Nonlinear Functional Analysis and Applications
Vol. 13, No. 2 (2008), pp. 277-289

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

APPROXIMATE ALGORITHM OF SOLUTIONS FOR GENERAL NONLINEAR FUZZY MULITVALUED QUASI-VARAIATIONAL INCLUSIONS WITH (G, η)-MONOTONE MAPPINGS

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Abstract. In this paper, a new class of general nonlinear fuzzy mulitvalued quasi-variational inclusions involving (G, η)-monotone mappings in Hilbert spaces is introduced and studied. By using the resolvent operator associated with (G, η)-monotone mappings, an existence theorem of solutions for this kind of fuzzy mulitvalued quasi-variational inclusions is established and a new iterative algorithm is suggested and discussed. The results presented in this paper generalize, improve, and unify some recent results in this field.

1. Introduction

Variational inclusions have wide applications to many fields including, for example, mechanics, physics, optimization and control, nonlinear programming, economics, and engineering sciences. For these reasons, various variational inclusions have been intensively studied in recent years. For details, we refer the reader to [1]–[15], [17]–[26] , and the references therein.

Chang and Zhou [3] introduced and investigated a class of variational inequalities for fuzzy mappings in 1989. Afterwards, Chang and Huang [4], Ding and Jong [5], Jin[14], Tain [18] and others studies Several kinds of variational inequalities (inclusions) for fuzzy mappings.

^{0}Received August 21, 2007. Revised April 7, 2008.
^{0}2000 Mathematics Subject Classification: 49J40; 47H06.
^{0}Keywords: General nonlinear fuzzy mulitvalued quasi-variational nclusions, (G, η)-monotone mapping, resolvent operator, convergence, iterative algorithm.

On the other hand, Monotonicity techniques were extended and applied in recent years because of their importance in the theory of variational inequalities, complementarity problems, and variational inclusions. In 2003, Huang and Fang [12] introduced a class of generalized monotone mappings, maximal η-monotone mappings, and defined an associated resolvent operator. Using resolvent operator methods, they developed some iterative algorithms to approximate the solution of a class of general variational inclusions involving maximal η-monotone operators. Huang and Fang's method extended the resolvent operator method associated with an η-subdifferential operator due to Ding and [6]. In [7], Fang and Huang introduced another class of generalized monotone operators, H-monotone operators, and defined an associated resolvent operator. They also established the Lipschitz continuity of the resolvent operator and studied a class of variational inclusions in Hilbert spaces using the resolvent operator associated with H-monotone operators. In a recent paper [8], Fang-Huang-Thompson further introduced a new class of generalized monotone operators, (H, η)-monotone operators, which provide a unifying framework for classes of maximal monotone operators, maximal η-monotone operators, and H-monotone operators. Just recently, using the generalized resolvent operators technique, Verma [19] studied the solvability of a class of nonlinear variational inclusions involving A-monotone mapping, which are more general mappings than the H-monotone operator. Zhang [24] introduced and studied generalized implicit variational-like inclusion problems involving G-η-monotone mappings in Banach spaces lately.

Inspired and motivated by recent research works in this field, in this paper, a new class of general nonlinear fuzzy mulitvalued quasi-variational inclusions involving (G, η)-monotone mappings in Hilbert spaces is introduced and studied. By using the resolvent operator associated with (G, η)-monotone mappings, an existence theorem of solutions for this kind of fuzzy mulitvalued mixed quasi-variational inclusions is established and a new iterative algorithm is suggested and discussed. The results presented in this paper generalize, improve, and unify some recent results in this field.

1.1. **General nonlinear fuzzy mulitvalued quasi-variational Inclusions.**

Let X is a real Hilbert space with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{F}(X)$ be a collection of all fuzzy sets over X. A mapping $\widehat{F}: X \to \mathcal{F}(X)$ is called a fuzzy mapping. For each $x \in X$, $\widehat{F}(x)$(denote it by $\widehat{F}_x$, in the sequel) is a fuzzy set on X and $\widehat{F}_x(y)$ is the membership function of y in $\widehat{F}_x$.

Let $\widehat{B} \in \mathcal{F}(X), q \in [0,1]$. Then the set

$$(\widehat{B})_q = \{x \in X : \widehat{B}(x) \geq q\}$$

is called a q-cut set of $\widehat{B}$.

