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ASYMPTOTIC STABILITY OF SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS

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Abstract. In this paper we investigate the asymptotic stability of solutions of the integral equations of Volterra type and its solvability in the space of continuous and bounded functions on R_+. The main tool used in our study is the technique associated with measures of noncompactness and a fixed point theorem.

1. Introduction

The theory of integral equations has many applications in describing numerous events and problems of the real world. For example, integral equations are often applicable in engineering, mathematical physics, economics and biology [11, 12]. It is a well known fact that the nonlinear quadratic equations are often encountered in various applications. It is worthwhile mentioning the applications of those equations in the theory of radiative transfer, kinetic theory of gases, in the traffic theory and in the theory of neutron transport, for instance. Especially the so-called quadratic integral equation of Chandrasekhar type can be very often encountered in several applications [1, 4, 10, 13, 15, 16]. Integral equations of such a type are also often an object of mathematical investigations [9, 12, 17].

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Fixed point theorems used in nonlinear functional analysis allows us, in general, to obtain existence theorems concerning investigated functional-operator equations. It is rather difficult to obtain characterizations of solutions of considered equations with the help of those theorems. In this paper we study the nonlinear quadratic integral equation of volterra type by using the measure of noncompactness and the Darbo fixed point theorem. The results generalise the previous results of [5, 6, 7, 8, 14].

2. Preliminaries

Let $(E, \|.\|)$ be an infinite dimensional Banach Space with zero element θ. Let $B(x, r)$ denote the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta, r)$. If X is a subset of E then $\overline{X}$, conv X denote the closure and convex closure of X, respectively. The family of all nonempty and bounded subsets of E is denoted by $\mathcal{M}_E$ and its subfamily consisting of all relatively compact sets is denoted by $\mathcal{N}_E$.

Definition 2.1. A mapping $\mu : \mathcal{M}_E \to R_+ = [0, +\infty)$ is said to be a measure of noncompactness in E if it satisfies the following conditions :

(1) The family $\ker\mu = \{X \in \mathcal{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathcal{N}_E$;
(2) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$;
(3) $\mu(\text{conv } X) = \mu(X)$;
(4) $\mu(\overline{X}) = \mu(X)$;
(5) $\mu(\lambda X + (1-\lambda)Y) \leq \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0, 1]$;
(6) If (X_n) is a sequence of sets from $\mathcal{M}_E$ such that $X_{n+1} \subset X_n$, $\overline{X}_n = X_n$ $(n = 1, 2, 3...)$ and if $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty.

The family $\ker\mu$ described in (1) is called the kernel of the measure of noncompactness μ.

A measure μ is said to be sublinear if it satisfies the following two conditions:

(7) $\mu(\lambda X) = |\lambda|\mu(X)$ for $\lambda \in R$;
(8) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.

Further facts concerning measures of noncompactness and its properties may be found in [3]. For our further purposes we will only need the following fixed point theorem due to Darbo.

Theorem 2.2. *Let Q be nonempty bounded closed convex subset of the space E and let $F : Q \to Q$ be a continuous operator such that $\mu(FX) \leq k\mu(X)$ for any nonempty subset X of Q, where $k \in [0, 1)$ is a constant. Then F has a fixed point in the set Q.*

Remark 2.3. Under the assumptions of the above theorem it can be shown that the set fix F of fixed points of F belonging to Q is a member of the kernel kerμ.

This observation allows us to characterize solutions of considered operator equations.

Let us consider the Banach space $BC(R_+)$ consisting of all real functions defined, bounded and continuous on R_+. The space $BC(R_+)$ is equipped with the standard norm

$$\|x\| = \sup\{|x(t)| : t \geq 0\}.$$

Now we recollect the construction of the measure of noncompactness which will be used in this paper [2].

