Nonlinear Functional Analysis and Applications
Vol. 13, No. 2 (2008), pp. 265-270

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

SOME PROPERTIES OF A SUBCLASS OF ANALYTIC FUNCTIONS

Khalida Inayat Noor[1] and Wasim Ul-Haq[2]

[1,2]Mathematics Department, COMSATS Institute of Information Technology,
Islamabad, Pakistan
e-mail: khalidanoor@hotmail.com

Abstract. In this paper, we define and study a certain subclass $\tilde{R}_k(\alpha, \beta, \gamma)$ of analytic functions in the open unit disc. Inclusion result, radius problem, invariance under certain integral operators and some other interesting properties for this class are investigated

1. Introduction

Let $\tilde{P}(\gamma)$ be the class of functions p analytic in the unit disc $E = \{Z : |z| < 1\}$ with $p(0) = 1$ and satisfying the condition

$$|\arg p(z)| < \frac{\gamma\pi}{2}, \quad 0 < \gamma \leq 1.$$

We note that $\tilde{P}(1) \equiv P$ is the class of functions with positive real part. We define the class $\tilde{P}_k(\gamma)$ as follows.

Definition 1.1. *Let p be analytic in E with $p(0) = 1$. Then $p \in \tilde{P}_k(\gamma), 0 < \gamma \leq 1, k \geq 2$, if and only if, there exists $p_1, p_2 \in \tilde{P}(\gamma)$ such that*

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z) \tag{1.1}$$

Let A be the class of functions f, analytic in E and be given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.2}$$

We introduce a subclass of A as follows.

[0]Received March 16, 2007. Revised August 9, 2007.
[0]2000 Mathematics Subject Classification: 30C45, 30C50.
[0]Keywords: Functions with positive real part, convex, convolution, integral operator.

Definition 1.2. *Let $f \in A$. Then, $f \in \tilde{R}_k(\alpha,\beta,\gamma), k \geq 2, \alpha,\beta \geq 0, \quad (\alpha+\beta) > 0, \quad 0 < \gamma \leq 1$ if and only if,*

$$\left\{\frac{\alpha}{\alpha+\beta} f'(z) + \frac{\beta}{\alpha+\beta}\left(zf'(z)\right)'\right\} \in \tilde{P}_k(\gamma).$$

The main object of this paper is to investigate the properties of the class $\tilde{R}_k(\alpha,\beta,\gamma)$. Some applications involving integral operators are also considered. To prove our main results, we need the follwing.

Lemma 1.3. [**2**]. *Let p be anlytic in E with $p(0) = 1$ and $p(z) \neq 0$ in E and suppose that*

$$\left|arg[p(z) + \beta_1 z p'(z)]\right| < \frac{\pi}{2}\left(\alpha_1 + \frac{2}{\pi} tan^{-1}\beta_1\alpha_1\right), \quad \alpha_1 > 0, \beta_1 > 0.$$

Then

$$|arg p(z)| < \frac{\alpha_1 \pi}{2} \quad for \quad z \in E.$$

Lemma 1.4. [**3**]. *If $p(z)$ is analytic in $E, p(0) = 1$ and $Re\{p(z)\} > \frac{1}{2}, \quad z \in E$, then for any function F, analytic in E, the function $p \star F$ takes the values in the convex hull of $F(E)$.*

2. Main Results

We prove the following results.

Theorem 2.1.

$$\tilde{R}_k(\alpha,\beta,\gamma) \subset \tilde{R}_k(\alpha, o, \gamma_1), \quad z \in E,$$

where

$$\left\{\gamma = \gamma_1 + \frac{2}{\pi} tan^{-1}\beta_1\gamma_1, \quad \beta_1 = \frac{\beta}{\alpha+\beta}\right\}. \tag{2.1}$$

Proof. Set

$$f'(z) = p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \quad z \in E$$

where $p(z)$ is analytic in E with $p(0) = 1$. Then

$$\left\{p(z) + \frac{\beta}{\alpha+\beta} z p'(z)\right\} \in \tilde{P}_k(\gamma).$$

This implies that

$$\left\{p_i(z) + \frac{\beta}{\alpha+\beta} z p_i'(z)\right\} \in \tilde{P}(\gamma), \quad i = 1, 2.$$

