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COMMON FIXED POINT THEOREMS IN GENERALIZED NORMED SPACES

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Abstract. In this paper, we introduce the new concept of generalized normed space and prove some common fixed point theorems for self mappings in these spaces. Our results extend some of the known results in literature.

1. Introduction

It is well known that the Banach contraction principle is a fundamental result in fixed point theory. After this classical result, fixed and common fixed point Theorem in different types of spaces have been developed. For example, Ultrametric spaces [2]. In this paper we introduce the new definitions of normed spaces and give some properties of it and we prove a common fixed point theorem.

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2. Preliminaries

In this section, first we define binary operation and give some examples.

A binary operation is a mapping $\diamond : [0,\infty) \times [0,\infty) \longrightarrow [0,\infty)$ which satisfy the following conditions:

(i) $\diamond$ is associative and commutative,
(ii) $\diamond$ is continuous,
(iii) $a \diamond 0 = a$ for all $a \in [0,\infty)$,
(iv) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,\infty)$.

Let $a, b \in [0,\infty)$. Five typical examples of $\diamond$ are:

(a) $a \diamond_1 b = \max\{a, b\}$
(b) $a \diamond_2 b = \sqrt{a^2 + b^2}$
(c) $a \diamond_3 b = a + b$
(d) $a \diamond_4 b = ab + a + b$
(e) $a \diamond_5 b = (\sqrt{a} + \sqrt{b})^2$.

For $a, b \in [0,\infty)$, it is easy to see that:

$$a \diamond_1 b \leq a \diamond_2 b \leq a \diamond_3 b,$$

and

$$a \diamond_3 b \leq \min\{a \diamond_4 b, a \diamond_5 b\}.$$

The following lemma show a binary operation which is induced of a self map on $[0,\infty)$.

Lemma 2.1. ([2]) *Let $f : [0,\infty) \longrightarrow [0,\infty)$ be a continuous, onto, and increasing map. If define $a \diamond b = f^{-1}(f(a) + f(b))$ for each $a, b \in [0,\infty)$, then $\diamond$ is a binary operation.*

Example 2.2. ([2]) $f : [0,\infty) \longrightarrow [0,\infty)$ defined by $f(x) = e^x - 1$. Then f is a continuous, onto and increasing map and $a \diamond b = Ln(e^a + e^b - 1)$ for $a, b \in [0,\infty)$ is a binary operation.

We have the following simple Lemma.

Lemma 2.3. ([2])

(i) *If $r, r' \geq 0$, then $r \leq r \diamond r'$.*
(ii) *If $\delta \in (0, r)$, there exist $\delta' \in (0, r)$ such that $\delta' \diamond \delta < r$.*
(iii) *For every $\varepsilon > 0$, there exists $\delta > 0$ such that $\delta \diamond \delta < \varepsilon$.*

Here afterwards, we assume that X is a vector space over field $\mathbb{R}$.

Now we introduce the new concept of a normed space which is generalized of ordinary normed space.

Definition 2.4. Let X be a vector space and $\diamond$ be a binary operation. A generalized normed on X is a function: $N : X \longrightarrow \mathbb{R}$ that satisfies the following properties:

(1) $N(x) \geq 0$ for each x in X,
(2) $N(x) = 0$ if and only if $x = 0$,
(3) $N(\alpha x) = |\alpha|^t N(x)$ for some $t \in (0, \infty)$, for each x in X and every $\alpha \in \mathbb{R}$,
(4) $N(x + y) \leq N(x) \diamond N(y)$, for each $x, y \in X$.

The $3-$*tuple* $(X, N, \diamond)$ is called a generalized normed space or a $G-$*normed* space.

