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GENERALIZED MIXED VARIATIONAL INCLUSIONS WITH FUZZY MAPPINGS

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Abstract. In this paper, we consider and study a class of generalized mixed variational inclusions with fuzzy mappings involving H-monotone operator in real Hilbert spaces. By using the resolvent operator technique, we construct the iterative algorithm for finding approximate solutions of generalized mixed variational inclusions with fuzzy mappings. Further, we prove the existence of solutions for this class of problems and discuss the convergence of iterative sequences generated by iterative algorithm.

1. Introduction

Variational inequality theory has become very effective and powerful tool for studying a wide range of problems arising in differential equations, mechanics, management sciences, operations research, contact problems in elasticity, general equilibrium problems in economics and transportation, optimization and control problems etc. Hassouni and Moudafi [11] introduced and studied a class of mixed type variational inequalities with single-valued mappings, which was called variational inclusions. Since then, many authors have obtained important extensions and generalizations of the results [11] in different directions,

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see [1, 8, 13, 14, 18, 19]. Verma [16, 17] introduced and studied some system of variational inequalities and iterative algorithms to compute approximate solutions. Fang and Huang [9, 10] introduced a class of H-monotone operators and studied a new class of variational inclusions involving H-monotone operators.

In 1989, Chang and Zhu [6] introduced and studied a class of variational inequality for fuzzy mappings. Recently, various classes of variational inequalities and inclusions for fuzzy mappings were considered by Chang [4], Chang and Huang [5], Ding and Park [7] and Lee et al. [14].

In this paper, under assumptions that H is strongly monotone continuous and single-valued, we first prove that a multivalued operator is H-monotone if and only if it is maximal monotone. Subsequently, we define the resolvent operator associated with a strongly H-monotone operator, prove its Lipschitz continuity and estimate its Lipschitz constant. Further we study the variational inclusions for fuzzy mappings with strongly H-monotone operators, and suggest and analyze some iterative algorithms for finding approximate solutions of generalized mixed variational inclusions for fuzzy mappings. Further we prove the existence of solutions for this class of problems and discuss the convergence of iterative sequences generated by these iterative algorithms.

2. Preliminaries

Throughout in this paper, we assume that $\mathcal{H}$ is a real Hilbert space endowed with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ respectively. Let $2^{\mathcal{H}}$ denotes the family of all nonempty subsets of $\mathcal{H}$, $CB(\mathcal{H})$ the family of all nonempty closed and bounded subsets of $\mathcal{H}$, $D(\cdot,\cdot)$ the Hausdorff metric on $CB(\mathcal{H})$ defined by

$$D(A,B)=\max\left\{\sup_{x\in A}\inf_{y\in B} d(x,y),\ \sup_{y\in B}\inf_{x\in A} d(x,y)\right\};\ A,B\in CB(\mathcal{H}).$$

We denote the collection of all fuzzy sets of $\mathcal{H}$ by $\mathcal{F}(\mathcal{H})=\{\mu:\mathcal{H}\to I=[0,1]\}$. A mapping $F:\mathcal{H}\to\mathcal{F}(\mathcal{H})$ is called a fuzzy mapping on $\mathcal{H}$. If F is a fuzzy mapping then $F(x)$ (in the sequel we shall denote by F_x) a fuzzy set on $\mathcal{H}$ and $F_x(y)$ is the membership function of y in F_x.

Let $M\in\mathcal{F}(\mathcal{H})$, $q\in[0,1]$. The set $(M)_q=\{x\in\mathcal{H}:\mu(x)\geq q\}$ is called a q-cut set of M. A fuzzy mapping $F:\mathcal{H}\to\mathcal{F}(\mathcal{H})$ is said to be closed if for any $x\in\mathcal{H}$ the function $F_x(y)$ is upper semicontinuous with respect to y, i.e., for any given point $y_0\in\mathcal{H}$ and any net $\{y_\alpha\}\subset\mathcal{H}$, when $y_n\to y_0$, we have $F_x(y_0)\geq\limsup_{\alpha} F_x(y_\alpha)$. Let $F,G,E:\mathcal{H}\to\mathcal{F}(\mathcal{H})$ be the closed fuzzy mappings satisfying the following condition:

Condition (S). There exist functions $a,b,c:\mathcal{H}\to[0,1]$ such that for all $x\in\mathcal{H}$, we have $(F_x)_{a(x)}$, $(G_x)_{b(x)}$, $(E_x)_{c(x)}\in CB(\mathcal{H})$ by $\tilde{F}(x)=(F_x)_{a(x)}$,

$\tilde{G}(x) = (G_x)_{b(x)}$ and $\tilde{E}(x) = (E_x)_{c(x)}$. In the sequel $\tilde{F}$, $\tilde{G}$ and $\tilde{E}$ are called the multivalued mappings induced by the fuzzy mappings F, G, E respectively. More precisely, let $N : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be single-valued mapping and $F, G, E : \mathcal{H} \to \mathcal{F}(\mathcal{H})$ the fuzzy mappings.

