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A NEW SYSTEM OF GENERALIZED NONLINEAR RELAXED CO-COERCIVE SET-VALUED VARIATIONAL INCLUSIONS IN BANACH SPACES

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Abstract. In this paper, we introduce and study a new system of generalized nonlinear relaxed co-coercive set-valued variational inclusions in Banach spaces. By using the resolvent operator technique for (A, η)-accretive mapping due to Lan-Cho-Verma, we construct some new iterative algorithms for approximating the solutions of the system of nonlinear relaxed co-coercive set-valued variational inclusions and prove the existence of the solutions for the system of nonlinear relaxed co-coercive set-valued variational inclusions and convergence of iterative sequences generated by the algorithm. The results presented in this paper improve and extend the previously known results in this area.

1. Introduction

Variational inclusions, as an important generalization of the classical variational inequality, has wide applications in a large variety of problems arising in mechanics, physics, optimization and control, economics, and transportation equilibrium, and engineering sciences, for details, we can refer to [1]-[30] and the references therein. Recently, Huang and Fang [15] were the first to introduce the generalized m-accretive mapping and give the definition of the

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resolvent operator for the generalized m-accretive mappings in Banach spaces. They also showed some properties of the resolvent operator for the generalized m-accretive mappings in Banach spaces. For further works, see Huang [12] and the references therein. Very recently, inspired and motivated by the works of [5], [7]-[9], [12], [16], [18], [27]. Lan et al. [20] introduced a new concept of (A, η)-accretive mappings, which generalizes the existing monotone or accretive operators, and studied some properties of (A, η)-accretive mappings and defined resolvent operators associated with (A, η)-accretive mappings. They also studied a class of variational inclusions using the resolvent operator associated with (A, η)-accretive mappings.

On the other hand, in [25], Verma introduced a new systems of nonlinear strongly monotone variational inequalities and studied the approximate of this system based on the projection method, and in [26], Verma discussed the approximate solvability of a system of nonlinear relaxed co-coercive variational inequalities in Hilbert spaces. Recently, Kim and Kim [19] introduced and studied a system of nonlinear mixed variational inequalities in Hilbert spaces, and obtained some approximate solvability results. In the recent paper [22], Lan et al. introduced and studied a new systems of generalized nonlinear variational inclusions in Banach spaces. They proved existence theorems of the solutions and convergence theorems of the generalized Mann iterative procedures with mixed errors for this system of variational inclusion in q-uniformly smooth Banach spaces. Some related works, we refer to [2, 6, 7, 17, 24].

Inspired and motivated by recent research works in this field, in this paper, we shall introduce and study a new system of generalized nonlinear relaxed co-coercive set-valued variational inclusions in Banach spaces. By using the resolvent operator technique for (A, η)-accretive mapping due to Lan-Cho-Verma, we construct some new iterative algorithms for approximating the solutions of the system of nonlinear relaxed co-coercive set-valued variational inclusions and prove the existence of the solutions for the system of nonlinear relaxed co-coercive set-valued variational inclusions and convergence of iterative sequences generated by the algorithm. The results presented in this paper improve and extend the previously known results in this area.

2. Preliminaries

Throughout this paper, we assume that X is a real Banach space with dual space X^*, $\langle \cdot, \cdot \rangle$ is the dual pair between X and X^*, and 2^X denote the family of all the nonempty subsets of X. The generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f^* \in X^* : \langle x, f^* \rangle = \|x\|^q, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$ for all $x \neq 0$

and J_q is single-valued if X^* is strictly convex, and if $X = H$, the Hilbert space,then J_2 becomes the identity mapping on H.

The modulus of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$ defined by

$$\rho_X(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\}.$$

A Banach space X is called uniformly smooth if

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0.$$

X is called q-uniformly smooth if there exists a constant $c > 0$, such that

$$\rho_X(t) \leq ct^q, \qquad q > 1.$$

Note that J_q is single -valued if X is uniformly smooth. In the study of characteristic inequalities in q-uniformly smooth Banach spaces, Xu [29] proved the following result:

Lemma 2.1. [29] *Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $C_q > 0$, such that for all $x, y \in X$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + C_q\|y\|^q.$$

For $i = 1, 2$, let $S, P, g, A_i : X \to X$ and $N_i, \eta_i : X \times X \to X$ be single-valued mappings, $T, Q : X \to 2^X$ be set-valued mappings and M_i be (A_i, η_i)-accretive mappings. For any given $\theta_i \in X$, we consider the following problem:

Find $x, y \in X$ such that $g(x) \in dom(M_1)$ and

$$\begin{cases} y - A_1(g(x)) - \lambda_1(N(S(y), v) - \theta_1) \in \lambda_1 M_1(g(x)), \forall v \in T(y), \\ x - A_2(y) - \lambda_2(N_2(P(x), u) - \theta_2) \in \lambda_2 M_2(y), \forall u \in Q(x), \end{cases} \tag{2.1}$$

where λ_1, λ_2 are two constants. Problem (2.1) is called a system generalized nonlinear relaxed co-coercive set-valued variational inclusions in Banach spaces.