Now, with $\gamma = \alpha_1 + \frac{2}{\pi} tan^{-1}\beta_1\alpha_1, \gamma_1 = \alpha_1, \beta_1 = \frac{\alpha}{\alpha+\beta}$, we apply Lemma 1.4 to have $p_i \in \tilde{P}(\gamma_1)$, $i = 1, 2$, $z \in E$. Consequently, $p \in \tilde{P}_k(\gamma_1)$ and $f \in \tilde{R}_k(\alpha, 0, \gamma_1)$ in E. This compltetes the proof. □

Theorem 2.2. *Let $f \in A$ and let*

$$\left\{ f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right\} \in \tilde{P}_k(\alpha + \frac{2}{\pi} tan^{-1}\frac{\alpha}{\mu}).$$

Then

$$\left(\frac{f(z)}{z} \right)^{\mu} \in \tilde{P}_k(\alpha), \quad \alpha > 0, \mu > 0 \quad and \quad z \in E.$$

Proof. Let. for $\mu > 0$, $\left(\frac{f(z)}{z} \right)^{\mu} = p(z)$, with $p(z)$ defined by (1.1). We note that $p(z)$ is analytic and $p(0) = 1$. Then

$$\left\{ p(z) + \frac{1}{\mu} zp'(z) \right\} = \left\{ f'(z) \left(\frac{f(z)}{z} \right)^{\mu-1} \right\} \in \tilde{P}_k(\alpha_1),$$

where $\alpha_1 = \left(\alpha + \frac{2}{\pi} tan^{-1}\frac{\alpha}{\mu} \right)$. This implies that, for $z \in E$ and $i = 1, 2$,

$$\left\{ p_i(z) = \frac{1}{\mu} zp_i'(z) \right\} \in \tilde{P}(\alpha_1),$$

and, using Lemma 1.3, we have $p_i \in \tilde{P}(\alpha)$. Therefore, by (1.1), $p \in \tilde{P}_k(\alpha)$ and this proves our result □

Theorem 2.3. *Let*

$$F(z) = I_{\mu,c}(f(z)) = \left[\frac{\mu + c}{z^c} \int_0^z f^{\mu}(t) t^{c-1} dt \right]^{\frac{1}{\mu}}, \tag{2.2}$$

where $\mu > 0, c + \mu > 0$ and $\left(\frac{I_{\mu,c}f(z)}{z} \right) \neq 0$ in E. Let, for $\alpha > 0$,

$$\left\{ f'(z) \left(\frac{f(z)}{z} \right))^{\mu-1} \right\} \in \tilde{P}_k(\alpha + \frac{2}{\pi} tan^{-1}(\frac{\alpha}{\mu + c}).$$

Then

$$\left\{ F'(z) \left(\frac{F(z)}{z} \right)^{\mu-1} \right\} \in \tilde{P}_k(\alpha), \quad z \in E.$$

Proof. Let

$$h(z) = F'(z)\left(\frac{F(z)}{z}\right)^{\mu-1}, \quad z \in E. \tag{2.3}$$

From (2.2) and (2.3), we have

$$\left\{h(z) + \frac{1}{\mu+c} z h'(z)\right\} = \left\{f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right\} \in \tilde{P}_k(\alpha + \frac{2}{\pi} tan^{-1}\frac{\alpha}{\mu+c})$$

and proceeding as before, we obtain the required result. □

Theorem 2.4. *Let* $f_1 \in \tilde{R}_2(\alpha,\beta,\gamma)$, $f_2 \in \tilde{R}_k(\alpha,\beta,\gamma)$, *and let* $\phi(z) = (f_1 \star f_2)(z))$, *where* $\star$ *denotes the convolution (Hadamard product). Then*

$$\frac{(z\phi'(z))'}{\phi'(z)} \in \tilde{P}_k(\lambda),$$

where $\lambda = (\gamma_1 + \eta)$ *and* γ_1, η *are given by (2.1) and (2.4) respectively.*

Proof. Since $f_2 \in \tilde{R}_k(\alpha,\beta,\gamma)$, it follows from Theorem 2.1, $f_2' \in \tilde{P}_k(\gamma_1)$, where γ_1 is as given by (2.1). Similarly $f_1' \in \tilde{P}(\gamma_1)$. Let

$$\begin{aligned} f_2'(z) &= p(z) = \left(\frac{k}{4}+\frac{1}{2}\right)p_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)p_2(z), \\ f_1'(z) &= h(z), \quad p_1, p_2, h \in \tilde{P}(\gamma_1). \end{aligned}$$