Let $\widehat{S},\widehat{T},\ \widehat{P}$, and $\widehat{Q}: X \to \mathcal{F}(X)$ be four fuzzy mappings satisfying the condition $(*)$:

(*) there exists four functions $a,b,c,d : X \to [0,1]$ such that for all $x \in X$, we have $(\widehat{S}_x)_{a(x)}, (\widehat{T}_x)_{b(x)},\ (\widehat{P}_x)_{c(x)} \in CB(X)$, and $(\widehat{Q}_x)_{d(x)} \in CB(X)$, where $CB(X)$ denotes the family of all nonempty bounded closed subsets of X.

By using the fuzzy mappings $\widehat{S},\widehat{T},\ \widehat{P}$, and $\widehat{Q}$, we can define four mulitvalued mappings $S,T,P,Q : X \to CB(X)$ by

$$S(x) = (\widehat{S}_x)_{a(x)}, \quad T(x) = (\widehat{T}_x)_{b(x)}, \quad P(x) = (\widehat{P}_x)_{c(x)}, \quad Q(x) = (\widehat{Q}_x)_{d(x)},$$

for each $x \in X$. In the sequel, S,T,P and Q are called the mulitvalued mappings induced by the fuzzy mappings $\widehat{S},\widehat{T},\ \widehat{P}$, and $\widehat{Q}$, respectively.

Let $G,f,g : X \to X$ and $\eta, N : X \times X \to X$ be single-valued mappings and let $\widehat{S},\widehat{T},\widehat{P},\widehat{Q} : X \to \mathcal{F}(X)$ be fuzzy mappings. Let $a,b,c,d : X \to [0,1]$ be four functions. Let $M : X \times X \to 2^X$ be a mulitvalued mapping such that for each given $t \in X$, $M(\cdot,t) : X \to 2^X$ is a (G,η)-monotone mapping and $range(f) \bigcap dom M(\cdot,t) \neq \emptyset$ for each $t \in X$. we consider the following problem:

Find $x,u,v,z,w \in X$ such that $\widehat{S}_x(u) \geq a(x), \widehat{T}_x(v) \geq b(x), \widehat{P}_x(z) \geq c(x), \widehat{Q}_x(w) \geq d(x)$ and

$$0 \in N(u,v) + M(f(x),z) - g(w). \tag{1.1}$$

Problem (1.1) is called a general nonlinear fuzzy mulitvalued quasi-variational inclusions involving (G,η)-monotone mappings(GNFMQ-VI involving (G,η)-monotone mappings).

If $S,T,P,Q : X \to CB(X)$ are mulitvalued mappings, we can define the fuzzy mappings $\widehat{S},\widehat{T},\widehat{P},\widehat{Q} : X \to \mathcal{F}(X)$ by

$$x \mapsto \chi_{S(x)}, x \mapsto \chi_{T(x)}, x \mapsto \chi_{P(x)}, x \mapsto \chi_{Q(x)},$$

where $\chi_{S(x)}, \chi_{T(x)}, \chi_{P(x)}$ and $\chi_{Q(x)}$ are the characteristic functions of S,T,P,Q, respectively. Taking $a(x) = b(x) = c(x) = d(x) = 1$ for all $x \in X$, then problem (1.1) equivalent to the following problem:

Find $x \in X, u \in S(x), v \in T(x), z \in P(x), w \in Q(x)$ such that

$$0 \in N(u,v) + M(f(x),z) - g(w), \tag{1.2}$$

which is called general nonlinear mulitvalued quasi-variational inclusions involving (G,η)-monotone mappings(GNMQ-VI involving (G,η)-monotone mappings).

For a suitable choice of G,g,f,η,N,M,S,T,P,Q and the space X, a number of known classes of variational inclusions and variational inequalities in [1], [3]–[7], [10]–[15], [17]–[22], [24]–[25] can be obtained as special cases of the generalized nonlinear mulitvalued quasi-variational Inclusions (1.2). Furthermore, these types of variational inclusions can enable us to study many important nonlinear problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, finance, structural, transportation, elasticity, and applied sciences in a general and unified framework. Let us recall some concepts and results.

1.2. **Preliminaries.**

Definition 1.1. *A mulitvalued mapping $S : X \to CB(X)$ is said to be*

(i) D-Lipschitz continuous if there exists a constant $\alpha > 0$ such that

$$D(S(x_1), S(x_2)) \leq \alpha \|x_1 - x_2\| \quad \forall x_1, x_2 \in X,$$

where $D(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$.