Find $u, x, y, z \in \mathcal{H}$ such that $F_u(x) \geq a(u),\ G_u(y) \geq b(u),\ E_u(z) \geq c(u),$

$$0 \in N(x, y) + M(u, z) \tag{2.1}$$

where $M : \mathcal{H} \times \mathcal{H} \to 2^{\mathcal{H}}$ is a multivalued mapping such that for each $t \in \mathcal{H}$, $M(\cdot, t) : \mathcal{H} \to 2^{\mathcal{H}}$ is maximal H-monotone, then the problem (2.1) is called *generalized mixed variational inclusion for fuzzy mappings.*

We need the following pertinent definitions and concepts.

Definition 2.1. *Let $T, H : \mathcal{H} \to \mathcal{H}$ be two single-valued operators, T is said to be*

(i) *monotone if*

$$\langle Tu - Tv,\, u - v\rangle \ \geq\ 0, \ \textit{for all } u, v \in \mathcal{H},$$

(ii) *strictly monotone if T is monotone and*

$$\langle Tu - Tv,\, u - v\rangle \ =\ 0 \ \Leftrightarrow\ u \ =\ v,$$

(iii) *strongly monotone if there exists some constant $r > 0$ such that*

$$\langle Tu - Tv,\, u - v\rangle \ \geq\ r\|u - v\|^2, \ \textit{for all } u, v \in \mathcal{H},$$

(iv) *strongly monotone with respect to H if there exists some constant $\gamma > 0$ such that*

$$\langle Tu - Tv,\, Hu - Hv\rangle \ \geq\ \gamma\|u - v\|^2, \ \textit{for all } u, v \in \mathcal{H},$$

(v) *Lipschitz continuous if there exists some constant $s > 0$ such that*

$$\|Tu - Tv\| \ \leq\ s\|u - v\|, \ \textit{for all } u, v \in \mathcal{H}.$$

Remark 2.2. *If T and H are Lipschitz continuous with constants s and τ respectively, and T is strongly monotone with respect to H with constant γ, then $\gamma \leq \tau s$.*

Definition 2.3. *A multivalued operator $M : \mathcal{H} \to 2^{\mathcal{H}}$ is said to be*

(i) *monotone if*

$$\langle x - y,\, u - v\rangle \ \geq\ 0, \ \textit{for all } u, v \in \mathcal{H}, \ x \in Mu, \ y \in Mv,$$

(ii) *strongly monotone if there exists some constant $\eta > 0$ such that*

$$\langle x - y,\, u - v\rangle \ \geq\ \eta\|u - v\|^2, \ \textit{for all } u, v \in \mathcal{H}, \ x \in Mu, \ y \in Mv,$$

(iii) *maximal monotone if M is monotone and $(I + \lambda M)(\mathcal{H}) = \mathcal{H}$ for all $\lambda > 0$, where I denotes the identity mapping on $\mathcal{H}$,*

(iv) *maximal strongly monotone if M is strongly monotone and*

$$(I + \lambda M)(\mathcal{H}) \;=\; \mathcal{H}, \text{ for all } \lambda > 0.$$

Remark 2.4. *A multivalued operator M is maximal monotone if and only if M is monotone and there is no other monotone operator whose graph properly contains the graph $Gr(M)$ of M, where*

$$Gr(M) \;=\; \{(u, x) \in \mathcal{H} \times \mathcal{H} : x \in M(u)\}.$$

Definition 2.5. [9, 10] *Let $H : \mathcal{H} \to \mathcal{H}$ be a single-valued operator and $M : \mathcal{H} \to 2^{\mathcal{H}}$ a multivalued operator. M is said to be*

(i) *H-monotone if M is monotone and $(H + \lambda M)(\mathcal{H}) = \mathcal{H}$ holds for every $\lambda > 0$,*

(ii) *strongly H-monotone if M is strongly monotone and $(H + \lambda M)(\mathcal{H}) = \mathcal{H}$ holds for every $\lambda > 0$.*

Remark 2.6. *If $H = I$, then the definition of I-monotone operator reduces to the maximal monotone operators. As a matter of fact, the class of H-monotone operator has close relation with the maximal monotone operator.*

Proposition 2.7. [9, 10] *Let $H : \mathcal{H} \to \mathcal{H}$ be a single-valued strictly monotone operator and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a H-monotone operator. Then M is maximal monotone.*