For this, let us fix a nonempty bounded subset X of $BC(R_+)$ and a positive number $T > 0$. For $x \in X$ and $\epsilon \geq 0$ denote by $\omega^T(x, \epsilon)$ the modulus of continuity of the function x on the interval $[0, T]$, i,e.,

$$\omega^T(x, \epsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T], \quad |t - s| \leq \epsilon\}.$$

Further, let us put

$$\omega^T(X, \epsilon) = \sup\left\{\omega^T(x, \epsilon) : x \in X\right\},$$

$$\omega_0^T(X) = \lim_{\epsilon \to 0} \omega^T(X, \epsilon), \qquad \omega_0(X) = \lim_{T \to \infty} \omega_0^T(X).$$

For a fixed number $t \geq 0$ we denote

$$X(t) = \{x(t) : x \in X\}$$

and

$$diam X(t) = \sup\{|x(t) - y(t)| : x, y \in X\}.$$

Finally, let us define the function μ on the family $\mathcal{M}_{BC(R_+)}$ by the formula

$$\mu(X) = \omega_0(X) + \lim_{t \to \infty} \sup diam X(t).$$

It can be shown [2] that the function μ is a sublinear measure of noncompactness on the space $BC(R_+)$. The kernel kerμ of this measure contains nonempty and bounded sets X such that functions from X are locally equicontinuous on R_+ and the thickness of the bundle formed by functions from X tends to zero at infinity. This property allows us to characterize solutions of the following integral equations and will be used in the next section.

3. Main result

Consider the following nonlinear functional-integral equations:

$$x(t) \;=\; g\Big(t, x(t)\Big) + h\Big(t, x(t)\Big) \int_0^t u\Big(t, s, x(s)\Big) ds, \quad t \geq 0, \tag{3.1}$$

and

$$x(t) \;=\; f\left(t, x(t) \int_0^t u\Big(t, s, x(s)\Big) ds\right), \quad t \geq 0. \tag{3.2}$$

We assume the following conditions for Eq.(3.1):

(H1) $g : R_+ \times R \to R$ is a continuous function $g(t, 0) \in BC(R_+)$;
(H2) $h : R_+ \times R \to R$ is a continuous function $h(t, 0) \in BC(R_+)$;
(H3) There exists a continuous function $m(t) : R_+ \to R_+$ such that

$$|g(t, x) - g(t, y)| \leq m(t)|x - y|$$

for all $x, y \in R,\ t \in R_+$;
(H4) There exists a continuous function $n(t) : R_+ \to R_+$ such that

$$|h(t, x) - h(t, y)| \leq n(t)|x - y|$$

for all $x, y \in R,\ t \in R_+$;
(H5) $u : R_+ \times R_+ \times R \to R$ is a continuous function such that

$$\lim_{t \to \infty} \int_0^t \Big|u\Big(t, s, x(s)\Big)\Big| ds = 0,$$

$$\lim_{t \to \infty} n(t) \int_0^t \Big|u\Big(t, s, x(s)\Big)\Big| ds = 0$$

uniformly with respect to $x \in BC(R_+)$;
(H6) There exists a constant $k \in [0, 1)$ such that

$$\sup_{t \geq 0} \left(m(t) + n(t) \int_0^t \Big|u\Big(t, s, x(s)\Big)\Big| ds \right) \leq k$$

for any $t \geq 0$.

Remark 3.1. The concept of the asymptotic stability of a solution $x = x(t)$ of Eq.(3.1) is understood in the following sense.

For any $\epsilon > 0$ there exist $T > 0$ and $r > 0$ such that if $x, y \in B_r$ and $x = x(t),\ y = y(t)$ are solutions of Eq.(3.1) then $|x(t) - y(t)| \leq \epsilon$ for $t \geq T$.

Theorem 3.2. *Assume (H1)-(H6) hold. Then Eq.(3.1) has at least one solution $x(t)$ which belongs to the space $BC(R_+)$ and is asymptotically stable on the interval R_+.*

Proof. Define the operator L on the space $BC(R_+)$ by the formula

$$(Lx)(t) = g\Big(t, x(t)\Big) + h\Big(t, x(t)\Big) \int_0^t u\Big(t, s, x(s)\Big) ds, \ t \geq 0.$$

Clearly, the function Lx is continuous on the interval R_+ for any function $x \in BC(R_+)$.