If $g = I$, the identity mapping, then problem (2.1) is equivalent to finding $x, y \in X$ such that

$$\begin{cases} y - A_1(x) - \lambda_1(N(S(y), v) - \theta_1) \in \lambda_1 M_1(x), \forall v \in T(y), \\ x - A_2(y) - \lambda_2(N_2(P(x), u) - \theta_2) \in \lambda_2 M_2(y), \forall u \in Q(x). \end{cases} \tag{2.2}$$

We remark that for suitable choices of the mappings η_1, η_2, S, T, P, Q, g, A_1, A_2, N_1, N_2, M_1, M_2 and the spaces X, problem (2.1) reduces to various of variational inclusions and variational inequalities, see for example, [1],[2],[11],[19], [22]-[26],[30] and the references therein.

Definition 2.1. A single-valued mapping $g : X \to X$ is said to be

(i) accretive if

$$\langle g(x) - g(y), J_q(x - y)\rangle \geq 0, \quad \forall x, y \in X;$$

(ii) r-strongly accretive if there exists a constant $r > 0$ such that

$$\langle g(x) - g(y), J_q(x - y)\rangle \geq r\|x - y\|^q, \quad \forall x, y \in X;$$

(iii) s-relaxed co-coercive if there exists a constant $s > 0$ such that

$$\langle g(x) - g(y), J_q(x - y)\rangle \geq (-s)\|g(x) - g(y)\|^q, \qquad \forall x, y \in X;$$

(iv) (α, ξ)-relaxed co-coercive if there exist constants $\alpha, \xi > 0$ such that

$$\langle g(x) - g(y), J_q(x - y)\rangle \geq (-\alpha)\|g(x) - g(y)\|^q + \xi\|x - y\|^q, \qquad \forall x, y \in X;$$

(v) t-Lipschitz continuous if there exists a constant $t > 0$ such that

$$\|g(x) - g(y)\| \leq t\|x - y\|, \quad \forall x, y \in X.$$

Definition 2.2. Let $S : X \to X$ and $N : X \times X \to X$ be single-valued mappings. N is said to be

(i) (a, b)-relaxed co-coercive with respect to S in first argument if there exist constants $a, b > 0$ such that

$$\begin{aligned}&\langle N(S(x_1), \cdot) - N(S(x_2), \cdot), J_q(x_1 - x_2)\rangle \\ &\geq (-a)\|N(S(x_1), \cdot) - N(S(x_2), \cdot)\|^q + b\|x_1 - x_2\|^q, \forall x_1, x_2 \in X.\end{aligned}$$

(ii) α-Lipschitz continuous with respect to the first argument if there exists a constant $\alpha > 0$ such that

$$\|N(x_1, \cdot) - N(x_2, \cdot)\| \leq \alpha\|x_1 - x_2\|, \quad \forall x_, x_2 \in X.$$

In a similar way, we can define Lipschitz continuity of N with respect to the second argument.

Definition 2.3. The mapping $T : X \to CB(X)$ is said to be ξ-H-Lipschitz continuous if there exists a constant $\xi > 0$ such that

$$H(T(x), T(y)) \leq \xi\|x - y\|, \forall x, y \in X.$$

Definition 2.4. The mapping $\eta : X \times X \to X$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau\|x - y\|, \forall x, y \in X.$$

Definition 2.5. Let $\eta : X \times X \to X$ and $A : X \to X$ be single-valued mappings. Then set-valued mapping $M : X \to 2^X$ is said to be

(i) accretive if

$$\langle u - v, J_q(x - y)\rangle \geq 0, \quad \forall x, y \in X, \quad u \in M(x), \quad v \in M(y);$$

(ii) η-accretive if

$$\langle u - v, J_q(\eta(x, y))\rangle \geq 0, \quad \forall x, y \in X, \quad u \in M(x), \quad v \in M(y);$$

(iii) strictly η-accretive if M is η-accretive and equality holds if and only if $x = y$;

