that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} \|w_n - u_{n+1}\|^2 \leq \|w_0 - u\|^2$$

$$\sum_{n=0}^{\infty} \|u_n - w_n\|^2 \leq \|u_0 - u\|^2$$

which implies that

$$\lim_{n\to\infty} \|w_n - u_{n+1}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|w_n - u_n\| = 0.$$

Thus

$$\lim_{n\to\infty} \|u_n - u_{n+1}\| = \lim_{n\to\infty} \|u_n - w_n\| + \lim_{n\to\infty} \|w_n - u_{n+1}\| = 0. \quad (2.15)$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequence $\{u_{n_j}\}$ converges to $\hat{u}$. Replacing u_{n+1} and w_n by u_{n_j} in (2.5) and (2.6) and taking the limit, we have

$$\langle T\hat{u}, v - \hat{u}\rangle \geq 0, \quad \forall v \in K.$$

This shows that $\hat{u} \in K$ is a solution of the variational inequality and

$$\|u_{n+1} - \hat{u}\|^2 \leq \|u_n - \hat{u}\|^2.$$

It follows from the above inequality that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\to\infty} u_n = \hat{u}$, from which it follows that $\lim_{n\to\infty} u_n = \hat{u}$. the required result. □

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