Applying our assumptions, we have the following estimate:

$$\begin{aligned}
& |(Lx)(t)| \\
\leq\ & \Big| g\Big(t, x(t)\Big) - g(t, 0) \Big| \\
& + \Big| h\Big(t, x(t)\Big) \int_0^t u\Big(t, s, x(s)\Big) ds - h(t, 0) \int_0^t u\Big(t, s, x(s)\Big) ds \Big| \\
& + \Big| g(t, 0) + h(t, 0) \int_0^t u\Big(t, s, x(s)\Big) ds \Big| \\
\leq\ & \left(m(t) + n(t) \int_0^t \Big| u\Big(t, s, x(s)\Big) \Big| ds \right) |x(t)| \\
& + |g(t, 0)| + |h(t, 0)| \int_0^t \Big| u\Big(t, s, x(s)\Big) \Big| ds. \\
\leq\ & k|x(t)| + |g(t, 0)| + |h(t, 0)| \int_0^t \Big| u\Big(t, s, x(s)\Big) \Big| ds.
\end{aligned}$$

Hence we derive that the function Lx is bounded on the interval R_+. Thus $Lx \in BC(R_+)$.

Moreover, from the above estimate we obtain

$$\|Lx\| \leq k\|x\| + A, \tag{3.3}$$

where we have denoted

$$A = \sup \left\{ |g(t, 0)| + |h(t, 0)| \int_0^t \Big| u\Big(t, s, x(s)\Big) \Big| ds : \ t \geq 0 \right\}.$$

Obviously in view of the assumptions (H1), (H2) and (H5) we have that $A < \infty$. Since $k < 1$, from Eq.(3.3), the operator L transforms B_r into itself for $r = A/(1-k)$.

Take $\epsilon > 0$, and $x, y \in B_r$ such that $\|x - y\| \leq \epsilon$. Then for $t \geq 0$, we get

$$\begin{aligned}
|(Lx)(t) - (Ly)(t)| \ \leq\ & \Big| g\Big(t, x(t)\Big) - g\Big(t, y(t)\Big) \Big| \\
& + \Big| h\Big(t, x(t)\Big) - h\Big(t, y(t)\Big) \Big| \int_0^t \Big| u\Big(t, s, x(s)\Big) \Big| ds \\
& + \Big| h\Big(t, y(t)\Big) \Big| \int_0^t \Big| u\Big(t, s, x(s)\Big) - u\Big(t, s, y(s)\Big) \Big| ds
\end{aligned}$$

$$\begin{aligned}
&\le \Big(m(t)+n(t)\int_0^t \Big|u\Big(t,s,x(s)\Big)\Big|ds\Big)\Big|x(t)-y(t)\Big| + \Big|h\Big(t,y(t)\Big) \\
&\quad -h(t,0)+h(t,0)\Big|\int_0^t \Big|u\Big(t,s,x(s)\Big)-u\Big(t,s,y(s)\Big)\Big|ds \\
&\le k\epsilon + \Big(rn(t)+|h(t,0)|\Big)\int_0^t \Big|u\Big(t,s,x(s)\Big)-u\Big(t,s,y(s)\Big)\Big|ds.
\end{aligned} \tag{3.4}$$

Next using assumptions (H2) and (H5), we choose a number $T>0$ such that for $t \ge T$ the following inequalities hold,

$$\left.\begin{aligned}
&rn(t)\int_0^t \Big|u\Big(t,s,x(s)\Big)\Big|ds \le (1-k)\epsilon/4, \\
&\sup\{|h(t,0)| : t\ge 0\}\int_0^t \Big|u\Big(t,s,x(s)\Big)\Big|ds \le (1-k)\epsilon/4.
\end{aligned}\right\} \tag{3.5}$$