(iv) r-strongly η-accretive if there exists a constant $r > 0$ such that

$$\langle u - v, J_q(\eta(x, y))\rangle \geq r\|x - y\|^q, \quad \forall x, y \in X, \quad u \in M(x), \quad v \in M(y);$$

(v) α-relaxed η-accretive if here exists a constant $m > 0$ such that

$$\langle u - v, J_q(\eta(x, y))\rangle \geq (-\alpha)\|x - y\|^q, \quad \forall x, y \in X, \quad u \in M(x), \quad v \in M(y).$$

In a similar way, we can define strictly η-accretivity and strongly η-accretivity of the single-valued mapping A.

Definition 2.6. Let $A : X \to X, \eta : X \times X \to X$ is two single-valued mappings. Then a set-valued mapping $M : X \to 2^X$ is called (A, η)-accretive if M is m-relaxed η-accretive and $(A + \lambda M)(X) = X$ for every $\lambda > 0$.

Remark 2.1. For appropriate and suitable choices of m, A, η and X, it is easy to see Definition 2.5 includes a number of definitions of monotone operators and accretive operators (see [20]).

In [20], Lan et al. showed that $(A + \rho M)^{-1}$ is a single-valued operator if $M : X \to 2^X$ be an (A, η)-accretive mapping and $A : X \to X$ be r-strongly η-accretive mapping. Based on this fact, we can define the resolvent operator $R^{\eta,M}_{\rho,A}$ associated with an (A, η)-accretive mapping M as follows:

Definition 2.7. Let $A : X \to X$ be a strictly η-accretive mapping and $M : X \to 2^X$ be an (A, η)-accretive mapping. The resolvent operator $R^{\eta,M}_{\rho,A} : X \to X$ is defined by

$$R^{\eta,M}_{\rho,A}(x) = (A + \rho M)^{-1}(x), \qquad \forall x \in X.$$

Lemma 2.2. [20] *Let $\eta : X \times X \to X$ be τ-Lipschitz continuous, $A : X \to X$ be r-strongly η-accretive mapping and $M : X \to 2^X$ be an (A, η)-accretive mapping. Then the resolvent operator $R^{\eta,M}_{\rho,A} : X \to X$ is $\frac{\tau^{q-1}}{r - \rho m}$-Lipschitz continuous, i. e.,*

$$\|R^{\eta,M}_{\rho,A}(x) - R^{\eta,M}_{\rho,A}(y)\| \leq \frac{\tau^{q-1}}{r - \rho m}\|x - y\|, \quad \forall x, y \in X,$$

where $\rho \in (0, \frac{r}{m})$ is a constant.

Lemma 2.3. [23] *Let (X, d) be a complete metric space. Suppose that $F : X \to CB(X)$ satisfies*

$$H(F(x), F(y)) \le td(x, y), \quad \forall x, y \in X,$$

where $t \in (0, 1)$ is a constant. Then the mapping F has a fixed point in X.

3. Existence theorems

In this section, we shall give the existence theorems of the solution of problems (2.1) and (2.2), respectively.

Lemma 3.1. *For any given $x, y \in X$, (x, y) is a solution of problem (2.1) if and only if*

$$\begin{cases} g(x) \in R^{\eta_1, M_1}_{\lambda_1, A_1}(y - \lambda_1(N_1(S(y), T(y)) - \theta_1)), \\ y \in R^{\eta_2, M_2}_{\lambda_2, A_2}(x - \lambda_2(N_2(P(x), Q(x)) - \theta_2)). \end{cases} \tag{3.1}$$

Proof. The proof directly follows from the definition of $R^{\eta_i, M_i}_{\lambda_i, A_i}$ for $i = 1, 2$ and so it is omitted. □

Theorem 3.1. *Let X be q-uniformly smooth Banach spaces. Let $g : X \to X$ be (a, b)-relaxed co-coercive and γ-Lipschitz continuous, $S, P : X \to X$ be μ-Lipschitz continuous and ξ-Lipschitz continuous, respectively. $T, Q : X \to CB(X)$ be ν-H-Lipschitz continuous and δ-H-Lipschitz continuous, respectively. Let $N_1 : X \times X \to X$ be γ_1-Lipschitz continuous and (α_1, β_1)-relaxed co-coercive with respect to S in the first argument, $N_2 : X \times X \to X$ be γ_2-Lipschitz continuous and (α_2, β_2)-relaxed co-coercive with respect to P in the first argument, N_i be t_i-Lipschitz continuous in the second argument, $\eta_i : X \times X \to X$ be τ_i-Lipschitz continuous, $A_i : X \to X$ be r_i-strongly η_i-accretive, $M_i : X \to 2^X$ be (A_i, η_i)-accretive mappings, for $i = 1, 2$. If there exist constants $\lambda_1 \in (0, \frac{r_1}{m_1})$ and $\lambda_2 \in (0, \frac{r_2}{m_2})$ such that*

