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FUZZY GENERALIZED VECTOR VARIATIONAL INEQUALITIES AND COMPLEMENTARITY PROBLEMS

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Abstract. The main objective of present paper is to study a class of fuzzy generalized vector variational inequalities and fuzzy generalized vector complementarity problems. We prove the existence of solutions for this kind of vector variational inequalities and discuss the relations between the solutions of the fuzzy generalized vector variational inequalities and the solutions of fuzzy generalized vector complementarity problems in Hausdorff topological vector spaces.

1. Introduction

Firstly Chang and Zhu [2] introduced and studied the concept of variational inequalities for fuzzy mappings and investigated existence theorems for some kinds of variational inequalities for fuzzy mappings. Chang [1] proved the coincidence theorems for fuzzy mappings and some existence theorems for more general variational inequalities for fuzzy mappings. Lee et al [13] obtained some existence theorems of certain variational inequalities for fuzzy mappings

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following the approach of Chang and Zhu [2] and using the result of Kim and Tan [11].

On the other hand, Lee et al [18] considered vector variational inequalities for fuzzy mappings, which were the fuzzy extensions of vector variational inequalities studied by Chen and Yang [6] and obtained some existence theorems of solutions for their inequalities for fuzzy mappings. Since then Chang et al [3,4] and Lee et al [14] studied several kinds of vector variational inequalities and vector quasi-variational inequalities for fuzzy mappings and proved some existence theorems of solutions for their inequalities.

In 1980, Giannessi [8] introduced a vector variational inequality in a finite dimensional Euclidean space. In the past ten years or so, Chen et al [5,6], Lai and Yao [12], Isac [10], Lee et al [17,18] and many others have intensively studied the vector variational inequalities. Very recently Isac [10] and Yang [20] considered vector complementarity problems and proved some existence theorems of the solutions for the vector complementarity problems.

In this paper, we considered the fuzzy generalized vector variational inequalities and fuzzy complementarity problems. We prove the existence of solutions for this kind of vector variational inequalities with fuzzy mappings and discuss the relations between the solutions of the fuzzy generalized vector variational inequalities and solutions of fuzzy generalized vector complementarity problems in Hausdorff topological vector spaces.

Let E be a nonempty subset of a vector space X and D a nonempty set. A function T from D into the collection $\mathcal{F}(E)$ of all fuzzy sets on E is called a fuzzy mapping. If $T : D \to \mathcal{F}(E)$ is a fuzzy mapping, then $T(x)$, $x \in D$ (denoted by T_x in the sequel) is a fuzzy set in $\mathcal{F}(E)$ and $T_x(y)$, $y \in E$ is the degree of membership of y in T_x. A fuzzy mapping $T : D \to \mathcal{F}(E)$ is said to be closed if for each $x \in D$ the function $y \to T_x(y)$ is u.s.c., i.e., for any given net $\{y_\alpha\} \subset D$ satisfying $y_\alpha \to y_0 \subset D$, $\limsup_\alpha T_x(y_\alpha) \leq T_x(y_0)$.

Let $A \in \mathcal{F}(E)$ and $\alpha \in (0,1]$.Then the set

$$(A)_\alpha \ = \ \{x \in E : \, A(x) \geq \alpha\}$$

is called an α-cut set of A.

The fuzzy mapping $T : D \to \mathcal{F}(E)$ is said to be convex if E is a convex subset of X and for any $x \in D$, $y, z \in E$ and $\alpha \in [0,1]$,

$$T_x(\alpha y + (1-\alpha)z) \ \geq \ \min\{T_x(y), T_x(z)\}.$$

Definition 1.1 (15)**.** *Let X and Y be two topological spaces and $F : X \to 2^Y$ be a multifunction. Then*

(1) *T is upper semi-continuous (briefly u.s.c.) at $x_0 \in X$ if for any open set N containing $T(x_0)$, there exists a neighborhood M of x_0 such that $T(M) \subset N$. T is u.s.c. if T is u.s.c. at every $x_0 \in X$.*
(2) *T is closed at $x \in X$ if for any net $\{x_\lambda\}$ in X such that $x_\lambda \to x$ and for any net $\{y_\lambda\}$ in Y such that $y_\lambda \to y$ and $y_\lambda \in T(x_\lambda)$ for any λ, we have $y \in T(x)$.*
(3) *T has a closed graph if the graph of T, $Gr(T) = \{(x, y) \in X \times Y : y \in T(x)\}$ is closed in $X \times Y$.*