Let us consider two cases:

(a) $t \ge T$. Then in view of Eq.(3.4) and Eq.(3.5), we obtain
$|(Lx)(t)-(Ly)(t)| \le k\epsilon+(1-k)\epsilon/4+(1-k)\epsilon/4+(1-k)\epsilon/4+(1-k)\epsilon/4 = \epsilon.$

(b) $t \le T$. In this case, let us consider the function $\omega = \omega(\epsilon)$ defined by the formula
$\omega(\epsilon)=\sup\{|u(t,s,x)-u(t,s,y)| : \ t,s\in[0,T],\ x,y\in[-r,r],\ |x-y|\le\epsilon\}.$

Taking into account the uniform continuity of the function $u = u(t,s,x)$ on the set $[0,T]\times[0,T]\times[-r,r]$, we deduce that $\omega(\epsilon)\to 0$ as $\epsilon \to 0$.

Thus in this case, by virtue of Eq.(3.5), we get

$$\begin{aligned}
|(Lx)(t)-(Ly)(t)| \le\ & k\epsilon + \Big(r\sup\{n(t) : t\in[0,T]\} \\
& + \sup\{|h(t,0)| : t\in[0,T]\}\Big)T\omega(\epsilon).
\end{aligned}$$

Finally linking cases (a) and (b) and keeping in mind the above established facts, we conclude that the operator L is continuous on the ball B_r.

Take a nonempty set $X \subset B_r$. Then, for $x,y\in X$, and for a fixed $t\ge 0$, calculating in the same way as in the proof of estimate Eq.(3.4), we obtain

$$\begin{aligned}
|(Lx)(t)-(Ly)(t)| \le\ & k|x(t)-y(t)| + \Big(rn(t)+|h(t,0)|\Big) \\
& \Big[\int_0^t \Big|u\Big(t,s,x(s)\Big)\Big|ds + \int_0^t \Big|u\Big(t,s,y(s)\Big)\Big|ds\Big].
\end{aligned}$$

Hence we can easily deduce,

$$\operatorname{diam}(LX)(t) \leq k\operatorname{diam}X(t) + \sup_{x,y\in X}\Big\{\Big(rn(t)+|h(t,0)|\Big)\Big[\int_0^t\Big|u\Big(t,s,x(s)\Big)\Big|ds+\int_0^t\Big|u\Big(t,s,y(s)\Big)\Big|ds\Big]\Big\}.$$

Now, taking into account our assumptions, we get

$$\lim_{t\to\infty}\sup\operatorname{diam}(LX)(t) \leq k\lim_{t\to\infty}\sup\operatorname{diam}X(t). \tag{3.6}$$

Further, let us fix arbitrarily numbers $T > 0$ and $\epsilon > 0$. Choose a function $x \in X$ and take $t, s \in [0, T]$ such that $|t - s| \leq \epsilon$. Without loss of generality, we may assume that $s < t$. Then, in view of our assumptions, we have