$$\begin{cases} l + h_1 h_2 < 1, \\ l = (1 - qb + qa\gamma^q + C_q\gamma^q)^{\frac{1}{q}}, qa\gamma^q + C_q\gamma^q < qb \\ h_1 = \frac{\tau_1^{q-1}}{r_1 - \lambda_1 m_1}[(1 - q\lambda_1\beta_1 + q\lambda_1\alpha_1\gamma_1^q\mu^q + C_q\lambda_1^q\gamma_1^q\mu^q)^{\frac{1}{q}} + \lambda_1 t_1 \nu], \\ h_2 = \frac{\tau_2^{q-1}}{r_2 - \lambda_2 m_2}[(1 - q\lambda_2\beta_2 + q\lambda_2\alpha_2\gamma_2^q\xi^q + C_q\lambda_2^q\gamma_2^q\xi^q)^{\frac{1}{q}} + \lambda_2 t_2 \delta], \end{cases} \tag{3.2}$$

where C_q is the same as in Lemma 2.1, then problem (2.1) has a solution (x^*, y^*).

Proof. For any given $\lambda_1, \lambda_2 > 0$, define a mapping $F : X \to CB(X)$ as follows:

$$\begin{aligned}
&F(x) = x - g(x) + R^{\eta_1,M_1}_{\lambda_1,A_1}(R^{\eta_2,M_2}_{\lambda_2,A_2}(x - \lambda_2(N_2(P(x), Q(x)) - \theta_2))\\
&-\lambda_1(N_1(S(R^{\eta_2,M_2}_{\lambda_2,A_2}(x - \lambda_2(N_2(P(x), Q(x)) - \theta_2))),\\
&T(R^{\eta_2,M_2}_{\lambda_2,A_2}(x - \lambda_2(N_2(P(x), Q(x))) - \theta_2)))) - \theta_1)), \quad \forall x \in X \qquad (3.3)
\end{aligned}$$

It follows from (3.3) and Lemma 3.1 that (x^*, y^*) is a solution of problem (2.1) if and only if there exists $x^* \in X$ such that $x^* \in F(x^*)$. Now we prove that F has a fixed point in X. In fact, for any given $x, y \in X$, $\varepsilon > 0$ and $a \in F(x)$, there exist $u \in Q(x)$ and $v \in T(w)$ such that $a = x - g(x) + R^{\eta_1,M_1}_{\lambda_1,A_1}(w - \lambda_1(N_1(S(w), v) - \theta_1))$, where $w = R^{\eta_2,M_2}_{\lambda_2,A_2}(x - \lambda_2(N_2(P(x), u) - \theta_2))$. Since $Q, T : X \to CB(X)$, it follows from Nadler [23] that there exist $u' \in Q(y), v' \in T(w')$ such that

$$\|u - u'\| \le (1 + \varepsilon)H(Q(x), Q(y)), \|v - v'\| \le (1 + \varepsilon)H(T(w), T(w')),$$

where $w' = R^{\eta_2,M_2}_{\lambda_2,A_2}(y - \lambda_2(N_2(P(y), u') - \theta_2))$.

Let $b = y - g(y) + R^{\eta_1,M_1}_{\lambda_1,A_1}(w' - \lambda_1(N_1(S(w'), v') - \theta_1))$. Thus, we obtain

$$\begin{aligned}
&\|a - b\|\\
&\le \|x - y - (g(x) - g(y))\| + \|R^{\eta_1,M_1}_{\lambda_1,A_1}(w - \lambda_1(N_1(S(w), v) - \theta_1))\\
&\quad -R^{\eta_1,M_1}_{\lambda_1,A_1}(w' - \lambda_1(N_1(S(w'), v') - \theta_1))\|\\
&\le \|x - y - (g(x) - g(y))\|\\
&\quad +\frac{\tau_1^{q-1}}{r_1 - \lambda_1 m_1}(\lambda_1\|N_1(S(w'), v) - N_1(S(w'), v')\|\\
&\quad +\|w - w' - \lambda_1(N_1(S(w), v) - N_1(S(w'), v)\|), \qquad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
&\|w - w'\|\\
&= \|R^{\eta_2,M_2}_{\lambda_2,A_2}(x - \lambda_2(N_2(P(x), u) - \theta_2))\\
&\quad -R^{\eta_2,M_2}_{\lambda_2,A_2}(y - \lambda_2(N_2(P(y), u') - \theta_2))\\
&\le \frac{\tau_2^{q-1}}{r_2 - \lambda_2 m_2}(\|x - y - \lambda_2(N_2(P(x), u) - N_2(P(y), u))\|\\
&\quad +\lambda_2\|N_2(P(y), u) - N_2(P(y), u')\|). \qquad (3.5)
\end{aligned}$$