Definition 1.2 (19)**.** *Let X and Y be two topological spaces and $T : X \to \mathcal{F}(Y)$ be a fuzzy mapping. Then T is a fuzzy mapping with closed fuzzy set-values if $T_x(y)$ is u.s.c. on $X \times Y$ as a real ordinary function.*

Definition 1.3. *A mapping $T : K \to Y$ is convex if for any $x_1, x_2 \in K$ and $t \in [0, 1]$*

$$f(tx_1 + (1-t)x_2) \ \leq_{C(x)} tf(x_1) + (1-t)f(x_2),$$

that is,

$$tf(x_1) + (1-t)f(x_2) - f(tx_1 + (1-t)x_2) \in C(x).$$

Lemma 1.4. *If A is a closed subset of a topological space X then the characteristic function χ_A of A is an u.s.c. real-valued function.*

Lemma 1.5 (16)**.** *Let K be a nonempty closed convex subset of a real Hausdorff topological vector space X, E be a nonempty closed convex subset of a real Hausdorff topological vector space Y and $\alpha : X \to (0, 1]$ be a lower semicontinuous function. Let $T : K \to \mathcal{F}(E)$ be a fuzzy mapping with $(T_x)_{\alpha(x)} \neq 0$ for any $x \in X$. Let $\tilde{T} : K \to 2^E$ be a multifunction defined by $\tilde{T}(x) = (T_x)_{\alpha(x)}$. If T is a fuzzy mapping with closed fuzzy set-values, then $\tilde{T}$ is closed multifunction.*

Proof. Let $\{x_\lambda\}$ be a net in K converging to x, $\{y_\lambda\}$ be a net in E converging to y and $y_\lambda \in \tilde{T}(x_\lambda) = (T_{x_\lambda})_{\alpha(x_\lambda)}$. Then $T_{x_\lambda}(y_\lambda) \geq \alpha(x_\lambda)$. Since $T_x(y)$ is u.s.c. on $X \times Y$ as a real ordinary function, $T_x(y) \geq \overline{\lim} T_{x_\lambda}(y_\lambda) \geq \underline{\lim} T_{x_\lambda}(y_\lambda) \geq \underline{\lim} \alpha(x_\lambda) \geq \alpha(x)$, which shows $y \in \tilde{T}(x)$. □

2. Fuzzy Generalized Vector Variational Inequalities for Fuzzy Mappings

Let X be a Hausdorff topological vector space, Y a topological vector space; $M(X, Y)$ and $L(X, Y)$ denote the set of all continuous mappings from X to Y and the set of all continuous linear mappings from X to Y respectively. Suppose that K is a nonempty closed convex subset of X, $\{C(x) : x \in K\}$ is a family of closed pointed and convex cones of Y with $\text{int}\, C(x) \neq \emptyset$. Let $T : X \to \mathcal{F}(M(X,Y))$ be a fuzzy mapping and $\alpha : X \to (0,1]$ be a function. We define a partial order $\leq_{C(x)}$ in Y with the convex cone $C(x)$ as, for $y_1, y_2 \in Y$

$$y_1 \leq_{C(x)} y_2 \text{ if and only if } y_2 - y_1 \in C(x).$$

A mapping $f : K \to Y$ is called convex if for any finite subset $\{x_1, x_2, \cdots, x_n\}$ of K and $\{\lambda_i\}_{i=1}^n \subset [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$, we have $f(\sum_{i=1}^n \lambda_i x_i) \leq_{C(x)} \sum_{i=1}^n \lambda_i f(x_i)$.

Let E be a nonempty subset of $X, T : E \to 2^X$ is called a KKM mapping if for any arbitrary finite subset $\{x_1, x_2, \cdots, x_n\}$ of E

$$Co\{x_1, x_2, \cdots, x_n\} \subset \bigcup_{i=1}^n T(x_i)$$

where $Co(A)$ denotes the convex hull of A.