(ii) β-strongly monotone if there exists a constant $\beta > 0$ such that

$$\langle u_1 - u_2, x_1 - x_2 \rangle \geq \beta \|x_1 - x_2\|^2 \quad \forall x_1, x_2 \in X, u_1 \in S(x_1), u_2 \in S(x_2),$$

Definition 1.2. *A single-valued mapping $\eta : X \times X \to X$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that*

$$\|\eta(x, y)\| \leq \tau \|x - y\|, \qquad \forall x, y \in X.$$

Definition 1.3. *Let $\eta : X \times X \to X$ is a single-valued mapping, and $M : X \to 2^X$ be a mulitvalued mapping. M is said to be:*

(i) η-monotone if

$$\langle u - v, \eta(x, y) \rangle \geq 0, \qquad \forall x, y \in X, u \in M(x), v \in M(y);$$

(ii) strictly η-monotone if

$$\langle u - v, \eta(x, y) \rangle \geq 0, \qquad \forall x, y \in X, u \in M(x), v \in M(y);$$

and equality holds if and only if $x = y$;

(iii) strongly η-monotone if there exists a constant $\delta > 0$ such that

$$\langle u - v, \eta(x, y) \rangle \geq \delta \|x - y\|^2, \qquad \forall x, y \in X, u \in M(x), v \in M(y);$$

(v) m-relaxed η-monotone if there exists a constant $m \geq 0$ such that

$$\langle u - v, \eta(x, y) \rangle \geq -m \|x - y\|^2, \qquad \forall x, y \in X, u \in M(x), v \in M(y);$$

Definition 1.4. *Let $S : X \to CB(X)$ be a mulivalued mapping, and $N, \eta : X \times X \to X$ be two single-valued mappings. $N(\cdot, \cdot)$ is said to be:*

(i) α-Lipschitz continuous with respect to the first argument if there exists a constant $\alpha > 0$ such that

$$\|N(x_1, \cdot) - N(x_2, \cdot)\| \leq \alpha \|x_1 - x_2\| \quad \forall x_i \in X, i = 1, 2.$$

(ii) $\eta -$ cocoercive in the fist argument with respect to S if there exists a constant $\sigma > 0$ such that

$$\langle N(u_1, \cdot) - N(u_2, \cdot), \eta(x, y) \rangle \geq \sigma \|N(u_1, \cdot) - N(u_2, \cdot)\|^2,$$

for all $x, y \in X, u_1 \in S(x), u_2 \in S(y)$;

(iii) η-strongly monotone in the fist argument with respect to S if there exists a constant $\gamma > 0$ such that

$$\langle N(u_1,\cdot) - N(u_2,\cdot), \eta(x,y)\rangle \geq \gamma\|x-y\|^2,$$

for all $x, y \in X, u_1 \in S(x), u_2 \in S(y)$.

In a similar way, we can define Lipschitz continuity of $N(\cdot,\cdot)$ with respect to the second argument.

Definition 1.5. *Let $\eta : X \times X \to X$ and $G : X \to X$ be two single-valued mappings. A mulivalued mapping $M : X \to 2^X$ is said to be (G,η)-monotone if M is m-relaxed η-monotone and $(G+\rho M)(X) = X$ for all $\rho \geq 0$.*

It is easy to see that the (G,η)-monotone mapping is more general than A-monotone, H-monotone mappings, (H,η)-monotone mappings and $g-\eta$-accretive mapping in Hilbet space([7], [13], [14], [15], [24]), and the Definition 1.5 reduces to the definition of the resolvent operator of a maximal η-monotone mapping as $G = I$([12]]). For details about these mappings, we refer the reader to [6]–[8], [12], [24]–[25] and the references therein.

2. The lemma and proposition of the resolvent operator.

Lemma 2.1. *([24]) If η is $\tau-$ Lipschitz continuous and N is $\eta-$ cocoercive in the fist argument with respect to S with constant σ, then*

$$\|N(u_1,\cdot) - N(u_2,\cdot)\| \leq \frac{\tau}{\sigma}\|x-y\|, \qquad \forall x, y \in X, u_1 \in S(x), u_2 \in S(y).$$

Lemma 2.2. *([24]) Let $\eta : X \times X \to X$ be a single-valued mapping satisfying $\eta(x,y) + \eta(y,x) = 0 (\forall x, y \in X)$, $G : X \to X$ be an r-strongly η-monotone single-valued mapping and $M : X \to 2^X$ be an (G,η)-monotone mapping. Then the mapping $(G+\rho M)^{-1}$ is single-valued, where $0 < \rho < \frac{r}{m}$.*

By Lemma 2.2, we can define the resolvent operator $R^{G,\eta}_{\rho,M}$ as follows.

Definition 2.3. *([24]) Let $\eta : X \times X \to X$ be a single-valued mapping, $G : X \to X$ be a strongly η-monotone single-valued mapping and $M : X \to 2^X$ be a (G,η)-monotone mapping. The resolvent operator $R^{G,\eta}_{\rho,M} : X \to X$ is defined by*

$$R^{G,\eta}_{\rho,M}(z) = (G+\rho M)^{-1}(z) \qquad \text{for all } z \in X,$$

where $\rho > 0$ is a constant.