Example 2.5. Let $(X, \|.\|)$ be a normed space, $a, b \in [0, \infty)$, and $x \in X$. If we define $\diamond : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$,

(i) $a \diamond b = a + b$, and N is defined by $N(x) = \|x\|$, then $(X, N, \diamond)$ is a $G-normed$ space for $t = 1$.
(ii) $a \diamond b = \sqrt{a^2 + b^2}$, and N is defined by $N(x) = \sqrt{\|x\|}$, then $(X, N, \diamond)$ is a $G-normed$ space for $t = \frac{1}{2}$.
(iii) $a \diamond b = (\sqrt{a} + \sqrt{b})^2$, and N is defined by $N(x) = \|x\|^2$, then $(X, N, \diamond)$ is a $G-normed$ space for $t = 2$.

Remark 2.6. It is easy to see that: every normed space is a $G-normed$ space and t in Definition 2.4 is unique.

Example 2.7. Let $X = \mathbb{R}^2$, if we define $\diamond : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ by $a \diamond b = (\sqrt[4]{a} + \sqrt[4]{b})^4$ for $a, b \in [0, \infty)$, and define $N : X \longrightarrow R$ by $N(x, y) = x^4 + y^4$ for $x, y \in \mathbb{R}$, then $(X, N, \diamond)$ is a $G-normed$ space for $t = 4$.

Definition 2.8. Let $(X, N, \diamond)$ be a $G-normed$ space, $r > 0$ and $A \subset X$.

(1) The set $B_N(x, r) = \{y \in X : N(x - y) < r\}$ is called an open ball centered at x and radius r.

(2) If for every $x \in A$ there exists $r > 0$ such that $B_N(x, r) \subset A$, then the subset A is called open subset of X.

(3) The subset A of X is said to be N-bounded if there exists $r > 0$ such that $N(x - y) < r$ for all $x, y \in A$.

(4) A sequence $\{x_n\}$ in X converges to x if $N(x_n - x) \to 0$ as $n \to \infty$ and write $\lim_{n\to\infty} x_n = x$. That is for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $N(x_n - x) < \epsilon$ for all $n \geq n_0$, then $\{x_n\}$ converges to x.

(5) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $N(x_n - x_m) < \epsilon$ for all $n, m \geq n_0$.

(6) The generalized normed space $(X, N, \diamond)$ is said to be generalized Banach space or $G - Banach$ space if every Cauchy sequence is convergent.

Let τ be the set of all open subsets $A \subseteq X$. It can be verified that τ is a topology on X, called a topology induced by generalized normed N.

Lemma 2.9. *Let $(X, N, \diamond)$ be a $G-$normed space.Then*

(i) $N(ax) \leq N(x)$ *for all real number a with $|a| \leq 1$.*

(ii) *if X is convex, then we get*

$$N(ax + (1-a)y) \leq N(x) \diamond N(y)$$

for all $x, y \in X$.

Proof. It follows immediately from Definition 2.4. □

Now we prove the following basic lemmas needed in the sequel.

Lemma 2.10. *Let $(X, N, \diamond)$ be a $G-$ normed space. If $r > 0$, then the ball $B_N(x, r)$ is open.*

Proof. Let $y \in B_N(x, r)$, so that we have $N(x - y) < r$. Put, $N(x - y) = \delta$ then by Lemma 2.3 there exists $\delta' > 0$ such that $\delta' \diamond \delta < r$. Now, we prove that $B_N(y, \delta') \subseteq B_N(x, r)$. For this, let $z \in B_N(y, \delta')$. By triangle inequality we have

$$N(x - z) \leq N(x - y) \diamond N(y - z) < \delta \diamond \delta' < r.$$

This implies that

$$B_N(y, \delta') \subseteq B_N(x, r).$$

Hence $B_N(x, r)$ is an open set. □

Lemma 2.11. *Every $G-$ normed space $(X, N, \diamond)$ is a Hausdorff space.*

Proof. Let $x, y \in X$ and $x \neq y$. If set $N(x - y) = r$ then for $0 < \delta < r$ by Lemma 2.3 there exists $0 < \delta' < r$ such that $\delta' \diamond \delta < r$. We prove that $B_N(x, \delta) \cap B_N(y, \delta') = \varnothing$. Let $z \in B_N(x, \delta) \cap B_N(y, \delta')$. Now, by triangle inequality, we get that

$$r = N(x - y) \leq N(x - z) \diamond N(z - y) < \delta \diamond \delta' < r,$$

which is a contradiction. Hence $(X, N, \diamond)$ is a Hausdorff space. □

Lemma 2.12. *Let $(X, N, \diamond)$ be a $G-$ normed space, then every convergent sequence in X is Cauchy in X.*