Now we consider the following *fuzzy generalized vector variational inequality*:
Find $y_0 \in K$, there exists $s_0 \in (T_{y_0})_{\alpha(y_0)}$ such that

$$\phi(p(s_0), x, y_0) \notin \text{-int}\, C(y_0), \text{ for all } x \in K, \tag{2.1}$$

where $\phi : \mathcal{F}(M(X,Y)) \times K \times K \to Y$ is a single-valued mapping and $p : \mathcal{F}(M(X,Y)) \to \mathcal{F}(M(X,Y))$ is a continuous mapping. We note that if p is an identity map and $T : X \to M(X,Y)$ is a mapping then we can define the fuzzy mapping $T : X \to \mathcal{F}(M(X,Y))$ by $x \to \chi_{T(x)}$, where $\chi_{T(x)}$ is the characteristic function of $T(x)$. Taking $\alpha(x) = 1$ for all $x \in X$, Problem (2.1) is equivalent to the following problem: Find $y_0 \in K$, there exists $s_0 \in T(y_0)$ such that

$$\phi(s_0, x, y_0) \notin \text{-int}\, C(y_0), \text{ for all } x \in K, \tag{2.2}$$

where $\phi : M(X,Y) \times K \times K \to Y$ is a single valued mapping and $p : M(X,Y) \to M(X,Y)$ is a continuous mapping. Problem (2.2) is called vector variational inequality and is studied by Huang and Gao [9].

Lemma 2.1 (7)**.** *Let X be a topological vector space and S a subset of X and $F : S \to 2^X$ a KKM mapping with closed values. If there exists an $x_0 \in S$ such that $F(x_0)$ is compact then*

$$\bigcap_{x \in S} F(x) \neq \emptyset.$$

Theorem 2.2. *Let X be a Hausdorff topological vector space, Y a topological vector space and $M(X,Y)$ the set of all continuous mappings from X to Y. Suppose that K is a nonempty closed convex subset of X, $\{C(x) : x \in K\}$ is a family of closed pointed and convex cones of Y with $intC(x) \neq \emptyset$, $W(x) = Y\backslash\{\text{-}intC(x)\}$ has a closed graph. Let a fuzzy mapping $T : X \to \mathcal{F}(M(X,Y))$ is an upper semi-continuous and closed set-valued mapping with compact values. Let $\phi : \mathcal{F}(M(X,Y)) \times K \times K \to Y$ be a continuous mapping such that*

(1) *there exists an upper semi-continuous function $\alpha : X \to (0,1]$ such that for any $x \in K$ the cut set $(T_x)_{\alpha(x)}$ is nonempty and $\bigcup_{x\in K}(T_x)_{\alpha(x)}$ is contained in some compact subset of $M(X,Y)$,*
(2) *$p : \mathcal{F}(M(X,Y)) \to \mathcal{F}(M(X,Y))$ is a continuous mapping,*
(3) *for each $x \in K$, there exists $s_0 \in (T_x)_{\alpha(x)}$ such that*

$$\phi(p(s_0), x, x) \notin \text{-}intC(x),$$

(4) *the mapping $\phi(p(s), x, y)$ is convex with respect to x to the convex cone $C(y)$ for any $y \in K$ and any $s \in (T_y)_{\alpha(y)}$,*
(5) *the mapping $\phi(p(s), x, y)$ is continuous with respect to (s, y).*
Moreover one of the following two assumptions is satisfied:
(6) *K is compact,*
(7) *there exists a compact subset D of K such that for each $x \in K\backslash D$, there exists $u \in D$ such that*

$$\phi(p(s), u, x) \in \text{-}intC(x) \text{ for all } s \in (T_x)_{\alpha(x)}.$$

Then there exists $y_0 \in K$ and $s_0 \in (T_{y_0})_{\alpha(y_0)}$ such that

$$\phi(p(s_0), x, y_0) \notin \text{-}intC(y_0) \text{ for all } x \in K.$$

Proof. Define a multifunction $\tilde{T} : K \to 2^{M(X,Y)}$ by $\tilde{T}(x) = (T_x)_{\alpha(x)}$. It follows from Lemma 1.5 and the condition (1) that $\tilde{T}$ is nonempty closed multifunction such that $\tilde{T}(K)$ is contained in semi compact subset of $M(X,Y)$. Assume that ϕ satisfies Assumption (3)–(5) and K is compact. Let

$$F(x) \triangleq \{y \in K : \exists\, s \in (T_y)_{\alpha(y)} \text{ such that } \phi(p(s), x, y) \notin \text{-int}\, C(y)\}$$

for each $x \in K$.