Let X be a real Banach space with norm $\|\cdot\|$, X^* the dual space of X and $\langle x, f\rangle$ denotes the value of $f \in X^*$ at $x \in X$. For $\kappa \in (-\infty, \infty)$, a fuzzy mapping $\tilde{A} : D(\tilde{A}) \subset X \to \tilde{F}(X)$ is said to be κ-accretive if for each $x, y \in D(\tilde{A})$ there exists $j(u - v) \in J(u - v)$ such that

$$\langle x - y,\, j(u - v)\rangle \;\geq\; \kappa\|u - v\|^2, \text{ for all } x \in \tilde{A}(u),\; y \in \tilde{A}(v), \tag{2.2}$$

where $J : X \to 2^{X^*}$ is the normalized duality mapping defined by

$$J(x) \;=\; \{f \in X^* :\; \langle x, f\rangle \;=\; \|x\|^2 \;=\; \|f\|^2\},$$

and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is an immediately consequence of Hahn-Banach Theorem that $J(x)$ is nonempty for each $x \in X$. Moreover it is known that J is single-valued if and only if X is smooth. For $\kappa > 0$ in inequality (2.2), $\tilde{A}$ is strongly accretive while for $\kappa = 0$, $\tilde{A}$ is simply accretive. In addition if the range of $I + \lambda\tilde{A}$ is precisely X for all $\lambda > 0$, where I is an identity mapping on X, then $\tilde{A}$ is said to be m-accretive. If $X = \mathcal{H}$, a real Hilbert space, then the definitions of strong accretivity and m-accretivity reduce to the strong, monotonicity and maximal monotonicity, respectively.

Proposition 2.8. [13] *Let X be a smooth Banach space, $A : D(\tilde{A}) \subset X \to 2^X$ a m-accretive and $S : D(S) \subset X \to X$ a continuous and strongly accretive with $\overline{D(\tilde{A})} \subset D(S)$, where $D(\tilde{A})$ and $D(S)$ are the domain of $\tilde{A}$ and S respectively, and $\overline{D(\tilde{A})}$ is the closure of $D(\tilde{A})$. Then for each $x \in X$, the equation $z \in Sx + \lambda \tilde{A}x$ has a unique solution x_λ for $\lambda > 0$.*

Corollary 2.9. *Let $\mathcal{H}$ be a real Hilbert space. Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be a maximal monotone multivalued operator and $H : \mathcal{H} \to \mathcal{H}$ a strongly monotone, continuous and single-valued operator. Then for each $z \in \mathcal{H}$ the equation $z \in Hx + \lambda Mx$ has a unique solution x_λ for $\lambda > 0$.*

Remark 2.10. *If $H : \mathcal{H} \to \mathcal{H}$ is a strongly monotone, continuous single-valued and $M : \mathcal{H} \to 2^{\mathcal{H}}$ a maximal monotone multivalued operator, then from Corollary 2.1, the operator $(H + \lambda M)^{-1}$ is single-valued. Hence we can define the resolvent operator $R^H_{M,\lambda} : \mathcal{H} \to \mathcal{H}$ as follows:*

$$R^H_{M,\lambda}(u) \;=\; (H + \lambda M)^{-1}(u) \text{ for all, } u \in \mathcal{H}. \tag{2.3}$$

Theorem 2.11. *Let $H : \mathcal{H} \to \mathcal{H}$ be a strongly monotone continuous and single-valued operator, then a multivalued operator $M : \mathcal{H} \to 2^{\mathcal{H}}$ is H-monotone if and only if M is maximal monotone.*

Proof. At first, let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be H-monotone. Since $H : \mathcal{H} \to \mathcal{H}$ is strongly monotone, H is strictly monotone. Thus it follows from Proposition 2.1 that M is maximal monotone.

Conversely, suppose that M is maximal monotone, then M is monotone. Note that H is strongly monotone, continuous and single-valued operator. Hence it follows from Corollary 2.1, that for each $z \in \mathcal{H}$, the equation $z \in Hx + \lambda Mx$ has a unique solution x_λ for $\lambda > 0$. This implies that $(H + \lambda M)(\mathcal{H}) = \mathcal{H}$ holds for every $\lambda > 0$. Therefore, M is H-monotone. ☐

Corollary 2.12. *Let $H : \mathcal{H} \to \mathcal{H}$ be a strongly monotone continuous and single-valued operator. Then a multivalued operator $M : \mathcal{H} \to 2^{\mathcal{H}}$ is strongly H-monotone if and only if M is maximal strongly monotone.*

Theorem 2.13. [10] *Let $H : \mathcal{H} \to \mathcal{H}$ be continuous and strongly monotone with constant γ. Let $M : \mathcal{H} \to 2^{\mathcal{H}}$ be maximal strongly monotone with constant*

η. Then the resolvent operator $R^H_{M,\lambda} : \mathcal{H} \to \mathcal{H}$ is Lipschitz continuous with constant $\frac{1}{\gamma + \lambda\eta}$, i.e.,

$$\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\| \ \leq \ \frac{1}{\gamma + \lambda\eta}\|u - v\|, \text{ for all } u, v \in \mathcal{H}.$$