By assumptions, we have

$$\begin{aligned}
&\|x - y - (g(x) - g(y))\|^q\\
&\le \|x - y\|^q - q\langle g(x) - g(y), J_q(x - y)\rangle + C_q\|g(x) - g(y)\|^q\\
&\le (1 - qb + qa\gamma^q + C_q\gamma^q)\|x - y\|^q, \qquad (3.6)
\end{aligned}$$

$$\begin{aligned}\|N_1(S(w),v)-N_1(S(w'),v')\| &\leq t_1\|v-v'\| \\ &\leq t_1(1+\varepsilon)H(T(w),T(w')) \\ &\leq t_1\nu(1+\varepsilon)\|w-w'\|, \end{aligned} \tag{3.7}$$

$$\begin{aligned}&\|w-w'-\lambda_1(N_1(S(w),v)-N_1(S(w'),v)\|^q \\ &\leq \|w-w'\|^q - q\lambda_1\langle N_1(S(w),v)-N_1(S(w'),v), J_q(w-w')\rangle \\ &\quad +C_q\lambda_1^q\|N_1(S(w),v)-N_1(S(w'),v)\|^q \\ &\leq (1-q\lambda_1\beta_1+q\lambda_1\alpha_1\gamma_1^q\mu^q+C_q\lambda_1^q\gamma_1^q\mu^q)\|w-w'\|^q, \end{aligned} \tag{3.8}$$

$$\begin{aligned}&\|x-y-\lambda_2(N_2(P(x),u)-N_2(P(y),u))\|^q \\ &\leq \|x-y\|^q - q\lambda_2\langle N_2(P(x),u)-N_2(P(y),u), J_q(x-y)\rangle \\ &\quad +C_q\lambda_2^q\|N_2(P(x),u)-N_2(P(y),u))\|^q \\ &\leq (1-q\lambda_2\beta_2+q\lambda_2\alpha_2\gamma_2^q\xi^q+C_q\lambda_2^q\gamma_2^q\xi^q)\|x-y\|, \end{aligned} \tag{3.9}$$

$$\begin{aligned}\|N_2(P(x),u)-N_2(P(y),u')\| &\leq t_2\|u-u'\| \\ &\leq t_2(1+\varepsilon)H(Q(x),Q(y)) \\ &\leq t_2\delta(1+\varepsilon)\|x-y\|. \end{aligned} \tag{3.10}$$

Combining (3.4)-(3.10), we have

$$\begin{aligned}&\|a-b\| \\ \leq\ & (1-qb+qa\gamma^q+C_q\gamma^q)^{\frac{1}{q}}\|x-y\| \\ & +\frac{\tau_1^{q-1}}{r_1-\lambda_1 m_1}[(1-q\lambda_1\beta_1+q\lambda_1\alpha_1\gamma_1^q\mu^q+C_q\lambda_1^q\gamma_1^q\mu^q)^{\frac{1}{q}} \\ & +\lambda_1 t_1\nu(1+\varepsilon)]\|w-w'\| \\ \leq\ & \{(1-qb+qa\gamma^q+C_q\gamma^q)^{\frac{1}{q}} \\ & +\frac{\tau_1^{q-1}}{r_1-\lambda_1 m_1}[(1-q\lambda_1\beta_1+q\lambda_1\alpha_1\gamma_1^q\mu^q+C_q\lambda_1^q\gamma_1^q\mu^q)^{\frac{1}{q}}+\lambda_1 t_1\nu(1+\varepsilon)] \\ & \times\frac{\tau_2^{q-1}}{r_2-\lambda_2 m_2}[(1-q\lambda_2\beta_2+q\lambda_2\alpha_2\gamma_2^q\xi^q+C_q\lambda_2^q\gamma_2^q\xi^q)^{\frac{1}{q}} \\ & +\lambda_2 t_2\delta(1+\varepsilon)]\}\|x-y\| \\ \leq\ & (l+h_1(\varepsilon)\cdot h_2(\varepsilon))\|x-y\| \\ \leq\ & h(\varepsilon)\|x-y\| \end{aligned} \tag{3.11}$$