Proof. Let $\{x_n\}$ be a sequence in X which converges to $x \in X$. For $\epsilon > 0$, by Lemma 2.3 we choose a $\delta > 0$ such that $\delta \diamond \delta < \epsilon$. Since $x_n \longrightarrow x$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we obtain that $N(x_n - x) < \delta$. Thus for every $n, m \geq n_0$, we have

$$N(x_n - x_m) \leq N(x_n - x) \diamond N(x - x_m) < \delta \diamond \delta < \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence. □

Lemma 2.13. *Let* $(X, N, \diamond)$ *be a* $G-$*normed space, then addition* $+ : X \times X \longrightarrow X$ *defined by* $+(x, y) = x+y$ *and scalar multiplication* $\cdot : \mathbb{R} \times X \longrightarrow X$ *defined by* $\cdot(\alpha, x) = \alpha \cdot x$ *are continuous.*

Proof. First we prove continuity of addition. Let $x_n \longrightarrow x$, $y_n \longrightarrow y$. By Lemma 2.3, for each $\epsilon > 0$ there exists $\delta > 0$ such that $\delta \diamond \delta < \epsilon$. Also, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \Longrightarrow N(x_n - x) < \delta,$$

and

$$n \geq n_0 \Longrightarrow N(y_n - y) < \delta.$$

By triangle inequality we have

$$N((x_n + y_n) - (x + y)) \leq N(x_n - x) \diamond N(y_n - y) < \delta \diamond \delta < \epsilon.$$

Now we prove that scalar multiplication is continuous. Let $\alpha_n \longrightarrow \alpha$ and $x_n \longrightarrow x$(which means that $\lim_{n \longrightarrow \infty} N(x_n - x) = 0$). Triangle inequality gives that

$$\begin{aligned} N(\alpha_n \cdot x_n - \alpha \cdot x) &= N(\alpha_n \cdot (x_n - x) + (\alpha_n - \alpha) \cdot x) \\ &\leq |\alpha_n|^t N(x_n - x) \diamond |\alpha_n - \alpha|^t N(x), \end{aligned}$$

hence

$$\limsup_{n \longrightarrow \infty} N(\alpha_n \cdot x_n - \alpha \cdot x) \leq \lim_{n \longrightarrow \infty} |\alpha_n|^t N(x_n - x) \diamond \lim_{n \longrightarrow \infty} |\alpha_n - \alpha|^t N(x) = 0.$$

Therefore

$$\lim_{n \longrightarrow \infty} \alpha_n . x_n = \alpha . x.$$

□

Corollary 2.14. *Let* $a \diamond b = \max\{a, b\}$, *then there is not any* $t \in (0, \infty)$ *such that* $N(\alpha \cdot x) = |\alpha|^t \cdot N(x)$.

Proof. Let there exists $t \in (0, \infty)$, then for $\alpha = 2$, we have

$$|2|^t \cdot N(x) = N(2x) = N(x + x) \leq N(x) \diamond N(x) = N(x)$$

that is $|2|^t \leq 1$, which is a contradiction. □

Henceforth, we assume that binary operation $\diamond$ on $[0,\infty)\times[0,\infty)$ satisfy the following properties:
(PI) : $\alpha\cdot(a\diamond b)=\alpha\cdot a\diamond\alpha\cdot b$ for every $\alpha\in\mathbb{R}^+$ and
(PII) : there exists $h\geq 0$ such that $1\diamond 1\diamond\cdots\diamond 1\leq n^h$, for every $n\in N$.

In the following example, we give some binary operations $\diamond$ with properties (PI) and $P(II)$.