Remark 2.4. *(i) When $\eta(x,y) = x - y$ and G=H for all $x, y \in X$, Definition 2.3 reduces to the definition of the resolvent operator of a H-monotone mapping, see [7].*

(ii) When $G = I$ and $\eta(x, y) = x - y$ for all $x, y \in X$, Definition 1.5 reduces to the definition of the resolvent operator of a maximal monotone mapping, see [13]. *When $G = H$, Definition 1.5 reduces to the definition of the resolvent operator of a (H, η)-monotone mapping, see* [8], [15], [24].

Lemma 2.5. *(*[24]*) Let $\eta : X \times X \to X$ be Lipschtiz continuous mapping with constant $\tau \geq 0$ satisfying $\eta(x, y) + \eta(y, x) = 0 (\forall x, y \in X)$ $G : X \to X$ be an r-strongly η-monotone mapping, and $M : X \to 2^X$ be a (G, η)-monotone mapping. Then the generalized resolvent operator $R^{G,\eta}_{\rho,M} : X \to X$ is $\tau/(r - m\rho)$-Lipschitz continuous, that is,*

$$\|R^{G,\eta}_{\rho,M}(x) - R^{G,\eta}_{\rho,M}(y)\| \leq \frac{\tau}{r - m\rho}\|x - y\| \qquad \text{for all } x, y \in X.$$

3. Approximation algorithms of solutions for the both GNFMQ-VI and GNMQ-VI problems (1.1)-(1.2).

3.1. **Approximation algorithm of solution for the GNFMQ-VI.**

We first transfer the GNFMQ-VI problem (1.1) into a fixed point problem.

Lemma 3.1. *(x, u, v, z, w) is a solution of problem (1.1) if and only if (x, u, v, z, w) satisfies the following relation*

$$f(x) = R^{G,\eta}_{\rho,M(\cdot,z)}(G(f(x)) + \rho g(w) - \rho N(u, v)), \tag{3.1}$$

where $u \in S(x), v \in T(x), z \in P(x), w \in Q(x)$ and $\rho > 0$ is a constant.

Proof. . By the definition of the resolvent operator $R^{G,\eta}_{\rho,M(\cdot,z)}$ of $M(\cdot, z)$, we have that (3.1) holds if and only if $u \in S(x), v \in T(x), z \in P(x)$ and $w \in Q(x)$ such that

$$G(f(x)) + \rho g(w) - \rho N(u, v) \in G(f(x)) + \rho M(f(x), z).$$

The above relations hold if and only if $u \in S(x), v \in T(x), z \in P(x)$ and $w \in Q(x)$ such that

$$0 \in N(u, v) + M(f(x), z) - g(w).$$

Hence (x, u, v, z, w) is a solution of problem (1.1) if and only if $u \in S(x), v \in T(x), z \in P(x)$ and $w \in Q(x)$ are such that (3.1) holds. □

Based on Lemma 3.1 and Nadler [16], we can develop a new approximate algorithm for solving problem (1.1) as follows:

Algorithm 3.2. . *Let $\widehat{S}, \widehat{T}, \widehat{P}, \widehat{Q} : X \to \mathcal{F}(x)$ be fuzzy mappings satisfying condition $(*)$ and $S, T, P, Q : X \to CB(X)$ be the mulitvalued mappings induced by the fuzzy mappings $\widehat{S}, \widehat{T}, \widehat{P}, \widehat{Q}$, respectively. Let $G, f, g : X \to X, \eta, N : X \times X \to X$ be single-valued mappings and let $M : X \times X \to 2^X$*

be such that for each fixed $t \in X$, $M(\cdot,t) : X \to 2^X$ be a (G,η)-monotone mapping and $range(f) \bigcap domM(\cdot,t) \neq \emptyset$.