where

$$h(\varepsilon)=l+h_1(\varepsilon)\cdot h_2(\varepsilon),\quad l=(1-qb+qa\gamma^q+C_q\gamma^q)^{\frac{1}{q}},$$

$$h_1(\varepsilon) = \frac{\tau_1^{q-1}}{r_1 - \lambda_1 m_1}[(1 - q\lambda_1\beta_1 + q\lambda_1\alpha_1\gamma_1^q\mu^q + C_q\lambda_1^q\gamma_1^q\mu^q)^{\frac{1}{q}} + \lambda_1 t_1\nu(1+\varepsilon)],$$
$$h_2(\varepsilon) = \frac{\tau_2^{q-1}}{r_2 - \lambda_2 m_2}[(1 - q\lambda_2\beta_2 + q\lambda_2\alpha_2\gamma_2^q\xi^q + C_q\lambda_2^q\gamma_2^q\xi^q)^{\frac{1}{q}} + \lambda_2 t_2\delta(1+\varepsilon)].$$

From (3.11), we know that

$$\sup_{a\in F(x)} d(a, F(y)) \le h(\varepsilon)\|x - y\|, \quad \forall x, y \in X. \tag{3.12}$$

Similarly, we have

$$\sup_{b\in F(y)} d(b, F(x)) \le h(\varepsilon)\|x - y\|, \quad \forall x, y \in X. \tag{3.13}$$

It follows from (3.12), (3.13) and the definition of Hausdorff metric that

$$H(F(x), F(y)) \le h(\varepsilon)\|x - y\|, \quad \forall x, y \in X.$$

Letting $\varepsilon \to 0$, we get

$$H(F(x), F(y)) \le h\|x - y\|, \quad \forall x, y \in X, \tag{3.14}$$

where $h = l + h_1h_2$, $l = (1 - qb + qa\gamma^q + C_q\gamma^q)^{\frac{1}{q}}$, $h_1 = \frac{\tau_1^{q-1}}{r_1-\lambda_1 m_1}[(1 - q\lambda_1\beta_1 + q\lambda_1\alpha_1\gamma_1^q\mu^q + C_q\lambda_1^q\gamma_1^q\mu^q)^{\frac{1}{q}} + \lambda_1 t_1\nu]$, $h_2 = \frac{\tau_2^{q-1}}{r_2-\lambda_2 m_2}[(1 - q\lambda_2\beta_2 + q\lambda_2\alpha_2\gamma_2^q\xi^q + C_q\lambda_2^q\gamma_2^q\xi^q)^{\frac{1}{q}} + \lambda_2 t_2\delta]$. It follows from (3.2) that $0 < h < 1$ and so by (3.14) and Lemma 2.3, we know that F has a fixed point in X, i.e., there exists a point $x^* \in X$ such that $x^* \in F(x^*)$. This completes the proof. □

If $g = I$, then Theorem 3.1 becomes the following existence theorem of the solution for problem (2.2).

Theorem 3.2. *Assume that $X, S, P, T, Q, \eta_i, A_i, N_i$ and M_i for $i = 1, 2$ are the same as in Theorem 3.1. If there exist constants $\lambda_1 \in (0, \frac{r_1}{m_1})$ and $\lambda_2 \in (0, \frac{r_2}{m_2})$ such that*

$$\begin{cases} h_1h_2 < 1, \\ h_1 = \frac{\tau_1^{q-1}}{r_1-\lambda_1 m_1}[(1 - q\lambda_1\beta_1 + q\lambda_1\alpha_1\gamma_1^q\mu^q + C_q\lambda_1^q\gamma_1^q\mu^q)^{\frac{1}{q}} + \lambda_1 t_1\nu], \\ h_2 = \frac{\tau_2^{q-1}}{r_2-\lambda_2 m_2}[(1 - q\lambda_2\beta_2 + q\lambda_2\alpha_2\gamma_2^q\xi^q + C_q\lambda_2^q\gamma_2^q\xi^q)^{\frac{1}{q}} + \lambda_2 t_2\delta], \end{cases} \tag{3.15}$$

where C_q is the same as in Lemma 2.1, then problem (2.2) has a solution (x^*, y^*).