Example 2.15. Let $a\diamond b=\max\{a,b\}$, $a\diamond b=\sqrt{a^2+b^2}$, $a\diamond b=a+b$ or $a\diamond b=(\sqrt{a}+\sqrt{b})^2$, then in each case, $\diamond$ satisfies properties (PI) and (PII).

The next example includes a binary operation $\diamond$ which does not satisfy above properties.

Example 2.16. Define $\diamond:[0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$ by $a\diamond b=a+b+ab$, for $a,b\in[0,\infty)$. Obviously $\diamond$ have not properties (PI) and (PII).

Lemma 2.17. *Let $(X,N,\diamond)$ be a $G-$normed space. If there exists a sequence $\{x_n\}$ in X such that*

$$N(x_n-x_{n+1})\leq kN(x_{n-1}-x_n)$$

for each $n\in\mathbb{N}$, and some $0<k<1$, then $\{x_n\}$ is a Cauchy sequence.

Proof. For every $n,m\in\mathbb{N}$, we have

$$\begin{aligned}
N(x_n-x_m) &\leq N(x_n-x_{n+1})\diamond N(x_{n+1}-x_{n+2})\diamond\cdots\diamond N(x_{m-1}-x_m)\\
&\leq k^nN(x_0-x_1)\diamond k^{n+1}N(x_0-x_1)\diamond\cdots\diamond k^{m-1}N(x_0-x_1)\\
&= k^nN(x_0-x_1)(1\diamond k\diamond k^2\diamond\cdots\diamond k^{m-n-1})\\
&\leq k^nN(x_0-x_1)(\underbrace{1\diamond 1\diamond 1\diamond\cdots\diamond 1}_{m-n})\\
&\leq k^nN(x_0-x_1)(\underbrace{1\diamond 1\diamond 1\diamond\cdots\diamond 1}_{m})\\
&\leq k^nN(x_0-x_1)\cdot m^h.
\end{aligned}$$

It is easy to see that for every $m\geq n$, there exists $s>0$ such that $m\leq n^s$. Thus

$$N(x_n-x_m)\leq k^nN(x_0-x_1)\cdot n^{hs}\longrightarrow 0.$$

Hence $\{x_n\}$ is a Cauchy sequence. □

3. Main Results

Let C be a nonempty convex subset of X and S,T self-map on C. For a couple of mapping S,T, we consider an Ishikawa scheme [1] which is defined by $x_0 \in C$

$$(IS)\qquad \begin{cases} y_n = (1-\beta_n)x_n + \beta_n S x_n, \ n \geq 0, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \ n \geq 0, \end{cases}$$

where the real sequences $\{\alpha_n\}, \{\beta_n\}$ satisfy

(i) $0 \leq \alpha_n \leq 1$, $0 \leq \beta_n \leq 1$, for $n \geq 0$

(ii) $\lim\limits_{n \longrightarrow \infty} a_n = \alpha > 0$.

Now we prove the main results of this paper.

Let Φ denote a family of mappings $\phi : [0,\infty) \longrightarrow [0,\infty)$ such that for each $\phi \in \Phi$,

(i) ϕ is continuous and increasing,

(ii) $\phi(t) < t$ for every $t > 0$.

An immediate example of such a function is: $\phi : [0,\infty) \longrightarrow [0,\infty)$ and $\phi(t) = kt, \ 0 < k < 1$.

Theorem 3.1. *Let $(X, N, \diamond)$ be a $G-$normed space and C a nonempty closed, convex subset of X and let S,T be two self-mappings of C satisfying the following condition:*

$$(\mathbf{A1}) : N(Sx - Ty) \leq \phi\left(\max \left\{ \begin{array}{c} N(x-y), \ N(x-Sx), \ N(y-Ty), \\ N(x-Ty), \ N(y-Sx) \end{array} \right\} \right)$$

for every $x, y \in C$, and $\phi \in \Phi$. Suppose that for some $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ of Ishikawa iterates converges to a point u and S is continuous in u, then u is a unique fixed point of S and T.