Step1. For any given $x_0 \in X, u_0 \in S(x_0), v_0 \in T(x_0), z_0 \in P(x_0)$ and $w_0 \in Q(x_0)$;

Step2. Letting

$$\left\{\begin{array}{l} x_{n+1} = (1-\lambda)x_n + \lambda[x_n - f(x_n) \\ +R^{G,\eta}_{\rho,M(\cdot,z_n)}(G(f(x_n)) + \rho g(w_n) - \rho N(u_n,v_n))] + \lambda e_n, \\ u_n \in S(x_n), \quad \|u_n - u_{n+1}\| \leq (1+(1+n)^{-1})D(S(x_n),S(x_{n+1})), \\ v_n \in T(x_n), \quad \|v_n - v_{n+1}\| \leq (1+(1+n)^{-1})D(T(x_n),T(x_{n+1})), \\ z_n \in P(x_n), \quad \|z_n - z_{n+1}\| \leq (1+(1+n)^{-1})D(P(x_n),P(x_{n+1})), \\ w_n \in Q(x_n), \quad \|w_n - w_{n+1}\| \leq (1+(1+n)^{-1})D(Q(x_n),Q(x_{n+1})), \end{array}\right. \tag{3.2}$$

we can get the iterative sequences $\{x_n\},\{u_n\},\{v_n\},\{z_n\}$ and $\{w_n\}$;

Step3. If $x_{n+1}, u_{n+1}, v_{n+1}, z_{n+1}$, and w_{n+1} satisfy (3.2) to sufficient accuracy, stop; otherwise, set n:=n+1 and return to Step 2;
where $n = 0,1,2,\cdots, 0 < \lambda < 1$ and $\rho > 0$ are both constants, $e_n \in X (n \geq 0)$ is an error to take into account a possible inexact computation of the approximation sequences.

3.2. **Approximation algorithm of solution for the GNMQ-VI.**

From Algorithm 3.2 and the condition $a(x) = b(x) = c(x) = d(x) = 1$ in the problems (1.2), we can get a algorithm for solving the GNMQ-VI problems (1.2) as follows:

Algorithm 3.3. . *For any given $x_0 \in X, u_0 \in S(x_0), v_0 \in T(x_0), z_0 \in P(x_0)$ and $w_0 \in Q(x_0)$, we can get the iterative sequences $\{x_n\},\{u_n\},\{v_n\},\{z_n\}$ and $\{w_n\}$ as follows:*

$$\left\{\begin{array}{l} x_{n+1} = (1-\lambda)x_n + \lambda[x_n - f(x_n) \\ +R^{G,\eta}_{\rho,M(\cdot,z_n)}(G(f(x_n)) + \rho g(z_n) - \rho N(u_n,v_n))] + \lambda e_n, \\ u_n \in S(x_n), \quad \|u_n - u_{n+1}\| \leq (1+(1+n)^{-1})D(S(x_n),S(x_{n+1})), \\ v_n \in T(x_n), \quad \|v_n - v_{n+1}\| \leq (1+(1+n)^{-1})D(T(x_n),T(x_{n+1})), \\ z_n \in P(x_n), \quad \|z_n - z_{n+1}\| \leq (1+(1+n)^{-1})D(P(x_n),P(x_{n+1})), \\ w_n \in Q(x_n), \quad \|w_n - w_{n+1}\| \leq (1+(1+n)^{-1})D(Q(x_n),Q(x_{n+1})), \end{array}\right. \tag{3.3}$$

where $n = 0,1,2,\cdots, 0 < \lambda < 1$ and $\rho > 0$ are both constants, $e_n \in X (n \geq 0)$ is an error to take into account a possible inexact computation of the approximation sequences.

Remark 3.4. : *If we choose suitable $G, f, g, \eta, N, S, T, P, Q$ and M, then Algorithm 3.3 can be degenerated to a number of algorithms involving many known algorithms which due to classes of variational inequalities, and variational inclusions (see, for example,* [1]–[7], [10]–[15], [17]–[25]*).*

4. Existence of the solutions and convergence the algorithms.

4.1. **Existence of the solution and convergence of the algorithm(3.2) for the GNFMQ-VI problem (1.1).**

In this section, we show the existence of solution for problem (1.1) and the convergence of the approximation sequences generated by Algorithm 3.2.

Theorem 4.1. *Let $\eta : X \times X \to X$ be a τ-Lipschtiz continuous mapping satisfying $\eta(x,y) + \eta(y,x) = 0 (\forall x, y \in X)$, $G : X \to X$ be an r-strongly η-monotone mapping and α-Lipschitz continuous, $f : X \to X$ be a ψ-strongly monotone and β-Lipschitz continuous mapping, and $g : X \to X$ be a ω-Lipschitz continuous. Let $\widehat{S}, \widehat{T}, \widehat{P}, \widehat{Q} : X \to \mathcal{F}(X)$ be fuzzy mappings satisfying condition $(*)$ and $S, T, P, Q : X \to CB(X)$ be mulivalued mappings induced by the fuzzy mappings $\widehat{S}, \widehat{T}, \widehat{P}, \widehat{Q}$, respectively. suppose that T, P, Q be D-Lipschitz continuous with constants ξ, ζ, φ, respectively. Let $N : X \times X \to X$ be ν-Lipschitz continuous with respect to the second argument and η-cocoercive in fist argument with respect to S with constant σ, respectively. Let $M : X \times X \to 2^X$ such that for each $t \in X$, $M(\cdot, t) : X \to 2^X$ be (G, η)-monotone mapping and $range(f) \bigcap dom M(\cdot, t) \neq \emptyset$. Suppose that for any $x, y, z \in X$*