Proof. Let u, v be any given points in $\mathcal{H}$. It follows from (2.3) that

$$R^H_{M,\lambda}(u) \ = \ (H + \lambda M)^{-1}(u)$$

and

$$R^H_{M,\lambda}(v) \ = \ (H + \lambda M)^{-1}(v).$$

This implies that

$$\frac{1}{\lambda}(u - H(R^H_{M,\lambda}(u))) \in M(R^H_{M,\lambda}(u))$$

and

$$\frac{1}{\lambda}(v - H(R^H_{M,\lambda}(v))) \in M(R^H_{M,\lambda}(v)).$$

Since M is strongly monotone, we have

$$\begin{aligned}
& \eta\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\|^2 \\
\leq \ & \frac{1}{\lambda}\langle u - H(R^H_{M,\lambda}(u)) - (v - H(R^H_{M,\lambda}(v))), \, R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\rangle \\
\leq \ & \frac{1}{\lambda}\langle u - v - (H(R^H_{M,\lambda}(u)) - H(R^H_{M,\lambda}(v))), \, R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
& \|u - v\| \, \|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\| \\
\geq \ & \langle u - v, \, R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\rangle \\
\geq \ & \langle H(R^H_{M,\lambda}(u)) - H(R^H_{M,\lambda}(v)), \, R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\rangle \\
& + \lambda\eta\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\|^2 \\
\geq \ & \gamma\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\|^2 + \lambda\eta\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\|^2 \\
= \ & (\gamma + \lambda\eta)\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\|^2
\end{aligned}$$

and hence

$$\|R^H_{M,\lambda}(u) - R^H_{M,\lambda}(v)\| \ \leq \ \frac{1}{\gamma + \lambda\eta}\|u - v\|$$

for all $u, v \in \mathcal{H}$. □

3. Existence and Convergence Theory

In this section, we develop the iterative algorithm, then we prove the existence and convergence of problem (2.1).

Theorem 3.1. *Let $\mathcal{H}$ be a real Hilbert space, $F, G, E : \mathcal{H} \to \mathcal{F}(\mathcal{H})$ the closed fuzzy mappings satisfying the Condition (S) and $\tilde{F}, \tilde{G}, \tilde{E} : \mathcal{H} \to CB(\mathcal{H})$ the multivalued mappings induced by F, G, E respectively. Let $H : \mathcal{H} \to \mathcal{H}$ and $N : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be the single-valued mappings, then the following statements are equivalent:*

(i) *$u \in \mathcal{H}$, $x \in \tilde{F}(u)$, $y \in \tilde{G}(u)$, $z \in \tilde{E}(u)$ are solution of the problem (2.1),*

(ii) *$u \in \mathcal{H}$, $x \in \tilde{F}(u)$, $y \in \tilde{G}(u)$, $z \in \tilde{E}(u)$ are the solution of the equation*

$$u = R^H_{M(\cdot,z),\lambda}(H(u) - \lambda N(x, y)), \tag{3.1}$$

where $\lambda > 0$ is a constant and $R^H_{M(\cdot,z),\lambda}$ is the resolvent operator associated with $M(\cdot, z)$.

Proof. $u \in \mathcal{H}$, $x \in \tilde{F}(u)$, $y \in \tilde{G}(u)$ and $z \in \tilde{E}(u)$ are solution of problem (2.1) if and only if

$$\begin{aligned} 0 &\in N(x, y) + M(u, z) \\ &\Leftrightarrow H(u) - \lambda N(x, y) \in H(u) + \lambda M(u, z) \\ &\Leftrightarrow H(u) - \lambda N(x, y) \in (H + \lambda M(\cdot, z))(u) \\ &\Leftrightarrow u = (H + \lambda M(\cdot, z))^{-1}(H(u) - \lambda N(x, y)) \\ &\Leftrightarrow u = R^H_{M(\cdot,z),\lambda}(H(u) - \lambda N(x, y)). \end{aligned}$$

□

Algorithm 3.2. *Assume that $N : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and $H : \mathcal{H} \to \mathcal{H}$ are two single-valued mappings. Let $F, G, E : \mathcal{H} \to \mathcal{F}(\mathcal{H})$ be closed fuzzy mappings satisfying Condition (S) and $\tilde{F}, \tilde{G}, \tilde{E} : \mathcal{H} \to CB(\mathcal{H})$ the multivalued mappings induced by fuzzy mappings F, G, E, respectively. For given $u_0 \in \mathcal{H}$, $x_0 \in \tilde{F}(u_0)$, $y_0 \in \tilde{G}(u_0)$ and $z_0 \in \tilde{E}(u_0)$, let*

$$u_1 = (1 - \mu)u_0 + \mu R^H_{M(\cdot,z_0),\lambda}[H(u_0) - \lambda N(x_0, y_0)].$$