We assert that $F(x)$ is closed. In fact for each fixed $x \in K$, if $\{y_n\} \subset F(x)$ and $y_n \to y_0 \in K$, then there exists $s_n \in \tilde{T}(y_n)$ such that $\phi(p(s_n), x, y_n) \notin$ -int $C(y_n)$.

Since K is a compact subset of X, T is upper semi-continuous and closed with compact values and $T(K)$ is compact then there exists a subnet $\{s_n\}$, denoted by $\{s_n\}$, such that $s_n \to s_0$ and $p(s_n) \to p(s_0)$. It follows from

Assumption (5) that

$$\phi(p(s_n), x, y_n) \to \phi(p(s_0), x, y_0),$$

and so $\phi(p(s_0), x, y_0) \in W(y_0)$ by the closed graph of W, which implies that

$$\phi(p(s_0), x, y_0) \notin \text{-int}\, C(y_0).$$

Hence $F(x)$ is closed and moreover compact. By Assumption (3) we know that

$$F(x) \neq \emptyset \text{ or each } x \in K.$$

Further, we shall prove that $F : K \to 2^K$ is a KKM mapping. Then there exists a finite subset $\{x_i\}_{i=1}^n$ such that $Co\{x_1, x_2, \cdots, x_n\} \not\subset \bigcup_{i=1}^n F(x_i)$. Then there exists at least one point $x = \sum_{i=1}^n \lambda_i x_i$, where $0 \leq \lambda_i \leq 1$ for each $1 \leq i \leq n$ and $\sum_{i=1}^n \lambda_i = 1$, such that

$$x \notin \bigcup_{i=1}^n F(x_i).$$

Hence for each $i = 1, 2, \cdots n$, $\phi(p(s), x_i, x) \in \text{-int}\, C(x)$ for all $s \in \tilde{T}(x)$ and it follows from Assumption (4) that

$$\phi(p(s), \sum_{i=1}^n \lambda_i x_i, x) \leq_{C(x)} \sum_{i=1}^n \lambda_i \phi(p(s), x_i, x)$$

that is,

$$\sum_{i=1}^n \lambda_i \phi(p(s), x_i, x) - \phi(p(s), x, x) \in C(x),$$

for all $s \in \tilde{T}(x)$. Since $\phi(p(s), x_i, x) \in \text{-int}\, C(x)$ and $C(x)$ is convex, it follows that

$$\phi(p(s), x, x) \in \text{-int}\, C(x) \text{ for all } s \in \tilde{T}(x),$$

which contradicts Assumption (3). Therefore $F : K \to 2^Y$ is a KKM mapping. From Lemma 2.1, we know that $\bigcap_{x \in K} F(x) \neq \emptyset$ which implies that there exists $y_0 \in K$ and $s_0 \in \tilde{T}(y_0)$ such that

$$\phi(p(s_0), x, y_0) \notin \text{-int}\, C(y_0) \text{ for all } x \in K.$$

Next we suppose that ϕ satisfies Assumption (3)–(5) and (7). Let

$$F(x) \overset{\Delta}{=} \{y \in D : \exists\, s \in (\tilde{T}_y) \text{ such that } \phi(p(s), x, y) \notin \text{-int}\, C(y)\}, \forall x \in K.$$

We can prove that $F : K \to 2^K$ is a KKM mapping with closed values. Now we show that $F(x) \neq \emptyset$ for each $x \in K$. It follows from Assumption (3) that $F(x) \neq \emptyset$ for each $x \in D$. On the other hand, for each fixed $z \in K\backslash D$, let

$$E_z = \overline{Co}\{D \cup \{z\}\}$$

where $\overline{Co}$ denotes the closed convex hull of a set. Since D is compact, E_z is also compact. For each $x \in E_z$ let

$$B(x) \stackrel{\Delta}{=} \{y \in E_z : \exists\, s \in \tilde{T}(y) \text{ such that } \phi(p(s), x, y) \notin \text{-int}\, C(y)\}.$$

