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A PERTURBATION OF NORMAL OPERATORS ON A HILBERT SPACE

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Abstract. Let $B(H)$ be the set of all bounded linear operators on a Hilbert space H. We will consider an approximately self-adjoint operator $S \in B(H)$ with $\|S^*(x) - S(x)\| \le \varepsilon\|x\|^p$ $(\forall x \in H)$, and an approximately normal operator $T \in B(H)$ satisfying $\|T^*T(x) - TT^*(x)\| \le \varepsilon\|x\|^p$ $(\forall x \in H)$ for some real numbers $\varepsilon \ge 0$ and p. We prove that an approximate self-adjoint (normal) operator is an exact self-adjoint (resp. normal) operator when $p \neq 1$. For $p = 1$, we give examples that such superstability results do not hold.

1. Introduction

It seems that the stability problem of functional equations had been first raised by S. M. Ulam (cf. [16, Chapter VI]). "For what metric groups G is it true that an ε-automorphism of G is necessarily near to a strict automorphism?

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(An ε-automorphism of G means a transformation f of G into itself such that $\rho(f(x \cdot y), f(x) \cdot f(y)) < \varepsilon$ for all $x, y \in G$.)"

D. H. Hyers [7, Theorem 1 and Corollary] gave an answer in the affirmative to the problem as follows.

Theorem A. *Suppose that* E_1 *and* E_2 *are two real Banach spaces and* $f\colon E_1 \to E_2$ *is a mapping. If there exists* $\varepsilon \geq 0$ *such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in E_1$*, then the limit*

$$T(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for each $x \in E_1$*, and* $T\colon E_1 \to E_2$ *is the unique additive mapping such that*

$$\|f(x) - T(x)\| \leq \varepsilon$$

for every $x \in E_1$*. If, in addition, the mapping* $\mathbb{R} \ni t \mapsto f(tx)$ *is continuous for each fixed* $x \in E_1$*, then* T *is linear.*

This result is called the *Hyers-Ulam stability* of the *additive* Cauchy equation $g(x+y) = g(x) + g(y)$. Here we note that Hyers calls any solution of this equation a "linear" function or transformation. Hyers considered only *bounded* Cauchy difference $f(x+y) - f(x) - f(y)$. T. Aoki [1] introduced unbounded one and generalized a result [7, Theorem 1] of Hyers obtaining the stability of additive mapping. Th.M. Rassias [11], who independently introduced the unbounded Cauchy difference, was the first to prove the stability of the linear mapping between Banach spaces. The concept of the Hyers-Ulam-Rassias stability was originated from Rassias's paper [11] for the stability of the linear mapping and its importance in the proof of further results in functional equations. Rassias generalized Hyers's Theorem as follows:

Theorem B. *Suppose that* E_1 *and* E_2 *are two real Banach spaces and* $f\colon E_1 \to E_2$ *is a mapping. If there exist* $\varepsilon \geq 0$ *and* $0 \leq p < 1$ *such that*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p)$$

for every $x, y \in E_1$*, then there is a unique additive mapping* $T\colon E_1 \to E_2$ *such that*

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{|2 - 2^p|} \|x\|^p$$

for every $x \in E_1$*. If, in addition, the mapping* $\mathbb{R} \ni t \mapsto f(tx)$ *is continuous for each fixed* $x \in E_1$*, then* T *is linear.*

This result is what is called, the Hyers-Ulam-Rassias stability of the linear mapping. The result of Hyers is just the case of Rassias's Theorem when $p = 0$. During the 27th International Symposium on Functional Equations, Rassias raised the problem whether a similar result to Theorem B holds for $1 \le p$. Z. Gajda [5, Theorem 2] proved that Theorem B is valid for $1 < p$. In the same paper [5, Example], he also gave an example that a similar result to the above does not hold for $p = 1$. Later, Th. M. Rassias and P. Šemrl [12, Theorem 2] gave another counter example for $p = 1$. Note that if $p < 0$, then $\|0\|^p$ is obviously meaningless. However, if we assume that $\|0\|^p$ means ∞, then the proof given in [11] also works for $x \neq 0$. Moreover, with minor changes in the proof, we see that the result is also valid for $p < 0$. Thus, the Hyers-Ulam-Rassias stability of the linear mapping holds for $p \in \mathbb{R} \setminus \{1\}$.