Since $x_0 \in \tilde{F}(u_0) \in CB(\mathcal{H})$, $y_0 \in \tilde{G}(u_0) \in CB(\mathcal{H})$, $z_0 \in \tilde{E}(u_0) \in CB(\mathcal{H})$ by Nadler [15] there exist $x_1 \in \tilde{F}(u_1)$, $y_1 \in \tilde{G}(u_1)$, $z_1 \in \tilde{E}(u_1)$ such that

$$\begin{aligned} F_{u_0}(x_0) &\geq a(u_0), \ \|x_0 - x_1\| \leq (1 + 1)D(\tilde{F}(u_0), \tilde{F}(u_1)), \\ G_{u_0}(y_0) &\geq b(u_0), \ \|y_0 - y_1\| \leq (1 + 1)D(\tilde{G}(u_0), \tilde{G}(u_1)), \\ E_{u_0}(z_0) &\geq c(u_0), \ \|z_0 - z_1\| \leq (1 + 1)D(\tilde{E}(u_0), \tilde{E}(u_1)), \end{aligned}$$

where $D(\cdot,\cdot)$ is a Hausdorff metric on $CB(\mathcal{H})$. Let

$$u_2 \;=\; (1-\mu)u_1 + \mu R^H_{M(\cdot,z_1),\lambda}[H(u_1) - \lambda N(x_1, y_1)].$$

By induction, we can define the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ as

$$u_{n+1} \;=\; (1-\mu)u_n + \mu R^H_{M(\cdot,z_n),\lambda}[H(u_n) - \lambda N(x_n, y_n)], \tag{3.2}$$

$$F_{u_n}(x_n) \;\geq\; a(u_n),\ \|x_n - x_{n+1}\| \;\leq\; (1+(n+1)^{-1})D(\tilde{F}(u_n),\, \tilde{F}(u_{n+1}))$$
$$G_{u_n}(y_n) \;\geq\; b(u_n),\ \|y_n - y_{n+1}\| \;\leq\; (1+(n+1)^{-1})D(\tilde{G}(u_n),\, \tilde{G}(u_{n+1}))$$
$$E_{u_n}(z_n) \;\geq\; c(u_n),\ \|z_n - z_{n+1}\| \;\leq\; (1+(n+1)^{-1})D(\tilde{E}(u_n),\, \tilde{E}(u_{n+1}))$$

where $\mu > 0$, $\lambda > 0$ for $n \geq 0$.

Definition 3.3. *Let $N : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and $H : \mathcal{H} \to \mathcal{H}$ be single-valued mappings and $\tilde{F}, \tilde{G} : \mathcal{H} \to CB(\mathcal{H})$ be multivalued mappings,*

(i) *$\tilde{F}$ is said to be strongly H-monotone with respect to H in the first argument of N, if there exists a constant $r > 0$ such that*

$$\langle N(x_1, y_1) - N(x_2, y_1),\, H(u_1) - H(u_2)\rangle \;\geq\; r\|u_1 - u_2\|^2$$

for all $u_i \in \mathcal{H}$ and $x_i \in \tilde{F}(u_i)$, $i = 1, 2$;

(ii) *$\tilde{G}$ is said to be relaxed H-monotone with respect to H in the second argument of N, if there exists a constant $\rho > 0$ such that*

$$\langle N(x_1, y_1) - N(x_1, y_2),\, H(u_1) - H(u_2)\rangle \;\geq\; -\rho\|u_1 - u_2\|^2$$

for all $u_i \in \mathcal{H}$ and $y_i \in \tilde{G}(u_i)$, $i = 1, 2$;

(iii) *$\tilde{F}$ is said to be Lipschitz continuous in the first argument of N, if there exists a constant $\beta > 0$ such that*

$$\|N(x_1, \cdot) - N(x_2, \cdot)\| \;\leq\; \beta\|u_1 - u_2\|$$

for all $u_i \in \mathcal{H}$ and $x_i \in \tilde{F}(u_i)$, $i = 1, 2$. In a similar way, we can define the Lipschitz continuity of N with respect to the second argument.

(iv) *$\tilde{F}$ is said to be Lipschitz continuous, if there exists a constant $\alpha > 0$ such that*

$$\hat{D}(\tilde{F}(u_1),\, \tilde{F}(u_2)) \;\leq\; \alpha\|u_1 - u_2\| \quad \textit{for all } u_i \in \mathcal{H}\ i = 1, 2.$$

Theorem 3.4. *Let $H : \mathcal{H} \to \mathcal{H}$ be a strongly monotone and Lipschitz continuous operator with constants γ and τ respectively. Let $N : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be the single-valued mapping. Assume that $F, G, E : \mathcal{H} \to \mathcal{F}(\mathcal{H})$ are closed fuzzy mappings satisfying Condition (S) and $\tilde{F}, \tilde{G}, \tilde{E} : \mathcal{H} \to CB(\mathcal{H})$ are the multivalued mappings induced by fuzzy mappings F, G, E respectively.*

(i) *$\tilde{F}, \tilde{G}, \tilde{E}$ are D-Lipschitz continuous with constants $\alpha, \zeta, \sigma > 0$ respectively,*

(ii) *$\tilde{F}$ is strongly monotone with respect to H in the first argument of N with constant $r > 0$,*

(iii) *$\tilde{G}$ is relaxed monotone with respect to H in the second argument of N with constant $\rho > 0$,*

(iv) *N is Lipschitz continuous with respect to first and second arguments with constants $\beta, \delta > 0$ respectively,*

(v) *for $t \in \mathcal{H}$, $M(\cdot, t) : \mathcal{H} \to 2^{\mathcal{H}}$ is maximal monotone with constant ξ such that*

$$\|R^H_{M(\cdot,z_1),\lambda}(w) - R^H_{M(\cdot,z_2),\lambda}(w)\| \ \leq \ \xi\|z_1 - z_2\| \tag{3.3}$$

for all $u_1, u_2, w \in \mathcal{H}$, $z_1 \in \tilde{G}(u_1)$, $z_2 \in \tilde{G}(u_2)$.