$$\|R^{G,\eta}_{\rho,M(\cdot,x)}(z) - R^{G,\eta}_{\rho,M(\cdot,y)}(z)\| \leq \delta \|x - y\|. \tag{4.1}$$

and there exists a constant $\rho \in (0, r/m)$ such that

$$\begin{cases} |\rho - \dfrac{\sigma\{\tau^4 - s\sigma[(1-l)r - q\tau]\}}{\tau^4 - \sigma^2 s^2}| \\ < \sigma \dfrac{\sqrt{\{\tau^4 - s\sigma[(1-l)r - q\tau]\}^2 - (\tau^4 - \sigma^2 s^2)\{\tau^4 - [(1-l)r - q\tau]^2\}}}{\tau^4 - \sigma^2 s^2}, \\ q = \beta\sqrt{\alpha^2 - 2r + \tau^2} + \tau(1 + \beta), \\ s = m(1-l) + \tau(\nu\xi + \omega\varphi), \tau^2 > s\sigma, \\ l = \delta\zeta + \sqrt{1 - 2\psi + \beta^2} < 1, \end{cases} \tag{4.2}$$

$$\lim_{n\to\infty} \|e_n\| = 0, \quad \sum_{n=1}^{\infty} \|e_n - e_{n-1}\| < \infty. \tag{4.3}$$

Then the iterative sequences $\{x_n\}, \{u_n\}, \{v_n\}, \{z_n\}$ and $\{w_n\}$ generated by Algorithm 3.2 converge strongly to x^*, u^*, v^*, z^* and w^*, respectively, and $(x^*, u^*, v^*, z^*, w^*)$ is a solution of problem (1.1).

Proof. . From Algorithm 3.2, Lemma 2.5, (4.1) and the r-strongly η-monotonicity and α-Lipschitz continuity of the G, we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \leq (1-\lambda)\|x_n - x_{n-1}\| + \lambda\|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| \\
&+\lambda\|e_n - e_{n-1}\| + \lambda\|R^{G,\eta}_{\rho,M(\cdot,z_n)}(G(f(x_n)) + \rho g(w_n) - \rho N(u_n, v_n)) \\
&-R^{G,\eta}_{\rho,M(\cdot,z_n)}(G(f(x_{n-1})) + \rho g(w_{n-1}) - \rho N(u_{n-1}, v_{n-1}))\| \\
&+\lambda\|R^{G,\eta}_{\rho,M(\cdot,z_n)}(G(f(x_{n-1})) + \rho g(w_{n-1}) - \rho N(u_{n-1}, v_{n-1})) \\
&-R^{G,\eta}_{\rho,M(\cdot,z_{n-1})}(G(f(x_{n-1})) + \rho g(w_{n-1}) - \rho N(u_{n-1}, v_{n-1}))\| \\
&\leq (1-\lambda)\|x_n - x_{n-1}\| + \lambda\|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| \\
&+\lambda\frac{\tau}{r - m\rho}\|G(f(x_n)) - G(f(x_{n-1})) - \rho(N(u_n, v_n) - N(u_{n-1}, v_{n-1})\| \\
&+\lambda\rho\frac{\tau}{r - m\rho}\|g(w_n) - g(w_{n-1})\| + \lambda\delta\|z_n - z_{n-1}\| + \lambda\|e_n - e_{n-1}\|. \\
&\leq (1-\lambda)\|x_n - x_{n-1}\| + \lambda\|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\| \\
&+\lambda\frac{\tau}{r - m\rho}(\|G(f(x_n)) - G(f(x_{n-1})) - \eta(f(x_n), f(x_{n-1})\| \\
&+\|\rho(N(u_n, v_n) - N(u_{n-1}, v_{n-1}) - \eta(f(x_n), f(x_{n-1})\|)) \\
&+\lambda\rho\frac{\tau}{r - m\rho}\|g(w_n) - g(w_{n-1})\| + \lambda\delta\|z_n - z_{n-1}\| + \lambda\|e_n - e_{n-1}\|. \qquad (4.4)
\end{aligned}$$