$$\begin{aligned}
&|(Lx)(t)-(Lx)(s)| \\
&\leq \Big|g\Big(t,x(t)\Big)-g\Big(s,x(s)\Big)\Big| \\
&\quad+\Big|h\Big(t,x(t)\Big)\int_0^t u\Big(t,\tau,x(\tau)\Big)d\tau-h\Big(s,x(s)\Big)\int_0^s u\Big(s,\tau,x(\tau)\Big)d\tau\Big| \\
&\leq m(t)|x(t)-x(s)|+\Big|g\Big(t,x(s)\Big)-g\Big(s,x(s)\Big)\Big| \\
&\quad+\Big|h\Big(t,x(t)\Big)-h\Big(s,x(s)\Big)\Big|\int_0^t\Big|u\Big(t,\tau,x(\tau)\Big)\Big|d\tau \\
&\quad+\Big|h\Big(s,x(s)\Big)\Big|\,\Big|\int_0^t u\Big(t,\tau,x(\tau)\Big)d\tau-\int_0^s u\Big(s,\tau,x(\tau)\Big)d\tau\Big| \\
&\leq \Big(m(t)+n(t)\int_0^t\Big|u\Big(t,\tau,x(\tau)\Big)\Big|d\tau\Big)|x(t)-x(s)| \\
&\quad+\Big|g\Big(t,x(s)\Big)-g\Big(s,x(s)\Big)\Big| \\
&\quad+\Big|h\Big(t,x(s)\Big)-h\Big(s,x(s)\Big)\Big|\int_0^t\Big|u\Big(t,\tau,x(\tau)\Big)\Big|d\tau \\
&\quad+\Big(n(s)|x(s)|+|h(s,0)|\Big)\int_s^t\Big|u\Big(t,\tau,x(\tau)\Big)\Big|d\tau \\
&\quad+\Big(n(s)|x(s)|+|h(s,0)|\Big)\int_0^s\Big|u\Big(t,\tau,x(\tau)\Big)-u\Big(s,\tau,x(\tau)\Big)\Big|d\tau \\
&\leq k|x(t)-x(s))|+\omega_r^T(g,\epsilon)+\omega_r^T(h,\epsilon)\int_0^t\Big|u\Big(t,\tau,x(\tau)\Big)\Big|d\tau \\
&\quad+\Big(n(s)r+|h(s,0)|\Big)\int_s^t\Big|u\Big(t,\tau,x(\tau)\Big)\Big|d\tau \\
&\quad+T\Big(n(s)r+|h(s,0)|\Big)\omega_r^T(u,\epsilon),
\end{aligned}$$

where we denoted

$$\omega_r^T(g,\epsilon) = \sup\{|g(t,x)-g(s,x)| : t,s\in[0,T], |t-s|\le\epsilon, |x|\le r\},$$
$$\omega_r^T(h,\epsilon) = \sup\{|h(t,x)-h(s,x)| : t,s\in[0,T], |t-s|\le\epsilon, |x|\le r\},$$
$$\omega_r^T(u,\epsilon) = \sup\{|u(t,\tau,x)-u(s,\tau,x)| : t,s,\tau\in[0,T], |t-s|\le\epsilon, |x|\le r\}.$$

Hence we obtain,

$$\begin{aligned}\omega^T(Lx,\epsilon) \le k\omega^T(x,\epsilon) + \omega_r^T(g,\epsilon) + \omega_r^T(h,\epsilon)\int_0^t \Big|u\Big(t,\tau,x(\tau)\Big)\Big|d\tau \\ + \epsilon r n(s)\sup\Big\{u\Big(t,\tau,x(\tau)\Big) : t,\tau\in[0,T], |x|\le r\Big\} \\ + \epsilon|h(s,0)|\sup\Big\{u\Big(t,\tau,x(\tau)\Big) : t,\tau\in[0,T], |x|\le r\Big\} \\ + T\sup\{n(s)r+|h(s,0)| : s\in[0,T]\}\,\omega_r^T(u,\epsilon).\end{aligned}$$

In view of our assumptions we infer that the function $g = g(t,x)$ and $h = h(t,x)$ is uniformly continuous on the set $[0,T]\times[-r,r]$ and the function $u = u(t,\tau,x)$ is uniformly continuous on $[0,T]\times[0,T]\times[-r,r]$.

Hence we deduce that $\omega_r^T(g,\epsilon)\to 0$, $\omega_r^T(h,\epsilon)\to 0$ and $\omega_r^T(u,\epsilon)\to 0$ as $\epsilon\to 0$. Consequently, from the above estimate we get

$$\omega_0^T(LX) \le k\omega_0^T(X)$$

and, further,

$$\omega_0(LX) \le k\omega_0(X). \tag{3.7}$$

Now, linking Eq.(3.6) and Eq.(3.7), and keeping in mind the definition of the measure of noncompactness μ in the space $BC(R_+)$, we obtain

$$\mu(LX) \le k\mu(X).$$

The conclusion of the theorem follows by the application of Theorem 2.2. □

Remark 3.3. Taking into account of the Remark 3.3 and the description of the kernel of the measure of noncompactness μ, we infer easily from the proof of Theorem 3.2 that any solution of Eq.(3.1) which belongs to the ball B_r is asymptotically stable in the earlier defined sense.