(vi)
$$(1-\mu)+\mu\xi\sigma+\frac{\mu}{\gamma+\lambda\eta}\sqrt{\tau^2 - 2\lambda(r-\rho) + \lambda^2(\beta\alpha+\delta\zeta)^2} < 1. \tag{3.4}$$

Then there exist $u \in \mathcal{H}$, $x \in \tilde{F}(u)$, $y \in \tilde{G}(u)$, $z \in \tilde{E}(u)$, which satisfy (2.1) and the sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by Algorithm 3.1 converge to u^*, x^*, y^* and z^* strongly in $\mathcal{H}$ respectively.

Proof. From (3.2)-(3.3) and Theorem 2.2, we have

$$\begin{aligned}
& \|u_{n+1} - u_n\| \qquad\qquad (3.5)\\
= \ & \|(1-\mu)u_n + \mu R^H_{M(\cdot,z_n),\lambda}[H(u_n) - \lambda N(x_n, y_n)] \\
& -(1-\mu)u_{n-1} - \mu R^H_{M(\cdot,z_{n-1}),\lambda}[H(u_{n-1}) - \lambda N(x_{n-1}, y_{n-1})]\| \\
\leq \ & (1-\mu)\|u_n - u_{n-1}\| \\
& +\mu\|R^H_{M(\cdot,z_n),\lambda}[H(u_n) - \lambda N(x_n, y_n)] - R^H_{M(\cdot,z_{n-1}),\lambda}[H(u_n) - \lambda N(x_n, y_n)]\| \\
& +\mu\|R^H_{M(\cdot,z_{n-1}),\lambda}[H(u_n) - \lambda N(x_n, y_n)] \\
& -R^H_{M(\cdot,z_{n-1}),\lambda}[H(u_{n-1}) - \lambda N(x_{n-1}, y_{n-1})]\| \\
\leq \ & (1-\mu)\|u_n - u_{n-1}\| + \mu\xi\|z_n - z_{n-1}\| \\
& +\frac{\mu}{\gamma+\lambda\eta}\|H(u_n) - H(u_{n-1}) - \lambda N(x_n, y_n) + \lambda N(x_{n-1}, y_{n-1})\| \\
\leq \ & (1-\mu)\|u_n - u_{n-1}\| + \mu\xi D(\tilde{G}u_n, \tilde{G}u_{n-1}) \\
& +\frac{\mu}{\gamma+\lambda\eta}\|H(u_n) - H(u_{n-1}) - \lambda(N(x_n, y_n) - N(x_{n-1}, y_{n-1}))\| \\
\leq \ & (1-\mu)\|u_n - u_{n-1}\| + \mu\xi\sigma(1+n^{-1})\|u_n - u_{n-1}\| \\
& +\frac{\mu}{\gamma+\lambda\eta}\|H(u_n) - H(u_{n-1}) - \lambda(N(x_n, y_n) - N(x_{n-1}, y_{n-1}))\|.
\end{aligned}$$

Since N is Lipschitz continuous with first and second arguments and $\tilde{F}$, $\tilde{G}$ are D-Lipschitz continuous, we have

$$\begin{aligned}\|N(x_n,y_n)-N(x_{n-1},y_n)\| &\leq \beta\|x_n-x_{n-1}\| \\ &\leq \beta(1+n^{-1})D(\tilde{F}(u_n),\tilde{F}(u_{n-1})) \\ &\leq \beta\alpha(1+n^{-1})\|u_n-u_{n-1}\| \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\|N(x_{n-1},y_n)-N(x_{n-1},y_{n-1})\| &\leq \delta\|y_n-y_{n-1}\| \\ &\leq \delta(1+n^{-1})D(\tilde{G}(u_n),\tilde{G}(u_{n-1})) \\ &\leq \delta\zeta(1+n^{-1})\|u_n-u_{n-1}\|. \end{aligned} \tag{3.7}$$