Let $B(H)$ be the set of all bounded linear operators on a Hilbert space H. It seems natural to consider stability problems for operators in $B(H)$. In fact, K. Fan and A. J. Hoffman [4] considered stability of self-adjoint operators in $B(H)$ for finite-dimensional H. Here and after, T^* denotes the adjoint of $T \in B(H)$. P. R. Halmos [6] pointed out that if H is a Hilbert space, which need not be of finite-dimensional, and if $S \in B(H)$ satisfies

$$\|S^*x - Sx\| \le \varepsilon\|x\| \qquad (\forall x \in H)$$

for some $\varepsilon \ge 0$, then $\tilde{S} = (S^* + S)/2 \in B(H)$ is a self-adjoint operator such that

$$\|Sx - \tilde{S}x\| \le \frac{\varepsilon}{2}\,\|x\| \qquad (\forall x \in H).$$

In this paper, we consider a perturbation of normal operators of the form

$$\|T^*Tx - TT^*x\| \le \varepsilon\|x\|^p \qquad (\forall x \in H),$$

where $\varepsilon \ge 0$ and $p \in \mathbb{R}$. For negative p, we assume that $\|0\|^p$ means ∞. We shall prove that "approximate normal operators" are exact ones when $p \neq 1$. Such stability phenomena are called superstability (cf. [2, 3]). We will also consider Hyers-Ulam stability of normal operators when $p = 1$.

2. Main results and examples

Before we consider stability of normal operators, we first prove superstability for self-adjoint operators. That is, "approximate self-adjoint" operators are exact self-adjoint operators.

Theorem 2.1. *If $S \in B(H)$ satisfies*

$$\|S^*x - Sx\| \le \varepsilon\|x\|^p \qquad (\forall x \in H) \tag{1}$$

for some $\varepsilon \ge 0$ and $p \in \mathbb{R} \setminus \{1\}$, then S is self-adjoint.

Proof. Take $x \in H \setminus \{0\}$ and fix $n \in \mathbb{N}$ arbitrarily. Put $s = |1-p|/(1-p)$. It follows from (1) that

$$\|S^*(n^s x) - S(n^s x)\| \le \varepsilon\|n^s x\|^p.$$

The linearity of S and S^* implies that

$$n^s\|S^*x - Sx\| \leq \varepsilon n^{sp}\|x\|^p,$$

and hence

$$\|S^*x - Sx\| \leq \varepsilon n^{s(p-1)}\|x\|^p.$$

Recall that $s(p-1) = -|1-p| < 0$. Since $n \in \mathbb{N}$ was arbitrary, we obtain $\|S^*x - Sx\| = 0$, and so $S^*x = Sx$: This is true for $x = 0$. We thus conclude that S is a self-adjoint operator. □

Example 1. In Theorem 2.1, we excluded the case where $p = 1$. We give an example to show that a similar result to Theorem 2.1 does not hold for $p = 1$. Let $\varepsilon > 0$. We define $S = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}$. Then $S^* = \begin{pmatrix} 0 & 0 \\ \varepsilon & 0 \end{pmatrix}$, and hence S is not self-adjoint. On the other hand, we get

$$\|S^*x - Sx\| = \varepsilon\|x\|$$

for every $x \in \mathbb{R}^2$. We thus conclude that Theorem 2.1 does not hold for $p = 1$ in general.

Next we prove superstability of normal operators. That is, approximate normal operators are exact normal ones.

Theorem 2.2. *If $T \in B(H)$ satisfies*

$$\|T^*Tx - TT^*x\| \leq \varepsilon\|x\|^p \qquad (\forall x \in H) \tag{2}$$

for some $\varepsilon \geq 0$ and $p \in \mathbb{R} \setminus \{1\}$, then T is normal.

Proof. Pick $x \in H \setminus \{0\}$ and fix $n \in \mathbb{N}$ arbitrarily. Put $s = |1-p|/(1-p)$. It follows from (2) that

$$\|T^*T(n^sx) - TT^*(n^sx)\| \leq \varepsilon\|n^sx\|^p.$$

The linearity of T and T^* implies that

$$\|T^*Tx - TT^*x\| \leq \varepsilon n^{s(p-1)}\|x\|^p.$$

Taking $n \to \infty$, we obtain $\|T^*Tx - TT^*x\| = 0$, and so we see that T is a normal operator. □

Example 2. A similar result to Theorem 2.2 does not hold for $p = 1$. Indeed, take $\varepsilon > 0$. If we define $T = \begin{pmatrix} 0 & \sqrt{\varepsilon} \\ 0 & 0 \end{pmatrix}$, then

$$T^*T = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon \end{pmatrix} \quad \text{and} \quad TT^* = \begin{pmatrix} \varepsilon & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, T is not normal. On the other hand, we obtain

$$\|T^*Tx - TT^*x\| = \varepsilon\|x\|$$

for every $x \in \mathbb{R}^2$. This shows that Theorem 2.2 need not be true for $p = 1$.

In the case where $p = 1$, we see by Example 2 that superstability need not hold. That is, there is an operator T such that T is not normal but that T satisfies

$$\|T^*T - TT^*\| \leq \varepsilon, \tag{3}$$

where $\|\cdot\|$ denotes the operator norm. One might ask whether Hyers-Ulam stability holds for normal operators. Here, we give an answer to this question in the negative in the following sense.