4. Iterative algorithms and convergence

In this section, we shall construct new Mann iterative algorithms to approximate the solution of problems (2.1) and (2.2) and discuss the convergence analysis of the algorithm.

Now, we give a Mann iterative algorithm for solving problem (2.1).

Algorithm 4.1. *For any given $x_0 \in X$, the generalized Mann iterative sequence $\{x_n\}$ and $\{y_n\}$ in X is defined as follows:*

$$\begin{cases} x_{n+1} \in (1-\alpha_n)x_n + \alpha_n[x_n - g(x_n) \\ +R^{\eta_1,M_1}_{\lambda_1,A_1}(y_n - \lambda_1(N_1(S(y_n),T(y_n)) - \theta_1))], \\ y_n \in R^{\eta_2,M_2}_{\lambda_2,A_2}(x_n - \lambda_2(N_2(P(x_n),Q(x_n)) - \theta_2)), \quad n = 0,1,2,\cdots, \end{cases} \tag{4.1}$$

where $\lambda_1, \lambda_2 > 0$ are constants, α_n is a sequence of real numbers such that $\alpha_n \in [0,1]$ and $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$.

If $g = I$, then Algorithm 4.1 reduces to the following algorithm for solving problem (2.2).

Algorithm 4.2. *For any given $x_0 \in X$, define the Mann iterative sequence $\{x_n\}$ and $\{y_n\}$ in X as follows:*

$$\begin{cases} x_{n+1} \in (1-\alpha_n)x_n + \alpha_n R^{\eta_1,M_1}_{\lambda_1,A_1}(y_n - \lambda_1(N_1(S(y_n),T(y_n)) - \theta_1))], \\ y_n \in R^{\eta_2,M_2}_{\lambda_2,A_2}(x_n - \lambda_2(N_2(P(x_n),Q(x_n)) - \theta_2)), \quad n = 0,1,2,\cdots, \end{cases} \tag{4.2}$$

where $\lambda_1, \lambda_2 > 0$ and α_n are the same as in Algorithm 4.1.

Theorem 4.1. *Let $X, S, P, T, Q, g, \eta_i, A_i, N_i$ and M_i for $i = 1,2$ are the same as in Theorm 3.1. If condition (3.2) of Theorem 3.1 hold, then the generalized Mann iterative sequence $\{x_n\}$ and $\{y_n\}$ defined by Algorithm 4.1 converge strongly to the solution (x^*, y^*) of problem (2.1).*

Proof. Let (x^*, y^*) be the solution of problem (2.1). It follows from Lemma 3.1 that

$$\begin{cases} g(x^*) = R^{\eta_1,M_1}_{\lambda_1,A_1}(y^* - \lambda_1(N_1(S(y^*),v^*) - \theta_1)), \forall v^* \in T(y^*), \\ y^* = R^{\eta_2,M_2}_{\lambda_2,A_2}(x^* - \lambda_2(N_2(P(x_2),u^*) - \theta_2)), \forall u^* \in Q(x^*). \end{cases} \tag{4.3}$$

Since $Q(x^*), Q(x_n), T(y^*), T(y_n) \in CB(X)$ for all $n \geq 0$, for any given $n \geq 0$ and $\varepsilon > 0$, it follows from Nadler [23] that there exist $u_n \in Q(x_n), v_n \in T(y_n)$

such that

$$\|u_n - u^*\| \le (1+\varepsilon)H(Q(x_n), Q(x^*)), \quad \|v_n - v^*\| \le (1+\varepsilon)H(T(y_n), T(y^*)).$$

From (4.1), (4.3) and the proof of (3.11), for all $v_n \in T(y_n)$ and $v^* \in T(y^*)$, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\| \\
&= \|(1-\alpha_n)x_n + \alpha_n(x_n - g(x_n) \\
&\quad + R^{\eta_1, M_1}_{\lambda_1, A_1}(y_n - \lambda_1(N_1(S(y_n), v_n) - \theta_1))) - x^*\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^* - (g(x_n) - g(x^*))\| \\
&\quad + \alpha_n\|R^{\eta_1, M_1}_{\lambda_1, A_1}(y_n - \lambda_1(N_1(S(y_n), v_n) - \theta_1)) \\
&\quad - R^{\eta_1, M_1}_{\lambda_1, A_1}(y^* - \lambda_1(N_1(S(x^*), v^*) - \theta_1))\| \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^* - (g(x_1) - g(x_2))\| \\
&\quad + \alpha_n \frac{\tau_1^{q-1}}{r_1 - \rho_1 m_1}(\|y_n - y^* - \lambda_1(N_1(S(y_n), v_n) - N_1(S(y^*), v_n))\| \\
&\quad + \lambda_1(N_1(S(y^*), v_n) - N_1(S(y^*), v^*))\|) \\
&\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n l\|x_n - x^*\| + \alpha_n h_1(\varepsilon)\|y_n - y^*\|. \qquad (4.4)
\end{aligned}$$