Now

$$\begin{aligned} \phi'(z) + z\phi''(z) &= \left(f_1' \star f_2'\right)(z) \\ &= \left(\frac{k}{4}+\frac{1}{2}\right)((h\star p_1)(z)) - \left(\frac{k}{4}-\frac{1}{2}\right)((h \star p_2)(z)) \end{aligned}$$

Applying Theorem 2.1, we have $\phi' \in \tilde{P}_k(\eta)$, where

$$\gamma_1 = \eta + \frac{2}{\pi} tan^{-1}\eta. \tag{2.4}$$

Let $H(z) = \frac{(z\phi'(z))'}{\phi'(z)}$. Then

$$\begin{aligned} |\arg H(z)| &\leq \left|\arg\left(z\phi'(z)\right)'\right| + \left|\arg|\phi'(z)|\right| \\ &< \frac{k}{2}\left(\frac{\gamma_1\pi}{2}\right) + \frac{k}{2}\left(\frac{\eta\pi}{2}\right) \\ &= \frac{k}{2}(\gamma_1+\eta)\frac{\pi}{2} = \frac{k}{4}\lambda\pi. \end{aligned}$$

This implies $H \in \tilde{P}_k(\lambda)$ and the proof is complete. □

Theorem 2.5. *Let $f \in \tilde{R}_k(\alpha,\beta,\gamma)$ and let ψ be a convex univalent function. Then $(\psi \star f) \in \tilde{R}_k(\alpha,\beta,\gamma)$ in E.*

Proof. Let

$$\begin{aligned}\frac{\alpha}{\alpha+\beta}(\psi\star f)'(z) &+ \frac{\beta}{\alpha+\beta}\left(z(\psi\star f)'\right)'(z)\\ &= \frac{\psi(z)}{z}\star\left[\frac{\alpha}{\alpha+\beta}f'(z)+\frac{\beta}{\alpha+\beta}\left(zf'(z)\right)'\right]\\ &= \frac{\psi(z)}{z}\star F(z), \quad F\in\tilde{P}_k(\gamma).\end{aligned}$$

We can write

$$\frac{\psi(z)}{z}\star F(z)=\left(\frac{k}{4}+\frac{1}{2}\right)\left(\frac{\psi(z)}{z}\star F_1(z)\right)-\left(\frac{k}{4}-\frac{1}{2}\right)\left(\frac{\psi(z)}{z}\star F_2(z)\right)$$

Since ψ is convex, $Re\{\frac{\psi(z)}{z}>\frac{1}{2}$ in E, see [1], and $F_i\in\tilde{P}(\gamma)$, $i=1,2$. Therefore, by Lemma 1.4, $\left(\frac{\psi(z)}{z}\star F_i\right)$ lies in the convex hull of $F_i(z)$. Since F_i, $i=1,2$, is analytic in E and $F_i(E)\subset\Omega\equiv\left\{w_i: \ |\arg w_i|<\frac{\gamma\pi}{2}\right\}$, it follows that $\frac{\psi(z)}{z}\star F_i(z)$ lies in Ω. It implies $\frac{\psi(z)}{z}\star F(z)\in\tilde{P}_k(\gamma)$ and consequently $(\psi\star f)\in\tilde{R}_k(\alpha,\beta,\gamma)$.

□

Applications of Theorem 2.5

We shall apply Theorem 2.5 to show that the class $\tilde{R}_k(\alpha,\beta,\gamma)$ is invariant under certain integral operators.

