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POINT DERIVATIONS ON SECOND DUALS AND UNITIZATION OF BANACH ALGEBRAS

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Abstract. In this paper, among other things, we show that every point derivation on a Banach algebra $\mathcal{A}$ is zero if and only if every point derivation on $\mathcal{A}^\sharp$ (the unitization of $\mathcal{A}$) is zero. So we show that every point derivation on $\mathcal{A}$ is zero if there are no non-zero point derivations on $\mathcal{A}^{**}$, the second dual of $\mathcal{A}$ with the first Arens product.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra. We set $\mathcal{A}^\flat = \mathcal{A} \oplus \mathbb{C}$ by product $(a, c)(b, c') = (ab + cb + c'a, cc')$. Then $\mathcal{A}^\flat$ is a Banach algebra with the following norm:

$$\|(a, c)\| = \|a\| + \|c\| \qquad (a \in \mathcal{A}\ ,\ c \in \mathbb{C}).$$

We set $\mathcal{A}^\sharp$ (the unitization of $\mathcal{A}$) to be $\mathcal{A}$ when $\mathcal{A}$ is unital, and to be $\mathcal{A}^\flat$ otherwise. Let $\mathcal{A}$ be a Banach algebra and let X be a Banach $\mathcal{A}$-module. Then X^* is a Banach $\mathcal{A}$-module if for every $a \in \mathcal{A}$, $x \in X$ and $x^* \in X^*$ we define

$$\langle x, ax^*\rangle = \langle xa, x^*\rangle, \qquad \langle x, x^*a\rangle = \langle ax, x^*\rangle.$$

Let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a Banach algebras homomorphism, then $\mathcal{B}$ is a $\mathcal{A}$-module by the following module actions

$$a.b = \varphi(a)b, \qquad b.a = b\varphi(a) \qquad\qquad (a \in \mathcal{A}, b \in \mathcal{B}).$$

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We denote $\mathcal{B}_\varphi$ the above $\mathcal{A}$-module. For a Banach algebra $\mathcal{A}$, $\mathcal{A}^{**}$ with the following product is a Banach algebra.

$$a'' \cdot b'' = w^*\text{-}\lim_\alpha \lim_\beta a_\alpha\, b_\beta$$

where $a'' = w^*\text{-}\lim_\alpha a_\alpha,\;\; b'' = w^*\text{-}\lim_\beta b_\beta$.

Let X is a Banach $\mathcal{A}$-module then a derivation from $\mathcal{A}$ into X is a continuous linear map $D : \mathcal{A} \longrightarrow X$, such that for every $a, b \in \mathcal{A}$, $D(ab) = D(a).b + a.D(b)$. Let $x \in X$, we define $\delta_x : \mathcal{A} \longrightarrow X$ as follows

$$\delta_x(a) = a.x - x.a \qquad (a \in \mathcal{A}),$$

then δ_x is a derivation, derivations of this form are called inner derivations. A Banach algebra $\mathcal{A}$ is amenable if $H^1(\mathcal{A}, X^*) = \{o\}$ for every $\mathcal{A}$-module X, where $H^1(\mathcal{A}, X^*)$ is the first cohomology group from $\mathcal{A}$ with coefficients in X^*. This definition was introduced by B. E. Johnson in [8]. $\mathcal{A}$ is weakly amenable if $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$. Bade, Curtis and Dales have introduced the concept of weak amenability for commutative Banach algebras [1]. For every character f on $\mathcal{A}$, every derivation from $\mathcal{A}$ into $\mathbb{C}_f$ is called a point derivation at f.

2. Main results

Let $\mathcal{A}$ be a Banach algebra. Then we say that

a) $\mathcal{A}$ is supper weakly amenable if for every Banach algebra $\mathcal{B}$ and every continuous homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, if d_φ is a (bounded) derivation from $\mathcal{A}$ into ${\mathcal{B}_\varphi}^*$, then the following condition holds

$$\langle d_\varphi(a), \varphi(b)\rangle + \langle d_\varphi(b), \varphi(a)\rangle = 0 \qquad (a, b \in \mathcal{A}) \qquad (1).$$

b) $\mathcal{A}$ is semiweakly amenable if every derivation $D : \mathcal{A} \longrightarrow \mathcal{A}^*$, by the following property

$$\langle D(a), b\rangle + \langle D(b), a\rangle = 0 \qquad (a, b \in \mathcal{A}) \qquad (2),$$

is an inner derivation (see [3]).