Theorem 2.3. *Let H be a Hilbert space with $\dim H \geq 2$. There is no constant $K \geq 0$ with the following property:*

($*$) *To each $\varepsilon \geq 0$ and $T \in B(H)$ satisfying (3) there corresponds a normal operator $N \in B(H)$ such that $\|T - N\| \leq K\varepsilon$.*

Proof. Suppose, on the contrary, that there is a constant $K \geq 0$ with ($*$). Let $\mathcal{N} \subset B(H)$ be the set of all normal operators. We first prove that ($*$) implies the following:

($\sharp$) $\inf\{\|T - N\| : N \in \mathcal{N}\} \leq K\|T^*T - TT^*\|$ holds for every $T \in B(H)$.

For if $T \in B(H)$, we put $\varepsilon_0 = \|T^*T - TT^*\|$. By hypothesis, there exists a normal operator $N_0 \in B(H)$ such that

$$\|T - N_0\| \leq K\varepsilon_0 = K\|T^*T - TT^*\|.$$

This implies that

$$\inf\{\|T - N\| : N \in \mathcal{N}\} \leq \|T - N_0\| \leq K\|T^*T - TT^*\|,$$

and so ($*$) $\Rightarrow$ ($\sharp$) is proved.

Since $\dim H \geq 2$, there exists $T_0 \in B(H) \setminus \mathcal{N}$. Put $T_n = n^{-1}T_0 \in B(H)$ for each $n \in \mathbb{N}$. By ($\sharp$) we have

$$\inf\{\|T_n - N\| : N \in \mathcal{N}\} \leq K\|{T_n}^*T_n - T_n{T_n}^*\|. \tag{4}$$

Since $n\mathcal{N} = \mathcal{N}$ for each $n \in \mathbb{N}$, we see that

$$\inf\{\|T_n - N\| : N \in \mathcal{N}\} = \frac{1}{n}\inf\{\|T_0 - N\| : N \in \mathcal{N}\}.$$

It follows from (4) that

$$\frac{1}{n}\inf\{\|T_0 - N\| : N \in \mathcal{N}\} \leq K\|{T_n}^*T_n - T_n{T_n}^*\| = \frac{K}{n^2}\|{T_0}^*T_0 - T_0{T_0}^*\|,$$

and so

$$\inf\{\|T_0 - N\| : N \in \mathcal{N}\} \leq \frac{K}{n}\|T_0^*T_0 - T_0T_0^*\|.$$

Letting $n \to \infty$, we obtain $\inf\{\|T_0 - N\| : N \in \mathcal{N}\} = 0$. Since $\mathcal{N}$ is closed, we conclude $T_0 \in \mathcal{N}$, in contradiction to $T_0 \in B(H) \setminus \mathcal{N}$. We thus proved that there is no constant $K \geq 0$ with ($*$). □

Example 3. If T is a (2×2)-matrix over $\mathbb{R}$, then there exists a (2×2)-matrix N_0, which is normal as an operator, such that

$$\|T^*T - TT^*\| = 2\|T - T^*\| \, \|T - N_0\|,$$

where $\|\cdot\|$ denotes the operator norm on $\mathbb{R}^2$. Indeed, put $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{R}$. A simple calculation shows that

$$T^*T - TT^* = (b - c) \begin{pmatrix} -b - c & a - d \\ a - d & b + c \end{pmatrix},$$

and hence

$$\|T^*T - TT^*\|^2 = |b - c|^2 \left\{(a - d)^2 + (b + c)^2\right\}.$$

We define a matrix N_0 by

$$N_0 = \frac{1}{2} \begin{pmatrix} a + d & b - c \\ c - b & a + d \end{pmatrix}.$$

It is easy to see that N_0 is normal. Moreover, we obtain

$$T - N_0 = \frac{1}{2} \begin{pmatrix} a - d & b + c \\ b + c & d - a \end{pmatrix},$$

and so

$$\|T - N_0\|^2 = \frac{1}{4} \left\{(a - d)^2 + (b + c)^2\right\}.$$

We thus obtain

$$\|T^*T - TT^*\|^2 = 4|b - c|^2 \, \|T - N_0\|^2 = 4\|T - T^*\|^2 \, \|T - N_0\|^2.$$

As a direct consequence, we get the following stability result: Suppose that T is a (2×2)-matrix over $\mathbb{R}$ satisfying

$$\|T^*T - TT^*\| \leq \varepsilon$$

for some $\varepsilon \geq 0$. If T is not self-adjoint, then there exists normal N_0 such that

$$\|T - N_0\| \leq \frac{\varepsilon}{2\|T - T^*\|}.$$

Here we notice that the constant $1/2\|T - T^*\|$ obviously depends on T, and so this stability result does not contradict Theorem 2.3.

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