Proof. Suppose first that, $Su = u$ for a point u in C. Then putting $x = y = u$ into inequality $(\mathbf{A1})$ gives $Tu = u$. For if $Tu \neq u$, then we get

$$\begin{aligned} N(u - Tu) &= N(Su - Tu) \\ &\leq \phi\left(\max \left\{ \begin{array}{c} N(u-u), \ N(u-Su), \ N(u-Tu), \\ N(u-Tu), \ N(u-Su) \end{array} \right\} \right) \\ &= \phi(N(u-Tu)) < N(u-Tu), \end{aligned}$$

which is a contradiction. Hence $Tu = u$.

Now let $\{x_n\}$ be a sequence of Ishikawa iterates with S and T such that

$$\lim_{n \longrightarrow \infty} x_n = u.$$

From (IS), we see that

$$\begin{aligned} u &= \lim_{n\longrightarrow\infty} x_{n+1} = \lim_{n\longrightarrow\infty}(1-\alpha_n)x_n + \lim_{n\longrightarrow\infty}\alpha_n T y_n \\ &= (1-\alpha)u + \alpha \lim_{n\longrightarrow\infty} T y_n. \end{aligned}$$

We get $\lim_{n\longrightarrow\infty} T y_n = u$. Since

$$N(u - Su) \leq N(u - Ty_n) \diamond N(Ty_n - Sx_n) \diamond N(Sx_n - Su). \tag{3.1}$$

step (I) : $\lim_{n\longrightarrow\infty} N(u - Ty_n) = 0$.
step (II) : Since S is continuous in u, we have $\lim_{n\longrightarrow\infty} N(Sx_n - Su) = 0$.
step (III) : Putting $x = x_n, y = y_n$ in (**A1**), we get

$$\begin{aligned} &N(Sx_n - Ty_n) \\ &\leq \phi\left(\max \left\{ \begin{array}{c} N(x_n - y_n),\ N(x_n - Sx_n),\ N(y_n - Ty_n), \\ N(x_n - Ty_n),\ N(y_n - Sx_n) \end{array} \right\} \right). \end{aligned} \tag{3.2}$$

We have

$$\begin{aligned} N(x_n - y_n) &= N(x_n - x_n + \beta_n x_n - \beta_n S x_n) \\ &= N(\beta_n(x_n - Sx_n)) \\ &\leq N(x_n - Sx_n) \\ &\leq N(x_n - Ty_n) \diamond N(Ty_n - Sx_n), \end{aligned}$$

hence

$$\lim_{n\longrightarrow\infty} N(x_n - y_n) \leq N(u - Su). \tag{3.3}$$

Also, we have

$$N(x_n - Sx_n) \leq N(x_n - Ty_n) \diamond N(Ty_n - Sx_n),$$

which gives

$$\lim_{n\longrightarrow\infty} N(x_n - Sx_n) \leq N(u - Su). \tag{3.4}$$

Moreover, we get

$$\begin{aligned} N(y_n - Ty_n) &= N((1-\beta_n)x_n + \beta_n S x_n - Ty_n)) \\ &\leq N(x_n - Ty_n) \diamond N(-\beta_n(x_n - Sx_n)) \\ &\leq N(x_n - Ty_n) \diamond N(x_n - Sx_n), \end{aligned}$$

hence

$$\lim_{n\longrightarrow\infty} N(y_n - Ty_n) \leq N(u - Su). \tag{3.5}$$

Also we have

$$\begin{aligned} N(y_n - Sx_n) &= N((1-\beta_n)x_n + \beta_n Sx_n - Sx_n)) \\ &= N((1-\beta_n)(x_n - Sx_n)) \\ &\le N(x_n - Sx_n) \\ &\le N(x_n - Ty_n) \diamond N(Ty_n - Sx_n), \end{aligned}$$

which gives

$$\lim_{n\longrightarrow\infty} N(y_n - Sx_n) \le \quad N(u - Su). \tag{3.6}$$

Substituting inequalities (3.3), (3.4), (3.5) and (3.6) into inequality (3.2) and letting $n \longrightarrow \infty$, we obtain