where $l = (1 - qb + qa\gamma^q + C_q\gamma^q)^{\frac{1}{q}}$, $h_1(\varepsilon) = \frac{\tau_1^{q-1}}{r_1 - \lambda_1 m_1}[(1 - q\lambda_1\beta_1 + q\lambda_1\alpha_1\gamma_1^q\mu^q + C_q\lambda_1^q\gamma_1^q\mu^q)^{\frac{1}{q}} + \lambda_1 t_1\nu(1+\varepsilon)]$.

Similarly, we obtain

$$\begin{aligned}
\|y_n - y^*\| &= \|R^{\eta_2, M_2}_{\lambda_2, A_2}(x_n - \lambda_2(N_2(P(x_n), u_n) - \theta_2)) - y^*\| \\
&\le h_2(\varepsilon)\|x_n - x^*\|, \qquad (4.5)
\end{aligned}$$

where $h_2(\varepsilon) = \frac{\tau_2^{q-1}}{r_2 - \lambda_2 m_2}[(1 - q\lambda_2\beta_2 + q\lambda_2\alpha_2\gamma_2^q\xi^q + C_q\lambda_2^q\gamma_2^q\xi^q)^{\frac{1}{q}} + \lambda_2 t_2\delta(1+\varepsilon)]$.

It It follows from (4.4) and (4.5) that

$$\begin{aligned}
\|x_{n+1} - x^*\| &\le (1 - \alpha_n + \alpha_n l + \alpha_n h_1(\varepsilon)h_2(\varepsilon))\|x_n - x^*\| \\
&= [1 - \alpha_n(1 - h(\varepsilon))]\|x_n - x^*\|, \qquad (4.6)
\end{aligned}$$

where $h(\varepsilon) = l + h_1(\varepsilon)h_2(\varepsilon)$. Let $\varepsilon \to 0$. Then we have $h_i(\varepsilon) \to h_i$ for $i = 1, 2$, $h(\varepsilon) \to h$, where $h = l + h_1 h_2$, $l = (1 - qb + qa\gamma^q + C_q\gamma^q)^{\frac{1}{q}}$, $h_1 = \frac{\tau_1^{q-1}}{r_1 - \lambda_1 m_1}[(1 - q\lambda_1\beta_1 + q\lambda_1\alpha_1\gamma_1^q\mu^q + C_q\lambda_1^q\gamma_1^q\mu^q)^{\frac{1}{q}} + \lambda_1 t_1\nu]$, $h_2 = \frac{\tau_2^{q-1}}{r_2 - \lambda_2 m_2}[(1 - q\lambda_2\beta_2 + q\lambda_2\alpha_2\gamma_2^q\xi^q + C_q\lambda_2^q\gamma_2^q\xi^q)^{\frac{1}{q}} + \lambda_2 t_2\delta]$.

It follows from (3.2) that $0 < h < 1$. By (4.6), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq [1 - (1-h)\alpha_n]\|x_n - x^*\| \\ &\leq (1 - (1-h)\alpha_n) \cdots (1 - (1-h)\alpha_0)\|x_0 - x^*\| \\ &= \prod_{j=0}^{n}(1 - (1-h)\alpha_j)\|x_0 - x^*\|. \end{aligned} \tag{4.7}$$

Since $0 < h < 1$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$,we have

$$\prod_{n=0}^{\infty}(1 - (1-h)\alpha_n) = \lim_{n\to\infty}\prod_{j=0}^{n}(1 - (1-h)\alpha_j) = 0,$$

which, hence, implies that $\{x_n\}$ converges strongly to x^*. This completes the proof. □

From Theorem 4.1, we can get the following Theorem 4.2.

Theorem 4.2. *Assume let $X, S, P, T, Q, \eta_i, A_i, N_i$ and M_i for $i = 1, 2$ are the same as in Theorem 3.2. If condition (3.15) of Theorem 3.2 hold, then the Mann iterative sequence $\{x_n\}$ and $\{y_n\}$ defined by Algorithm 4.2 converge strongly to the solution (x^*, y^*) of problem (2.2).*

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