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is weakly amenable if and only if $\mathcal{A}$ is supper weakly amenable and semiweakly amenable.*

Proof. It is Theorem 3.3 of [3]. □

Example 2.2. *a) Let $\mathcal{A} = \mathbb{C}$ by the product $ab = 0, (a, b \in \mathbb{C})$. Then $\mathcal{A}$ is semiweakly amenable but $\mathcal{A}$ is not supper weakly amenable [3].*

b) Let $S = \{t, 0\}$ by products $t0 = 0t = t^2 = 0^2 = 0$, then $l^1(S)$ is semiweakly amenable non-supper weakly amenable [3].

c) Let G be a locally compact topological group, then $M(G)$ is supper weakly amenable if and only if G is discrete [3].

d) Let E be Banach space without approximation property, then $\mathcal{N}(E)$ is supper weakly amenable, non-semiweakly amenable Banach algebra [3].

Theorem 2.3. *Let $\mathcal{A}$ be a Banach algebra. There are no non-zero point derivations on $\mathcal{A}^\sharp$ if and only if there are no non-zero point derivations on $\mathcal{A}$.*

Proof. Let $f : \mathcal{A} \longrightarrow \mathbb{C}$ be a character on $\mathcal{A}$. Then we define $f^\sharp : \mathcal{A}^\sharp \longrightarrow \mathbb{C}$ as follows

$$f^\sharp((a,c)) = f(a) + c \qquad (a \in \mathcal{A}, c \in \mathbb{C}).$$

For every $a, b \in \mathcal{A}$ and $c, c' \in \mathbb{C}$, we have

$$\begin{aligned} f^\sharp((a,c)(b,c')) &= f^\sharp((ab + cb + c'a, cc')) \\ &= f(ab + cb + c'a) + cc' \\ &= f(a)f(b) + cf(b) + c'(f(a) + cc' \\ &= (f(a) + c)(f(b) + c') \\ &= f^\sharp((a,c))f^\sharp((b,c')). \end{aligned}$$

Thus $f^\sharp$ is a character on $\mathcal{A}^\sharp$. Let now $D : \mathcal{A} \longrightarrow \mathbb{C}_f$ be a point derivation at f. We define $D^\sharp : \mathcal{A}^\sharp \longrightarrow \mathbb{C}$ by

$$D^\sharp(a,c) = D(a) \qquad (a \in \mathcal{A}, c \in \mathbb{C}).$$

For every $a, b \in \mathcal{A}$ and $c, c' \in \mathbb{C}$, we have

$$\begin{aligned} D^\sharp((a,c)(b,c')) &= D^\sharp((ab + cb + c'a, cc')) = D(ab + cb + c'a) \\ &= D(a)f(b) + f(a)D(b) + cD(b) + c'D(a) \\ &= D(a)(f(b) + c') + D(b)(f(a) + c) \\ &= D^\sharp(a,c)f^\sharp(b,c') + f^\sharp(a,c)D^\sharp(b,c'). \end{aligned}$$

This means that $D^\sharp$ is a point derivation at $f^\sharp$. Thus there is $c_1 \in \mathbb{C}$ such that $D^\sharp = \delta_{c_1}$. It is easy to show that $D = \delta_{c_1}$. For the converse, let $f : \mathcal{A}^\sharp \longrightarrow \mathbb{C}$ be a character on $\mathcal{A}^\sharp$ and let $D : \mathcal{A}^\sharp \longrightarrow \mathbb{C}_f$ be a point derivation at f. Set $f_1 = f \mid_{\mathcal{A}}$ and $D_1 = D \mid_{\mathcal{A}}$. Then it is easy to show that f_1 is a character on $\mathcal{A}$, and that D_1 is a point derivation at f_1. Thus there is a $c' \in \mathbb{C}$ such that $D_1 = \delta_{c'}$. For every $a \in \mathcal{A}$ and $c \in \mathbb{C}$, we have

$$\begin{aligned} D(a,c) &= D(a,0) + D(0,c) = D_1(a) \\ &= \delta_{c'}(a) = f_1(a)c' - c'f_1(a) \\ &= f(a,c)c' - c'f(a,c) \\ &= \delta_{c'}((a,c)). \end{aligned}$$

So the proof is complete. □

We know that the weak amenability of $\mathcal{A}^\sharp$ dose not implies the weak amenability of $\mathcal{A}$ [9]. But by above theorem we give the following result.

Corollary 2.4. *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}^\sharp$ is weakly amenable, then there are no non-zero point derivations on $\mathcal{A}$.*

Let $\mathcal{A}^{**}$ be the second dual of $\mathcal{A}$ with the first Arens product. Then amenability of $\mathcal{A}^{**}$ implies the amenability of $\mathcal{A}$ (see for example Proposition 2.8.59 of [2]). So weak amenability of $\mathcal{A}^{**}$ implies the weak amenability of $\mathcal{A}$ if one of the following conditions holds (see [4], [5] , [6] and [7]).

a) $\mathcal{A}$ is a left ideal in $\mathcal{A}^{**}$.

b) $\mathcal{A}$ is a dual Banach algebra.

c) $\mathcal{A}$ is Arens regular and every derivation from $\mathcal{A}$ into its dual is weakly compact.

d) $\mathcal{A}$ is a right ideal in $\mathcal{A}^{**}$ and $\mathcal{A}^{**}\mathcal{A} = \mathcal{A}^{**}$.