$$\lim_{n\longrightarrow\infty} N(Sx_n - Ty_n) \le \phi(N(u - Su)).$$

By inequality (3.1), we get

$$N(u - Su) \le \phi(N(u - Su)).$$

Hence $Su = u$. Because if $u \neq Su$, then we get

$$N(u - Su) \le \phi(N(u - Su)) < \quad N(u - Su),$$

which is a contradiction. Thus

$$Tu = Su = u.$$

For uniqueness of u, suppose that υ is another common fixed point of T, S. Putting $x = u, y = \upsilon$ in (**A1**) we get

$$\begin{aligned} N(u - \upsilon) &= N(Su - T\upsilon) \\ &\le \phi\left(\max \left\{ \begin{array}{c} N(u-\upsilon),\ N(u - Su),\ N(\upsilon - T\upsilon), \\ N(u - T\upsilon),\ N(\upsilon - Su) \end{array} \right\} \right) \\ &= \phi(N(u - \upsilon)) < N(u - \upsilon), \end{aligned}$$

which is a contradiction. Hence we have $u = \upsilon$. □

Corollary 3.2. *Let* $(X, N, \diamond)$ *be a* $G-$ *normed space and* C *a nonempty closed, convex subset of* X *and let* S, T *be two self-mappings of* C *satisfying the following condition:*

$$(\mathbf{A2}) : N(Sx - Ty) \le k\left(\max \left\{ \begin{array}{c} N(x-y),\ N(x - Sx),\ N(y - Ty), \\ N(x - Ty),\ N(y - Sx) \end{array} \right\} \right),$$

for every $x, y \in C$ *and* $0 < k < 1$. *Suppose that for some* $x_0 \in C$, *the sequence* $\{x_n\}_{n=0}^{\infty}$ *of Ishikawa iterates converges to a point* u *and* S *is continuous in* u. *Then* u *is a unique fixed point of* S *and* T.

Proof. It follows by Theorem 3.1, if we define: $\phi(t) = kt \ for \ 0 < k < 1$. □

Corollary 3.3. *Let $(X, \|\cdot\|)$ be an ordinary normed space and C a nonempty closed, convex subset of X and let S, T be two self-mappings of C satisfying the following condition:*

$$(\mathbf{A3}):\ \|Sx - Ty\| \leq \phi\left(\max\left\{ \begin{array}{c} \|x-y\|,\ \|x-Sx\|,\ \|y-Ty\|, \\ \|x-Ty\|,\ \|y-Sx\| \end{array} \right\} \right),$$

for every $x, y \in C$, and $\phi \in \Phi$. Suppose that for some $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ of Ishikawa iterates converges to a point u and S is continuous in u. Then u is a unique fixed point of S and T.

Proof. It follows By Theorem 3.1 and Remark 2.6. □

Theorem 3.4. *Let $(X, N, \diamond)$ be a $G-$normed space and C a nonempty closed, convex subset of X and let T be a self-mapping of C such that the following condition holds:*

$$(\mathbf{B1}):\ N(Tx - Ty) \leq \phi(\max\{N(x-y),\ N(y-Ty),\ N(x-Tx)\}),$$

for every $x, y \in C$ and $\phi \in \Phi$. Let sequence $\{x_n\}$ be generated by

$$x_0 \in C,\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \tag{3.7}$$

where the real sequence $\{\alpha_n\}$ satisfies $0 \leq \alpha_n \leq 1$, and

$$\lim_{n \longrightarrow \infty} \alpha_n = \alpha > 0 \quad \textit{for} \quad n \geq 0.$$

Suppose that for some $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a point u, Then u is a unique fixed point of T.