Therefore there exists $y_1 \in E_z$ and $s_1 \in \tilde{T}(y_1)$ such that

$$\phi(p(s_1), x, y_1) \notin \text{-int}\, C(y_1), \tag{2.3}$$

for all $x \in E_z$. Moreover, we assert that $y_1 \in D$. In fact if $y_1 \in E_z\backslash D \subset K\backslash D$, it follows from Assumption (7) that there exists $u \in D$ satisfying $\phi(p(s), u, y_1) \in \text{-int}\, C(y_1)$ for all $s \in \tilde{T}(y_1)$, which contradicts (2.3). So $y_1 \in D$ and this implies $F(x) \neq \emptyset$ for each $x \in E_z$. Especially $F(z) \neq \emptyset$. Since $z \in K\backslash D$ is arbitrary, we have

$$\begin{aligned} F(x) \quad &= \{y \in D : \exists\, s \in \tilde{T}(y) \text{ such that} \\ &\phi(p(s), x, y) \notin \text{-int} C(y)\} \neq \emptyset, \text{ for all } x \in K. \end{aligned}$$

Since $F(x)$ is closed for each $x \in D$ and D is compact. From Assumption (7), there exists $u \in D$ such that $F(u) \subset D$ and hence $F(u)$ is compact. Therefore from Lemma 2.1, we have $\bigcap_{x \in K} F(x) \neq \emptyset$ which implies that there exists $y_0 \in K$ and $s_0 \in \tilde{T}(y_0)$ such that $\phi(p(s_0), x, y_0) \notin \text{-int}\, C(y_0)$ for all $x \in K$.

This completes the proof. □

Corollary 2.3. *Let X, Y, K, T and $\mathcal{F}(M(X, Y))$ be the same as in Theorem 2.1. Assume that P is a closed, pointed and convex cone in Y such that $\text{int}\, P \neq \emptyset$ and $\phi : \mathcal{F}(M(X, Y)) \times K \times K \to Y$ is a mapping such that*

(1) *there exists a lower semi-continuous function $\alpha : X \to (0, 1]$ such that for any $x \in K$ the cut set $(T_x)_{\alpha(x)}$ is nonempty and $\bigcup_{x \in K} (T_x)_{\alpha(x)}$ is contained in some compact subset of $M(X, Y)$.*
(2) *$p : \mathcal{F}(M(X, Y)) \to \mathcal{F}(M(X, Y))$ is a continuous mapping,*
(3) *for each $x \in K$ there exists $s_0 \in \tilde{T}(x)$ such that*

$$\phi(p(s_0), x, x) \notin \text{-}int\, P,$$

(4) *the mapping $\phi(p(s), x, y)$ is convex with respect to x,*

(5) *the mapping* $\phi(p(s), x, y)$ *is continuous with respect to* (s, y).

Moreover, one of the following two assumptions is satisfied:

(6) *there exists a compact subset* D *of* K *and* $u \in D$ *such that for all* $x \in K \backslash D$ *and* $s \in \tilde{T}(x)$,

$$\phi(p(s), u, x) \in \text{-}int P,$$

(7) K *is compact.*

Then there exists $y_0 \in K$ *and* $s_0 \in \tilde{T}(y_0)$ *such that*

$$\phi(p(s_0), x, y_0) \notin \text{-}int P \text{ for all } x \in K.$$

Lemma 2.4 (5). *Let* (X, P) *be an ordered topological vector space with a closed pointed and convex cone* P *such that* $int P \neq \emptyset$. *Then for any* $y, z \in X$, *we have*

(i) $y - z \in \text{-}int P$ *and* $y \notin \text{-}int P$ *imply* $z \in \text{-}int P$,

(ii) $y - z \in P$ *and* $y \notin int P$ *imply* $z \notin int P$.

If we replace $M(X, Y)$ by $L(X, Y)$, then from Theorem 2.2 we have the following.