By the ψ-strongly monotonicity and β-Lipschitz continuity of the f, we can get

$$\begin{aligned}
&\|x_n - x_{n-1} - (f(x_n) - f(x_{n-1}))\|^2 \\
&= \|x_n - x_{n-1}\|^2 - 2\langle f(x_n) - f(x_{n-1}), x_n - x_{n-1}\rangle + \|f(x_n) - f(x_{n-1})\|^2 \\
&\leq (1 - 2\psi + \beta^2)\|x_n - x_{n-1}\|^2. \qquad (4.5)
\end{aligned}$$

By the ψ-strongly monotonicity and β-Lipschitz continuity of the f, and ν-Lipschitz continuity with respect to the second argument of the N, we have

$$\begin{aligned}
&\|G(f(x_n)) - G(f(x_{n-1})) - \eta(f(x_n), f(x_{n-1}))\|^2 \\
&\leq \|G(f(x_n)) - G(f(x_{n-1}))\|^2 + \|\eta(f(x_n), f(x_{n-1}))\|^2 \\
&-2\langle G(f(x_n)) - G(f(x_{n-1})), \eta(f(x_n), f(x_{n-1}))\rangle \\
&\leq \beta^2(\alpha^2 - 2r + \tau^2)\|x_n - x_{n-1}\|^2. \qquad (4.6)
\end{aligned}$$

By using η-cocoercive in fist argument with respect to the S with constant σ of the N, the ψ-strongly monotonicity and β-Lipschitz continuity of the f, Lemma 2.1 and [24], we can get

$$\begin{aligned}
&\|\rho(N(u_n,v_n)-N(u_{n-1},v_{n-1}))-\eta(f(x_n),f(x_{n-1}))\|\\
&\le \|\rho(N(u_n,v_n)-N(u_{n-1},v_n))-\eta(f(x_n),f(x_{n-1}))\|\\
&+\|\rho(N(u_{n-1},v_n)-N(u_{n-1},v_{n-1}))\|\\
&\le \rho\nu\xi(1+n^{-1})\|x_n-x_{n-1}\|\\
&+\|\rho(N(u_n,v_n)-N(u_{n-1},v_n))-\eta(f(x_n),f(x_{n-1}))\|.\\
&\le (\rho\nu\xi(1+n^{-1})+(\frac{\tau}{\sigma}\sqrt{\rho^2-2\sigma\rho+\sigma^2}+\tau(1+\beta)))\|x_n-x_{n-1}\|.
\end{aligned} \tag{4.7}$$

By the the conditions, we have

$$\|g(w_n)-g(w_{n-1})\|\le\omega\|w_n-w_{n-1}\|\le\omega\varphi(1+n^{-1})\|x_n-x_{n-1}\| \tag{4.8}$$

and

$$\|z_n-z_{n-1}\|\le\zeta(1+n^{-1})\|x_n-x_{n-1}\|. \tag{4.9}$$

From (4.4)$\sim$(4.9), It follows that

$$\begin{aligned}
\|x_{n+1}-x_n\| &\le (1-\lambda+\lambda h_n)\|x_n-x_{n-1}\|+\lambda\|e_n-e_{n-1}\|\\
&= \theta_n\|x_n-x_{n-1}\|+\lambda\|e_n-e_{n-1}\|.
\end{aligned}\tag{4.10}$$

where

$$\begin{aligned}
&\theta_n=1-\lambda+\lambda h_n,\\
&h_n=\sqrt{1-2\psi+\beta^2}+\frac{\tau}{r-m\rho}((\nu\xi+\omega\varphi)\rho(1+n^{-1})+\tau(1+\beta)\\
&+\beta\sqrt{\alpha^2-2r+\tau^2}+\frac{\tau}{\sigma}\sqrt{\rho^2-2\rho\sigma+\sigma^2})+\delta\zeta(1+n^{-1}).
\end{aligned}$$

Letting

$$\begin{aligned}
&\theta=1-\lambda+\lambda h,\\
&h=\sqrt{1-2\psi+\beta^2}+\frac{\tau}{r-m\rho}((\nu\xi+\omega\varphi)\rho+\tau(1+\beta)\\
&+\beta\sqrt{\alpha^2-2r+\tau^2}+\frac{\tau}{\sigma}\sqrt{\rho^2-2\rho\sigma+\sigma^2})+\delta\zeta.
\end{aligned}$$

we have that $h_n\to h$ and $\theta_n\to\theta$ as $n\to\infty$. It follows from condition (4.2) and $0<\lambda<1$ that $0<\theta<1$ and hence there exists $N_0>0$ and $\theta_*\in(\theta,1)$ such that $\theta_n<\theta_*$ for all $n\ge N_0$. Therefore, by (4.10), we have

$$\|x_{n+1}-x_n\|\le\theta_*\|x_n-x_{n-1}\|+\lambda\|e_n-e_{n-1}\|,\forall n\ge N_0.$$