Let $f\in\tilde{R}_k(\alpha,\beta,\gamma)$. Then $f_i,$, $i=1,2,3,4$, also belongs to $\tilde{R}_k(\alpha,\beta,\gamma)$, where

$$\begin{aligned}f_1(z) &= (\phi_1\star f)(z)) = \int_0^z \frac{f(t)}{t}dt, \qquad \phi_1(z)=-\log(1-z)\\ f_2(z) &= (\phi_2\star f)(z) = \frac{2}{z}\int_0^z f(t)dt, \qquad \phi_2(z)=\frac{-2}{z}\left[z+\log(1-z)\right]\\ f_3(z) &= (\phi_3\star f)(z)) = \int_0^z \frac{f(t)-f(xt)}{t-xt}dt, \qquad \phi_3(z)=\frac{1}{1-x}\log\frac{1-xz}{1-z},\\ & \quad |x|\le 1, x\ne 1,\\ f_4(z) &= (\phi_4\star f)(z)) = \frac{1+c}{z^c}\int_0^z t^{c-1}f(t)dt, \phi_4(z)=\sum_{n=1}^{\infty}\frac{1+c}{n+c}z^n, Rec>0.\end{aligned}$$

The proof of this statement is immediate since each ϕ_i, $i=1,2,3,4$ is convex in E.

We now study the converse of Theorem 2.1 as follows.

Theorem 2.6. *Let* $f \in \tilde{R}_k(\alpha, o, \gamma)$. *Then* $f \in \tilde{R}_k(\alpha, \beta, \gamma)$ *for* $|z| < r_{\beta_1}$, *where*

$$r_{\beta_1} = \frac{1}{2\beta_1 + \sqrt{4\beta^2{}_1 - 2\beta_1 + 1}}, \quad \beta_1 \neq \frac{1}{2}, \quad \beta_1 = \frac{\beta}{\alpha + \beta}. \tag{2.5}$$

Proof. Let

$$\begin{aligned} \phi_{\beta_1}(z) &= (1-\beta_1) f'(z) + \beta_1 \left(z f'(z)\right)' \\ &= \frac{\alpha}{\alpha+\beta} f'(z) + \frac{\beta}{\alpha+\beta}\left(z f'(z)\right)'. \end{aligned}$$

Then

$$\begin{aligned} \phi_{\beta_1}(z) &= \frac{\psi_{\beta_1}(z)}{z} \star f'(z) \\ &= \left(\frac{k}{4} + \frac{1}{2}\right)\left(\frac{\psi_{\beta_1}(z)}{z} \star p_1(z)\right) - \left(\frac{k}{4} - \frac{1}{2}\right)\left(\frac{\psi_{\beta_1}(z)}{z} \star p_2(z)\right) \end{aligned} \tag{2.6}$$

where

$$\psi_{\beta_1}(z) = (1-\beta_1)\frac{z}{1-z} + \beta_1 \frac{z}{(1-z)^2}.$$

Now $\psi_{\beta_1}(z)$ is convex for $|z| < r_{\beta_1}$, which implies $Re\{\frac{\psi_\beta(z)}{z}\} > \frac{1}{2}$ for $|z| < r_{\beta_1}$, and r_{β_1} is given by (2.5). Therefore, from Lemma 1.4, $\left(\frac{\psi_{\beta_1}}{z} \star p_i\right)$, $i = 1, 2$ lies in the convex hull of $p_i(E)$ in $|z| < r_{\beta_1}$. Since p_i is analytic in E and

$$p_i(E) \subset \Omega \equiv \left\{w_i : |\arg w_i| < \frac{\gamma\pi}{2}\right\}, \quad \left(\frac{\psi_{\beta_1}(z)}{z} \star p_i(z)\right)$$

lies in Ω for $|z| < r_{\beta_1}$. It implies that $\left(\frac{\psi_{\beta_1}}{z} \star p\right) \in \tilde{P}_k(\gamma)$ for $|z| < r_{\beta_1}$ and consequently $f \in \tilde{R}_k(\alpha, \beta, \gamma)$ for $|z| < r_{\beta_1}$, r_{β_1} given by (2.5). □

Acknowledgement. The authors wish to express deep gratitude to Dr. S. M. Junaid Zaidi, Rector, CIIT, for his support and providing excellent research facilities. This research is supported by the Higher Education Commission, Pakistan, through research grant No: 1-28/HEC/HRD/2005/90.

References

[1] P. L. Duren, *Univalent Functions,* Springer-Verlag, New York, 1983.

[2] A. Y. Lashin, *Applications of Nunokawa's theorem,* J. Ineq. Pure Appl. Math., **5**(2004), 1-5, Article 111.

[3] S. Singh and R. Singh, *Convolution propertis of a class of starlike functions,* Proc. Amer. Math. Soc., **106**(1989), 145-152.