Theorem 2.5. *Let* X, Y, K, C, T *and* W *be the same as in Theorem 2.2. Assume that* $\phi : \mathcal{F}(L(X, Y)) \times K \times K \to Y$ *is a mapping such that*

(1) *there exists a lower semi-continuous function* $\alpha : X \to (0, 1]$ *such that for any* $x \in K$, *the cut set* $(T_x)_{\alpha(x)}$ *is nonempty and* $\bigcup_{x \in K} (T_x)_{\alpha(x)}$ *is contained some compact subset of* $L(X, Y)$,

(2) $p : \mathcal{F}(M(X, Y)) \to \mathcal{F}(M(X, Y))$ *is a continuous mapping,*

(3) *there exists a mapping* $g : \mathcal{F}(L(X, Y)) \times K \times K \to Y$ *such that*

$$g(p(s), x, y) - \phi(p(s), x, y) \in \text{-}int C(x) \text{ for all } (s, x, y) \in L(X, Y) \times K \times K;$$

(4) *the set* $F(y) = \{x \in K \ \exists \ s \in \tilde{T}(y) \text{ such that } g(p(s), x, y) \in \text{-}int C(y)\}$ *is convex for all* $y \in K$,

(5) *there exists* $s \in \tilde{T}(x)$ *such that*

$$g(p(s), x, x) \notin \text{-}int C(x) \text{ forall } x \in K,$$

(6) *the mapping* $\phi(p(s), x, y)$ *and* $g(p(s), x, y)$ *are continuous with respect to* (s, y),

(7) *there exists a nonempty compact and convex subset* D *of* K *and* $u \in D$ *such that*

$$\phi(p(s), u, x) \in \text{-}int C(x) \text{ for all } x \in K \backslash D \text{ and } s \in \tilde{T}(x).$$

Then there exists $y_0 \in K$ *and* $s_0 \in \tilde{T}(y_0)$ *such that*

$$\phi(p(s_0), x, y_0) \notin \text{-}int C(y_0) \text{ for all } x \in K.$$

Proof. Define a multifunction $\tilde{T} : K \to 2^{L(X,Y)}$ by $\tilde{T}(x) = (T_x)_{\alpha(x)}$. From Lemma 1.5 and the condition (1) that $\tilde{T}$ is a nonempty closed multifunction such that $\tilde{T}(K)$ is contained in some compact subset of $L(X,Y)$. Let

$$F_1(x) = \{y \in D : \exists\, s \in \tilde{T}(y) \text{ such that } g(p(s),x,y) \notin \text{-int}\, C(y)\},$$
$$F_2(x) = \{y \in D : \exists\, s \in \tilde{T}(y) \text{ such that } \phi(p(s),x,y) \notin \text{-int}\, C(y)\},$$

for all $x \in K$. Since $g(p(s),\cdot,y)$ and $\phi(p(s),\cdot,y)$ are continuous and $W(y) = Y\backslash\{\text{-int}\, C(y)\}$ has a closed graph. It is easy to prove $F_1(y)$ and $F_2(y)$ is closed for each $y \in K$. From Theorem 2.2, we can prove F_1 is a KKM mapping. Further we shall prove $F_1(y) \subset F_2(y)$. For each $x \in F_1(y)$, there exists $s \in \tilde{T}(y)$ such that $g(p(s),x,y) \in \text{-int}C(y)$. From Assumption (3) and Lemma 2.4(i), for some $s \in \tilde{T}(y)$,

$$\phi(p(s),x,y) \notin \text{-int}\, C(y),$$

which implies that $x \in F_2(y)$. So $F_1(y) \subset F_2(y)$, and hence $F_2(y)$ is also a KKM mapping. From Theorem 2.2 and Assumption (7) we show that $F_2(y) \neq \emptyset$ for all $y \in K$ and there exists $u \in K$ such that $F_2(u)$ is compact. Hence the desired conclusion follows from Lemma 2.1. This completes the proof. □

3. A Fuzzy Generalized Vector Complementarity Problems

Let X be a Hausdorff topological vector space, Y a topological vector space, $\tilde{T} : X \to 2^{L(X,Y)}$, a mapping defined by $\tilde{T}(x) = (T_x)_{\alpha(x)}$, $p : \mathcal{F}(L(X,Y)) \to \mathcal{F}(L(X,Y))$ a continuous mapping, P a closed, pointed and convex cone in X such that $\text{int}\, P \neq \emptyset$ and $\{C(x) : x \in P\}$ a family of closed pointed and convex cones in Y such that $\text{int}\, C(x) \neq \emptyset$.