Next, we will consider Eq.(3.2) under the following assumptions:

(H1) $f : R_+\times R\to R$ is a continuous function $f(t,0)\in BC(R_+)$;

(H2) There exists a continuous function $m(t) : R_+\to R_+$ such that

$$|f(t,x)-f(t,y)| \le m(t)|x-y|$$

for all $x,y\in R,\ t\in R_+$;

(H3) $u : R_+ \times R_+ \times R \to R$ is a continuous function such that

$$\lim_{t\to\infty} m(t) \int_0^t \Big| u\Big(t, s, x(s)\Big)\Big| ds = 0$$

uniformly with respect to $x \in BC(R_+)$;

(H4) There exists a constant $c \in [0, 1)$ such that

$$m(t) \int_0^t \Big| u\Big(t, s, x(s)\Big)\Big| ds \leq c$$

for any $t \geq 0$.

Theorem 3.4. *Assume (H1)-(H4) hold. Then Eq.(3.2) has at least one solution $x(t)$ which belongs to the space $BC(R_+)$ and is asymptotically stable on the interval R_+.*

Proof. Define the operator F on the space $BC(R_+)$ by the formula

$$(Fx)(t) = f\left(t, x(t) \int_0^t u\Big(t, s, x(s)\Big) ds\right), \ t \geq 0.$$

Clearly, the function Fx is continuous on the interval R_+ for any function $x \in BC(R_+)$.
From the assumptions, we have the following estimate:

$$\begin{aligned} |(Fx)(t)| &\leq \left| f\left(t, x(t) \int_0^t u\Big(t, s, x(s)\Big) ds\right) - f(t, 0)\right| + |f(t, 0)| \\ &\leq m(t)|x(t)| \int_0^t \Big| u\Big(t, s, x(s)\Big)\Big| ds + |f(t, 0)| \\ &\leq c|x(t)| + |f(t, 0)|. \end{aligned}$$

Hence we derive that the function Fx is bounded on the interval R_+. Thus $Fx \in BC(R_+)$.

Moreover, from the above estimate we obtain

$$\|Fx\| \leq c\|x\| + D, \tag{3.8}$$

where we have denoted

$$D = \sup\{|f(t, 0)| : \ t \geq 0\}.$$

Since $c < 1$, this implies $F(B_\nu) \subset B_\nu$ for $\nu = D/(1-c)$. Let $\epsilon > 0$, and $x, y \in B_\nu$ such that $\|x - y\| \le \epsilon$. Then for $t \ge 0$, we get

$$\begin{aligned}
|(Fx)(t) - (Fy)(t)| &\le m(t)\Big|x(t)\int_0^t u\Big(t,s,x(s)\Big)ds - y(t)\int_0^t u\Big(t,s,y(s)\Big)ds\Big| \\
&\le m(t)|x(t)-y(t)|\int_0^t \Big|u\Big(t,s,x(s)\Big)\Big|ds \\
&\quad + m(t)|y(t)|\int_0^t \Big|u\Big(t,s,x(s)\Big)ds - u\Big(t,s,y(s)\Big)\Big|ds \\
&\le c\epsilon + \nu m(t)\int_0^t \Big|u\Big(t,s,x(s)\Big)ds - u\Big(t,s,y(s)\Big)\Big|ds.
\end{aligned}$$

As in the proof of Theorem 3.2, we can obtain that the operator F is continuous on the ball B_ν and

$$\lim_{t\to\infty}\sup \operatorname{diam}(FX)(t) \le c \lim_{t\to\infty}\sup \operatorname{diam} X(t). \tag{3.9}$$