Since $\tilde{F}$ and $\tilde{G}$ are Lipschitz continuous with constants $\alpha,\zeta>0$ respectively and $\tilde{F}$ is strongly monotone with respect to H in the first argument of N with constant $r>0$. $\tilde{G}$ is relaxed monotone with respect to H in the second argument of N with constant $\rho<0$ and H is Lipschitz continuous with respect to constant $\tau>0$, we have

$$\begin{aligned}&\|H(u_n)-H(u_{n-1})-\lambda(N(x_n,y_n)-N(x_{n-1},y_{n-1}))\|^2 \\ \leq\ &\|H(u_n)-H(u_{n-1})\|^2 \\ &-2\lambda\langle N(x_n,y_n)-N(x_{n-1},y_{n-1}),\ H(u_n)-H(u_{n-1})\rangle \\ &+\lambda^2\|N(x_n,y_n)-N(x_{n-1},y_{n-1})\|^2 \\ \leq\ &\tau^2\|u_n-u_{n-1}\|^2-2\lambda\langle N(x_n,y_n)-N(x_{n-1},y_n),\ H(u_n)-H(u_{n-1})\rangle \\ &-2\lambda\langle N(x_{n-1},y_n)-N(x_{n-1},y_{n-1}), Hu_n-Hu_{n-1}\rangle \\ &+\lambda^2\|N(x_n,y_n)-N(x_{n-1},y_{n-1})\|^2 \\ \leq\ &\tau^2\|u_n-u_{n-1}\|^2-2\lambda r\|u_n-u_{n-1}\|^2+2\lambda\rho\|u_n-u_{n-1}\|^2 \\ &+\lambda^2(1+n^{-1})^2(\beta\alpha+\delta\zeta)^2\|u_n-u_{n-1}\|^2 \\ \leq\ &(\tau^2-2\lambda(r-\rho)+\lambda^2(1+n^{-1})^2(\beta\alpha+\delta\zeta)^2\|u_n-u_{n-1}\|^2. \end{aligned} \tag{3.8}$$

From (3.5)-(3.8), we have

$$\begin{aligned}&\|x_{n+1}-x_n\| \\ \leq\ &(1-\mu)\|u_n-u_{n-1}\|+\mu\xi\sigma(1+n^{-1})\|u_n-u_{n-1}\| \\ &+\frac{\mu}{\gamma+\lambda\eta}\sqrt{\tau^2-2\lambda(r-\rho)+\lambda^2(1+n^{-1})^2(\beta\alpha+\delta\zeta)^2}\|u_n-u_{n-1}\| \end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\leq [(1-\mu)+\mu\xi\sigma(1+n^{-1}) \\
&\quad +\frac{\mu}{\gamma+\lambda\eta}\sqrt{\tau^2-2\lambda(r-\rho)+\lambda^2(1+n^{-1})^2(\beta\alpha+\delta\zeta)^2}]\|u_n-u_{n-1}\| \\
&\leq \theta_n\|u_n-u_{n-1}\|,
\end{aligned}$$

where,

$$\begin{aligned}
\theta_n &= (1-\mu)+\mu\xi\sigma(1+n^{-1}) \\
&\quad +\frac{\mu}{\gamma+\lambda\eta}\sqrt{\tau^2-2\lambda(r-\rho)+\lambda^2(1+n^{-1})^2(\beta\alpha+\delta\zeta)^2}.
\end{aligned}$$

Letting $n\to\infty$, we have $\theta_n\to\theta$, where

$$\theta = (1-\mu)+\mu\xi\sigma+\frac{\mu}{\gamma+\lambda\eta}\sqrt{\tau^2-2\lambda(r-\rho)+\lambda^2(\beta\alpha+\delta\zeta)^2}<1.$$

From condition (3.4), we know that $\theta<1$. Hence $\theta_n<1$ for sufficiently large n. Therefore $\{u_n\}$ is a Cauchy sequence and we can suppose that $u_n\to u\in\mathcal{H}$.

Now we prove that $x_n\to x^\star\in\tilde{F}(u^\star)$, $y_n\to y^\star\in\tilde{G}(u^\star)$ and $z_n\to z^\star\in\tilde{E}(u^\star)$. From Algorithm 3.1, we have

$$\begin{aligned}
\|x_n-x_{n+1}\| &\leq (1+(n+1)^{-1})D(\tilde{F}(u_n),\,\tilde{F}(u_{n-1})) \\
&\leq (1+(n+1)^{-1})\alpha\|u_n-u_{n-1}\|,
\end{aligned}$$

$$\begin{aligned}
\|y_n-y_{n+1}\| &\leq (1+(n+1)^{-1})D(\tilde{G}(u_n),\,\tilde{G}(u_{n-1})) \\
&\leq (1+(n+1)^{-1})\zeta\|u_n-u_{n-1}\|,
\end{aligned}$$

and

$$\begin{aligned}
\|z_n-z_{n+1}\| &\leq (1+(n+1)^{-1})D(\tilde{E}(u_n),\,\tilde{E}(u_{n-1})) \\
&\leq (1+(n+1)^{-1})\sigma\|u_n-u_{n-1}\|.
\end{aligned}$$