Definition 3.1. *Let $\{C(x) : x \in P\}$ be a family of closed pointed and convex cones in Y and P a closed pointed and convex cone in X. Then $\tilde{T} : X \to 2^{L(X,Y)}$ is generally positive on P related to $C(\cdot)$ if for any fixed $x \in P$ and for all $s \in \tilde{T}(x)$, we have*

$$\langle s, y\rangle \in C(x), \text{ for all } y \in P.$$

Now we consider the *fuzzy generalized vector complementarity problems.*

Find $y_0 \in P$ and $s_0 \in \tilde{T}(y_0)$ such that

$$\begin{aligned} \langle p(s_0), \theta(x,y_0)\rangle &\notin \text{-int}\, C(y_0) \text{ for all } x \in P, \\ \langle p(s_0), \theta(y_0,y_0)\rangle &\notin \text{int}\, C(y_0). \end{aligned} \tag{3.1}$$

Theorem 3.2. *Let P be a closed pointed and convex cone in X such that $\text{int}P \neq \emptyset$ and $\{C(x) : x \in P\}$ a family of closed, pointed and convex cones in Y such that $\text{int}C(x) \neq \emptyset$. Suppose that $\theta : P \times P \to P$ is a continuous mapping* *such that* $\theta(P,y) = P$ *for all* $y \in P$ *and* $\tilde{T} : X \to 2^{L(X,Y)}$ *is a mapping such that* $\tilde{T}(x) = (T_x)_{\alpha(x)}$, *where* $\alpha : X \to (0,1]$. *Let* $p : \mathcal{F}(L(X,Y)) \to \mathcal{F}(L(X,Y))$ *be a continuous mapping. If* $y_0 \in P$ *and* $s_0 \in \tilde{T}(y_0)$ *satisfy the following fuzzy generalized vector variational inequality*

$$\langle p(s_0), \theta(x,y_0) - \theta(y_0,y_0)\rangle \notin \text{-}int\, C(y_0) \text{ for all } x \in P, \tag{3.2}$$

then $y_0 \in P$ *and* $s_0 \in \tilde{T}(y_0)$ *satisfy problem (3.1). Moreover if* $\tilde{T}$ *is generally positive on* P *related to* $C(\cdot)$ *then* $y_0 \in P$ *and* $s_0 \in \tilde{T}(y_0)$ *are the solution of problem (3.1) if and only if* $y_0 \in P$ *and* $s_0 \in \tilde{T}(y_0)$ *are the solution of problem (3.2).*

Proof. Define a multifunction $\tilde{T} : X \to 2^{L(X,Y)}$ by $\tilde{T}(x) = (T_x)_{\alpha(x)}$. From the Definition 3.1, if $y_0 \in P$ and $s_0 \in \tilde{T}(y_0)$ satisfy problem (3.2) then there exists $x \in P$ such that $\theta(x,y_0) = 0$. Hence it follows from (3.2) that

$$\langle p(s_0), \theta(y_0,y_0)\rangle \notin \operatorname{int} C(x).$$

On the other hand since $\theta(y,y_0) + \theta(y_0,y_0) \in P$ and $\theta(P,y) = P$ for each $y \in P$ then there exists $x \in P$ such that $\theta(y,y_0) + \theta(y_0,y_0) = \theta(x,y_0)$.

From (3.1), we get

$$\langle p(s_0), \theta(x,y_0) - \theta(y_0,y_0)\rangle = \langle p(s_0), \theta(y,y_0)\rangle \notin \text{-}\operatorname{int} C(y_0) \text{ for each } y \in P.$$

Therefore $y_0 \in P$ and $s_0 \in \tilde{T}(y_0)$ satisfy problem (3.1).

Now suppose that $\tilde{T}$ is a generally positive on P related to $C(\cdot)$ and $y_0 \in P$ and $s_0 \in \tilde{T}(y_0)$ satisfy problem (3.1). Hence $s_0 \in \tilde{T}(y_0)$,

$$\langle p(s_0), y\rangle \ \in \ C(y_0), \text{ for all } y \in P \text{ and so,}$$

$$\langle p(s_0), \theta(x,y_0)\rangle = \langle p(s_0), \theta(y_0,y_0) - (\theta(y_0,y_0) - \theta(x,y_0))\rangle \ \in \ C(y_0), \text{ for all } x \in P.$$