Without loss of generality we assume

$$\|x_{n+1}-x_n\|\le\theta_*\|x_n-x_{n-1}\|+\lambda\|e_n-e_{n-1}\|,\forall n\ge 1,$$

Hence, for any $m > n > 0$, we have

$$\|x_m - x_n\| \leq \sum_{i=n}^{m-1} \|x_{i+1} - x_i\| \leq \sum_{i=n}^{m-1} \theta_*^i \|x_1 - x_0\| + \sum_{i=n}^{m-1} \sum_{j=1}^{i} \theta_*^{i-j} \lambda \|e_j - e_{j-1}\|.$$

It follows from condition (4.3) that $\|x_m - x_n\| \to 0$, as $n \to \infty$, and so $\{x_n\}$ is a Cauchy sequence in X. Let $x_n \to x^*$ as $n \to \infty$. By the Lipschitz continuity of S, T, P and Q, we obtain

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq (1 + n^{-1}) D(S(x_{n+1}), S(x_n)) \leq \gamma(1 + n^{-1}) \|x_{n+1} - x_n\|, \\
\|v_{n+1} - v_n\| &\leq (1 + n^{-1}) D(T(x_{n+1}), T(x_n)) \leq \xi(1 + n^{-1}) \|x_{n+1} - x_n\|, \\
\|z_{n+1} - z_n\| &\leq (1 + n^{-1}) D(P(x_{n+1}), P(x_n)) \leq \zeta(1 + n^{-1}) \|x_{n+1} - x_n\|, \\
\|w_{n+1} - w_n\| &\leq (1 + n^{-1}) D(Q(x_{n+1}), Q(x_n)) \leq \varphi(1 + n^{-1}) \|x_{n+1} - x_n\|.
\end{aligned}$$

It follows that $\{u_n\}$, $\{v_n\}, \{z_n\}$ and $\{w_n\}$ are also Cauchy sequences in X. We can assume that $u_n \to u^*$, $v_n \to v^*, z_n \to z^*$ and $w_n \to w^*$, respectively. Note that $u_n \in S(x_n)$, we have

$$\begin{aligned}
d(u^*, S(x^*)) &\leq \|u^* - u_n\| + d(u_n, S(x^*)) \\
&\leq \|u^* - u_n\| + D(S(x_n), S(x^*)) \\
&\leq \|u^* - u_n\| + \gamma \|x_n - x^*\| \to 0 (n \to \infty).
\end{aligned}$$

Hence $d(u^*, S(x^*)) = 0$ and therefore $u^* \in S(x^*)$. Similarly, we can prove that $v^* \in T(x^*), z^* \in P(x^*)$ and $w^* \in Q(x^*)$.

By the Lipschitz continuity of S, T, P and Q and Lemma 2.5, condition (4.1) and $\lim_{n\to\infty} \|e_n\| = 0$, we have

$$x^* = (1 - \lambda)x^* + \lambda[x^* - f(x^*) + R^{G,\eta}_{\rho, M(\cdot, z^*)}(G(f(x^*)) + \rho g(w^*) - \rho N(u^*, v^*))].$$

By Lemma 3.1, we know that $(x^*, u^*, v^*, z^*, w^*)$ is a solution of problem (1.1). This completes the proof. □

From Theorem 4.1, we have the following theorem.

4.2. **Existence of the solution and convergence of the algorithm(3.3) for the GNMQ-VI problem (1.2).**

Theorem 4.2. *Let $G, f, g, \eta, S, N, M, X$ be the same as in Theorem 4.1, and $T, P, Q : X \to CB(X)$ be D-Lipschitz continuous with constants ξ, ζ, φ, respectively. Let $N : X \times X \to X$ be ν-Lipschitz continuous with respect to the second argument and η-cocoercive in fist argument with respect to S with constant σ, respectively. If conditions (4.1)∼(4.3) of Theorem 4.1 hold, then the iterative sequences $\{x_n\}, \{u_n\}, \{v_n\}, \{z_n\}$ and $\{w_n\}$ generated by Algorithm 3.3 converge strongly to x^*, u^*, v^*, z^* and w^*, respectively, and $(x^*, u^*, v^*, z^*, w^*)$ is a solution of the problem (1.2).*

Remark 4.3. *For a suitable choice of the mappings $G, g, \eta, N, M, S, T, P, Q$, we can obtain several known results* [1], [3]–[7], [10]–[15], [17]–[22], [24], [25] *as special cases of Theorem 4.2.*

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