Similarly for supper weak amenability we have the following Theorem.

Theorem 2.5. *Let $\mathcal{A}$ be a Banach algebra with one of the conditions a), b), c) or d) as above. Let $\mathcal{A}^{**}$ be supper weakly amenable, then $\mathcal{A}$ is supper weakly amenable.*

Proof. Step I. Let $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a derivation, then D has an extension $\bar{D} : \mathcal{A}^{**} \longrightarrow (\mathcal{A}^{**})^*$ in which $\bar{D}$ is a derivation (see [4], [5], [6] and [7]). Since $\mathcal{A}^{**}$ is supper weakly amenable, then for every $a, b \in \mathcal{A}$, we have

$$\langle D(a), b\rangle + \langle D(b), a\rangle = \langle \bar{D}(\hat{a}), \hat{b}\rangle + \langle \bar{D}(\hat{b}), \hat{a}\rangle = 0.$$

Step II. Now let $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ be a Banach algebra homomorphism and let $d_\varphi : \mathcal{A} \longrightarrow {\mathcal{B}_\varphi}^*$ be a derivation. Set $D = d_\varphi \otimes \varphi : \mathcal{A} \longrightarrow \mathcal{A}^*$ as follows

$$\langle D(a), b\rangle = \langle d_\varphi(a), \varphi(b)\rangle \qquad (a, b \in \mathcal{A}).$$

For every $a, b, c \in \mathcal{A}$, we have

$$\begin{aligned}\langle D(ab), c\rangle &= \langle d_\varphi(ab), \varphi(c)\rangle \\ &= \langle d_\varphi(a)\varphi(b), \varphi(c)\rangle + \langle \varphi(a)d_\varphi(b), \varphi(c)\rangle \\ &= \langle d_\varphi(a), \varphi(b)\varphi(c)\rangle + \langle d_\varphi(b), \varphi(c)\varphi(a)\rangle \\ &= \langle d_\varphi(a), \varphi(bc)\rangle + \langle d_\varphi(b), \varphi(ca)\rangle \\ &= \langle D(a), bc\rangle + \langle D(b), ca\rangle \\ &= \langle D(a)b + aD(b), c\rangle.\end{aligned}$$

Therefore D is a derivation. Then by step I, we have

$$\langle d_\varphi(a), \varphi(b)\rangle + \langle d_\varphi(b), \varphi(a)\rangle = \langle D(a), b\rangle + \langle D(b), a\rangle = 0 \qquad (a, b \in \mathcal{A}).$$

So $\mathcal{A}$ is supper weakly amenable. □

Theorem 2.6. *Let $\mathcal{A}$ be a Banach algebra. If there are no no-zero point derivations on $\mathcal{A}^{**}$, then there are no no-zero point derivations on $\mathcal{A}$.*

Proof. Let $\varphi : \mathcal{A} \longrightarrow \mathbb{C}$ be a character on $\mathcal{A}$. Then it is easy to show that $\varphi'' : \mathcal{A}^{**} \longrightarrow \mathbb{C}$ is a character of $\mathcal{A}^{**}$. Let $d_\varphi : \mathcal{A} \longrightarrow \mathbb{C}_\varphi$ be a point derivation at φ. We have to show that $(d_\varphi)''$ is a point derivation at φ''. Let $a'', b'' \in \mathcal{A}^{**}$ then there are nets (a_α) and (b_β) in $\mathcal{A}$ such that converge respectively to a'' and b'' in the *weak**- topology of $\mathcal{A}^{**}$. We have

$$\begin{aligned}(d_\varphi)''(a''b'') &= w^*\text{-}\lim_\alpha \lim_\beta d_\varphi(a_\alpha b_\beta) \\ &= w^*\text{-}\lim_\alpha \lim_\beta d_\varphi(a_\alpha)\varphi(b_\beta) + w^*\text{-}\lim_\alpha \lim_\beta \varphi(a_\alpha)d_\varphi(b_\beta) \\ &= (d_\varphi)''(a'').\varphi''(b'') + \varphi''(a'').(d_\varphi)''(b'').\end{aligned}$$

This means that $(d_\varphi)'' : \mathcal{A}^{**} \longrightarrow \mathbb{C}$ is a point derivation at φ''. Thus $d_\varphi = 0$. □

Theorem 2.7. *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}^{**}$ is supper weakly amenable, then*

a) $\mathcal{A}$ is essential.

b) There are no no-zero continuous point derivations on $\mathcal{A}$.

Proof. a) By Theorem 3.2 of [3], $\mathcal{A}^{**}$ is essential. Then $\mathcal{A}$ is essential [6].
b) By Theorem 3.2 of [3], there are no no-zero continuous point derivations on $\mathcal{A}^{**}$. Then by Theorem 2.6, every point derivation on $\mathcal{A}$ is zero. □

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