It follows that $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are Cauchy sequences in $\mathcal{H}$. Let $x_n\to x^\star$, $y_n\to y^\star$, and $z_n\to z^\star$ respectively. Now we will show that $x^\star\in\tilde{F}(u^\star)$, $y^\star\in\tilde{G}(u^\star)$, $z^\star\in\tilde{E}(u^\star)$. In fact noting $x_n\in\tilde{F}(u_n)$, we have

$$\begin{aligned}
D(x^\star,\,\tilde{F}(u^\star)) &= \inf\{\|x^\star-w\|:\,w\in\tilde{F}(u^\star)\} \\
&\leq \|x^\star-x_n\|+d(x_n,\,\tilde{F}(u^\star)) \\
&\leq \|x^\star-x_n\|+D(\tilde{F}(u_n),\,\tilde{F}(u^\star)) \\
&\leq \|x^\star-x_n\|+\alpha\|u_n-u^\star\|\to 0 \text{ as } n\to\infty.
\end{aligned}$$

Hence $d(x^\star, \tilde{F}(u^\star)) = 0$ and therefore $x^\star \in \tilde{F}(u^\star)$. Similarly we can prove that $y^\star \in \tilde{G}(u^\star)$ and $z^\star \in \tilde{E}(u^\star)$. Since

$$u_{n+1} = (1-\mu)u_n + \mu R^H_{M(\cdot,z_n),\lambda}[H(u_n) - \lambda N(x_n, y_n)],$$

$(u^\star, x^\star, y^\star, z^\star)$ is a solution of (2.1), from Lemma 3.1. This completes the proof. □

References

[1] S. Adly, *Perturbed algorithm and sensitivity analysis for a general class of variational inclusions*, J. Math. Anal. Appl. **201**(1996), 609–630.

[2] R.P. Agarwal, M.F. Khan, Donal O. Regan and Salahuddin, *On generalized multivalued nonlinear variational mappings*, Adv. Nonlinear Var. Inequal. **8**(1)(2005), 41–55.

[3] R. Ahmad, Salahuddin and S. Hussain, *Generalized nonlinear variational inclusions for fuzzy mappings*, Math. **32**(6)(2001), 943–947.

[4] S.S. Chang, *Coincidence theorem and fuzzy variational inequalities for fuzzy mappings*, Fuzzy Sets and Systems **62**(1994), 359–368.

[5] S.S. Chang, and N.J. Huang, *Generalized complementarity problems for fuzzy mappings*, Fuzzy Sets and Systems **55**(1993), 227–234.

[6] S.S. Chang, and Y.G. Zhu, *On variational inequalities for fuzzy mappings*, Fuzzy Sets and Systems **32**(1989), 359–367.

[7] X.P. Ding and J.Y. Park, *A new class of generalized nonlinear implicit quasivariational inclusions with fuzzy mappings*, J. Comput. Appl. Math. **138**(2002), 243–257.

[8] X.P. Ding and J.L. Luo, *Perturbed proximal point algorithms for generalized quasivariational like inclusions*, J. Comput. Appl. Math. **210**(2000), 153–165.

[9] Y.P. Fang and N.J. Huang, *H-monotone operator and resolvent operator technique for variational inclusions*, Appl. Math. Comput. **145**(2003), 795–803.

[10] Y.P. Fang and N.J. Huang, *H-accretive operator and resolvent operator technique for solving variational inclusions in Banach spaces*, Appl. Math. Lett. **17**(2004), 647–653.

[11] A. Hassouni and A. Moudafi, *A perturbed algorithm for variational inequalities*, J. Math. Anal. Appl. **185**(1994), 706–712.

[12] N. J. Huang, *A new completely general class of variational inclusions with noncompact valued mappings*, Comput. Math. Appl. **35**(10)(1998), 9–14.

[13] J.S. Jung and C.H. Morales, *The Mann process for perturbed m-accretive operators in Banach spaces*, Nonlinear Anal. **46**(2001), 231–243.

[14] G.M. Lee, D.S. Kim and S.J. Cho, *Generalized vector variational inequality and Fuzzy extensions*, Appl. Math. Lett. **6**(1993), 47–51.

[15] S.B. Nadler, Jr., *Multivalued contraction mappings*, Pacific J. Math. **30**(1969), 475-488. co-variational inequalities with fuzzy mappings in Fuzzy Math. **14**(1)(2006), 135-149.

[16] R.U. Verma, *Projection method, algorithms and a new system of nonlinear variational inequalities*, Comput. Math. Appl. **41**(7-8)(2001), 1025–1031.

[17] R.U. Verma, *Generalized system for relaxed coercive variational inequalities and projection methods*, J. Optim. Theory Appl. **12**(1)(2004), 203–210.

[18] H.K. Xu and J.H. Kim, *Convergence of descent methods for variational inequalities*, J. Optim Theory Appl. **119**(1)(2003), 185-201.

[19] L.C. Zeng, *Perturbed proximal point algorithm for generalized nonlinear set-valued mixed quasivariational inclusions*, Acta. Math. Sinica **47**(1)(2004), 11–18.