For any $T > 0$, $\epsilon > 0$, choose a function $x \in X$ and take $t, s \in [0, T]$ such that $|t - s| \le \epsilon$. Without loss of generality, we may assume that $s < t$. Then, in view of our assumptions, we have

$$\begin{aligned}
|(Fx)(t) - Fx)(s)| &\le m(t)\Big|x(t)\int_0^t u\Big(t,\tau,x(\tau)\Big)d\tau - x(s)\int_0^s u\Big(s,\tau,x(\tau)\Big)d\tau\Big| \\
&\le m(t)|x(t)-x(s)|\int_0^t \Big|u\Big(t,\tau,x(\tau)\Big)\Big|d\tau \\
&\quad + m(t)|x(s)|\int_s^t \Big|u\Big(t,\tau,x(\tau)\Big)\Big|d\tau \\
&\quad + m(t)|x(s)|\int_0^s \Big|u\Big(t,\tau,x(\tau)\Big) - u\Big(s,\tau,x(\tau)\Big)\Big|d\tau \\
&\le c|x(t)-x(s)| + \nu m(t)\int_s^t \Big|u\Big(t,\tau,x(\tau)\Big)\Big|d\tau \\
&\quad + \nu m(t)\int_0^s \Big|u\Big(t,\tau,x(\tau)\Big) - u\Big(s,\tau,x(\tau)\Big)\Big|d\tau.
\end{aligned}$$

Hence

$$\begin{aligned}
&\omega^T(Fx,\epsilon) \\
&\quad\le c\,\omega^T(x,\epsilon) + \epsilon\nu m(t)\sup\Big\{\Big|u\Big(t,\tau,x(\tau)\Big)\Big| : t,\tau\in[0,T], |x|\le\nu\Big\} \\
&\qquad + \nu m(t) T \sup\left\{|u(t,\tau,x) - u(s,\tau,x)| : s,t,\tau\in[0,T], |s-t|\le\epsilon, |x|\le\nu\right\}.
\end{aligned}$$

Since $u = u(t, \tau, x)$ is uniformly continuous on the set $[0, T] \times [0, T] \times [-\nu, \nu]$, we deduce that

$$\sup\{|u(t,\tau,x)-u(s,\tau,x)| : s,t,\tau\in[0,T], |s-t|\le\epsilon, |x|\le\nu\}\to 0 \text{ as } \epsilon\to 0.$$

Hence, from the above estimate we obtain

$$\omega_0(FX)\le c\omega_0(X). \tag{3.10}$$

Now, from Eq.(3.9) and Eq.(3.10), we get

$$\mu(FX)\le c\mu(X).$$

The conclusion of the theorem follows by the application of Theorem 2.2. □

4. Examples

Consider the following functional-integral equations:

$$\begin{aligned}
x(t) &= \frac{1}{(1+t^2)}x(t)+\cos\Big(tx(t)\Big)\int_0^t \frac{\ln(1+s|x(s)|)}{(1+t^4)\Big(1+x^2(s)\Big)}ds,\\
x(t) &= \frac{\ln(1+t)}{(1+t)}\sin x(t)+\operatorname{arctg}\Big(t^2x(t)\Big)\int_0^t \frac{s\exp(-t-x^2(s))}{(1+s^2)}ds,\\
x(t) &= \exp(-t)x(t)+x(t)\int_0^t \frac{t}{4+t^2}\frac{\exp(-ts)}{\Big(1+x^2(s)\Big)}ds,\\
x(t) &= (\sin t)x(t)+x(t)\int_0^t \frac{s|x(s)|}{3+t^5}\exp\Big(-t-sx^2(s)\Big)ds.
\end{aligned}$$

The first two examples satisfy the assumptions of Theorem 3.2 and the next two examples satisfy the assumptions of Theorem 3.4.

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