Since

$$\langle p(s_0), \theta(y_0,y_0)\rangle \ \notin \ \operatorname{int} C(y_0),$$

it follows from Lemma 2.2(ii) that

$$\langle p(s_0), \theta(y_0,y_0) - \theta(x,y_0)\rangle \ \notin \ \operatorname{int} C(y_0), \text{ for all } x \in P,$$

i.e.,

$$\langle p(s_0), \theta(x,y_0) - \theta(y_0,y_0)\rangle \ \notin \ \text{-int}C(y_0), \text{ for all } x \in P.$$

Therefore $y_0 \in P$ and $s_0 \in \tilde{T}(y_0)$ satisfy problem (3.2). This completes the proof. □

References

[1] S. S. Chang, *Coincidence theorems and variational inequalities for fuzzy mappings*, Fuzzy Sets and Systems, **61**(1994), 359–368.

[2] S. S. Chang and Y. g. Zhu, *On variational inequalities for fuzzy mappings*, Fuzzy Sets and Systems, **32**(1989), 359–367.

[3] S. S. Chang, G. M. Lee and B. S. Lee, *Vector quasi-variational inequalities for fuzzy mappings (I)*, Fuzzy Sets and Systems, **87**(1997), 307–315.

[4] S. S. Chang, G. M. Lee and B. S. Lee, *Vector quasi-variational inequalities for fuzzy mappings (II)*, Fuzzy Sets and Systems, **102**(1999), 333–344.

[5] G. Y. Chen, *Existence of solutions for a vector variational inequality : An extension of the Hartman Stampacchia theorem*, J. Optim Th. Appl., **74**(3)(1992), 445–456.

[6] G. Y. Chen and X. Q. Yang, *The vector complementarity problems and its equivalence with weak minimal element in ordered spaces*, J. Math. Anal. Appl., **153**(1990), 136–158.

[7] K. Fan, *A generalization of Tychonoff's fixed point theorem*, Math. Ann., **142**(1961), 305–310.

[8] F. Giannessi, *Theorems of alternative, quadratic programs and complementarity problems: In Variational Inequalities and Complementarity Problems",* Edited by R.W. Cottle, F. Giannessi and J.L. Lions, John Wiley and Sons, New York, 1980.

[9] N. J. Huang and C. J. Gao, *Some generalized vector variational inequalities and complementarity problems for multivalued mappings*, Appl. Math. Lett., **16**(2003), 1003–1010.

[10] G. Isac, *A special variational inequality and the implicit complementarity problems*, J. Fac. Sci. Univ. Tokyo, **37**(1990), 109–127.

[11] W. K. Kim and K. K. Tan, *A variational inequalities in non-compact sets and its applications*, Bull. Austral. Math. Soc., **46**(1992), 139–148.

[12] T.C. Lai and J.C. Yao, *Existence results for VVIP*, Appl. Math. Lett., **9**(3)(1996), 17–19.

[13] B. S. Lee, G. M. Lee, S. J. Cho and D. S. Kim, *A variational inequalities for fuzzy mappings*, Proc. of Fifth International Fuzzy Systems Association World Congress, Seoul **93** (1)(1993), 326–329.

[14] B. S. Lee, G. M. Lee and D. S. Kim, *Generalized vector-valued variational inequalities and fuzzy extensions*, J. Korean Math. Soc., **33**(1996), 609–624.

[15] B. S. Lee, G. M. Lee and D. S. Kim, *Generalized vector variational-like inequalities on locally convex Hausdorff topological vector spaces*, Indian J. Pure Appl. Math., **28**(1997), 33–41.

[16] B. S. Lee and D. Y. Jung, *A fuzzy extension of Siddiqi et al.'s results for vector variational-like inequalities*, Indian J. Pure Appl. Math., **34**(10)(2003), 1495–1502.

[17] B. S. Lee and S. J. Lee, *Vector variational type inequality for set-valued mappings*, Appl. Math. Lett., **13**(3)(2000), 57–62.

[18] G. M. Lee, D. S. Kim, B. S. Lee and S. J. Cho, *Generalized vector variational inequality and fuzzy extensions*, Appl. Math. Lett., **6**(1993), 47–51.

[19] G. M. Lee, D. S. Kim and B. S. Lee, *Vector variational inequalities for fuzzy mappings*, Nonlinear Analysis Forum, **4**(1999), 119–129.

[20] X. Q. Yang, *Vector complementarity and minimal element problems*, J. Optim Theory Appl., **77**(1993), 483–495.