Proof. From (3.7), we see that

$$\begin{aligned} u &= \lim_{n \longrightarrow \infty} x_{n+1} = \lim_{n \longrightarrow \infty} (1-\alpha_n)x_n + \lim_{n \longrightarrow \infty} \alpha_n T x_n \\ &= (1-\alpha)u + \alpha \lim_{n \longrightarrow \infty} T x_n, \end{aligned}$$

hence we get $\lim_{n \longrightarrow \infty} T x_n = u$. Putting $x = x_n,\ y = u$ in $(\mathbf{B1})$, we have

$$N(Tx_n - Tu) \leq \phi(\max\{N(x_n - u),\ N(u - Tu),\ N(x_n - Tx_n)\}).$$

Letting $n \longrightarrow \infty$, we get

$$\begin{aligned} N(u - Tu) &\leq \phi(\max\{N(u-u),\ N(u-Tu),\ N(u-u)\}) \\ &= \phi(N(u - Tu)). \end{aligned}$$

Hence we get $Tu = u$.

Suppose υ is another common fixed point of T. Putting $x = u,\ y = \upsilon$ in (**B1**), we have

$$\begin{aligned} N(u-\upsilon) &= N(Tu - T\upsilon) \\ &\leq \phi(\max\{N(u-\upsilon),\ N(\upsilon - T\upsilon),\ N(u-Tu)\}) \\ &= \phi(N(u-\upsilon)). \end{aligned}$$

Therefore, we get $u = \upsilon$. Thus u is the unique common fixed point of T. □

Corollary 3.5. *Let $(X, N, \diamond)$ be a $G-$normed space and C a nonempty closed, convex subset of X and let T be a self-mapping of C such that the following condition holds:*

$$(\mathbf{B2}) : N(Tx - Ty) \leq k(\max\{N(x-y),\ N(y-Ty),\ N(x-Tx)\}),$$

for every $x, y \in C$ and $0 < k < 1$. Let sequence $\{x_n\}$ be generated by

$$x_0 \in C,\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,$$

where the real sequence $\{\alpha_n\}$ satisfies $0 \leq \alpha_n \leq 1$, and

$$\lim_{n\longrightarrow\infty} \alpha_n = \alpha > 0 \quad for \quad n \geq 0.$$

Suppose that for some $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a point u, then u is a unique fixed point of T.

Corollary 3.6. *Let $(X, \|\cdot\|)$ be an ordinary normed space and C a nonempty closed, convex subset of X and let T be a self-mapping of C such that the following condition holds:*

$$(\mathbf{B3}) :\ \|Tx - Ty\| \leq \phi(\max\{\|x-y\|,\ \|y-Ty\|,\ \|x-Tx\|\}),$$

for every $x, y \in C$ and $\phi \in \Phi$. Let sequence $\{x_n\}$ be generated by

$$x_0 \in C,\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,$$

where the real sequence $\{\alpha_n\}$ satisfies $0 \leq \alpha_n \leq 1$, and $\lim\limits_{n\longrightarrow\infty} \alpha_n = \alpha > 0$ for $n \geq 0$.

Suppose that for some $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a point u, then u is a unique fixed point of T.

Proof. Result follows By Theorem 3.4 and Remark 2.6. □

Example 3.7. Let $X = \mathbb{R}$ and $a \diamond b = \sqrt{a^2 + b^2}$ for all $a, b \in \mathbb{R}^+$. Let $N(x) = \sqrt{|x|}$ for every $x \in X$. If define $T : [0,3] \longrightarrow [0,3]$ by $T(x) = \frac{2x+1}{3}$. Also define sequence $\{x_n\}$ by $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ n = 1, 2, \cdots$, where $\alpha_n = \frac{1}{2}$. Then we have $x_{n+1} = \frac{5}{6}x_n + \frac{1}{6}$. It is easy to see that $\lim\limits_{n\longrightarrow\infty} x_n = 1$.

Also

$$N(Tx - Ty) \leq k(\max\{N(x-y),\ N(y-Ty),\ N(x-Tx)\})$$

for $\sqrt{\frac{2}{3}} \leq k < 1$. That is all conditions of Corrollary 3.5 are holds. Hence 1 is a unique fixed point of T.

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