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PERIODIC BOUNDARY VALUE PROBLEMS OF FIRST ORDER ORDINARY CARATHEODORY AND DISCONTINUOUS DIFFERNTIAL EQUATIONS

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Abstract. In this paper, an existence theorem for the periodic boundary value problems of first order Caratheéodory and discontinuous differential equations is proved in Banach algebras under the mixed generalized Lipschitz and Carathéodory conditions. The existence theorems for extremal solutions are also proved under certain monotonicity conditions.

1. Introduction

Let $\mathbb{R}$ denote the real line. Given a closed and bounded interval $J = [0, T]$ in $\mathbb{R}$, consider the periodic boundary value problems (in short PBVP) of first order ordinary differential equations

$$\begin{cases} \dfrac{d}{dt}\Big[\dfrac{x(t) - k(t, x(t))}{f(t, x(t))}\Big] = g(t, x(t)) \quad \text{a. e. } t \in J \\ \qquad\qquad x(0) = x(T), \end{cases} \tag{1.1}$$

where $f : J \times \mathbb{R} \to \mathbb{R} - \{0\}$ and $g, k : J \times \mathbb{R} \to \mathbb{R}$.

By a *solution* of PBVP (1.1) we mean a function $x \in AC(J, \mathbb{R})$ that satisfies

(i) the function $t \mapsto \Big(\dfrac{x(t) - k(t, x(t))}{f(t, x(t))}\Big)$ is absolutely continuous on J, and

(ii) x satisfies the equations in (1.1),

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where $AC(J, \mathbb{R})$ is the space of absolutely continuous real-valued functions on J.

First order ordinary differential equations (ODE) with periodic boundary value conditions are considered in many works. See Bernfeld and Lakshmikantham [1], Ladde *et al.* [18], Omari and Zanolin [21] and the references therein. The study of periodic boundary value problems of nonlinear first order differential equations with discontinuous nonlinearity has been exploited in the works of Heikkilä and Lakshmikantham [17]. But the study of periodic boundary value problems of ordinary differential equations in Banach algebras involving Carathéodory as well as discontinuous nonlinearity has not been made so far in the literature. The study of initial value problems of nonlinear differential equations in Banach algebras is initiated in the recent works of Dhage [4] and Dhage and O'Regan [11] and discussed the existence theory for first order differential equations. The study of such equations has been further exploited in the works of Dhage [2,3] and Dhage *et al.* [12] for various aspects of the solutions. In this paper, we deal with the periodic boundary value problems of nonlinear first order Carathéodory and discontinuous differential equations in Banach algebras and discuss the existence as well as existence results for extremal solutions under the mixed Lipschitz, Carathéodory and monotonic conditions. The main tools used in the study are the hybrid fixed point theorems to be developed in this paper itself. We claim that the nonlinear differential equation as well as the existence results of this paper are new to the literature on the theory of nonlinear ordinary differential equations.

Our method of study is to convert the PBVP (1.1) into an equivalent integral equation and apply the hybrid fixed point theorems of next section 2 under suitable conditions on the nonlinearities f, k and g involved it. In the following section 2 we give some preliminaries and the auxiliary results needed in the sequel.

2. Hybrid fixed point theory

Let X be a Banach algebra with the norm $\|\cdot\|$. A mapping $A : X \to X$ is called $\mathcal{D}$-Lipschitz if there exists a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying

$$\|Ax - Ay\| \leq \psi(\|x - y\|) \tag{2.1}$$

for all $x, y \in X$ with $\psi(0) = 0$. In the special case, when $\psi(r) = \alpha r$ $(\alpha > 0)$, A is called a Lipschitz with a Lipschitz constant α. In particular, if $\alpha < 1$, A is called a contraction with the contraction constant α. Further, if $\psi(r) < r$ for all $r > 0$, then A is called a nonlinear $\mathcal{D}$-contraction on X. For convenience, we call the function ψ to be a $\mathcal{D}$-function of A on X.

An operator $T : X \to X$ is called compact if $\overline{T(S)}$ is a compact subset of X for any $S \subset X$. Similarly, $T : X \to X$ is called totally bounded if T maps a bounded subset of X into a relatively compact subset of X. Finally, $T : X \to X$ is called completely continuous operator, if it is continuous and totally bounded operator on X. It is clear that every compact operator is totally bounded, but the converse may not be true. However, these two notions are equivalent on the bounded subsets of X.

2.1. **Fixed point theory in Banach spaces.** The following nonlinear alternative is fundamental and has been used extensively in the theory of differential and integral equations for proving the existence results under certain compactness conditions.

Theorem 2.1. [14] *Let K be a convex subset of a normed linear space E, U an open subset of K with $0 \in U$, and $N : \overline{U} \to K$ a continuous and compact map. Then either*

(a) *N has a fixed point in $\overline{U}$; or,*
(b) *there is an element $u \in \partial U$ such that $u = \lambda N u$ for some real number $\lambda \in (0, 1)$, where ∂U is the boundary of U.*

Before presenting the main results of this section, we give some preliminaries needed in the sequel.

The Kuratowskii measure of noncompactness α of a bounded set S in X is a nonnegative real number $\alpha(S)$ defined by

$$\alpha(S) = \inf\Big\{r > 0 : S = \bigcup_{i=1}^{n} S_i, \text{ and } \operatorname{diam}(S_i) \le r, \ \forall i\Big\}. \tag{2.2}$$

The function α enjoys the following properties:

(α_1) $\alpha(S) = 0 \Longleftrightarrow S$ is precompact.
(α_2) $\alpha(S) = \alpha(\overline{S}) = \alpha(\overline{co}\, S)$, where $\overline{S}$ and $\overline{co}\, S$ denote respectively the closure and the closed convex hull of S.
(α_3) $S_1 \subset S_2 \Rightarrow \alpha(S_1) \le \alpha(S_2)$
(α_4) $\alpha(S_1 \cup S_2) = \max\{\alpha(S_1), \alpha(S_2)\}$.
(α_5) $\alpha(\lambda S) = |\lambda| \alpha(S), \forall \lambda \in \mathbb{R}$.
(α_6) $\alpha(S_1 + S_2) \le \alpha(S_1) + \alpha(S_2)$.

The details of measures of noncompactness and their properties appear in Deimling [13] and Zeidler [22].

Definition 2.1. A mapping $T : X \to X$ is called α-condensing, if for any bounded subset S of X, $T(S)$ is bounded and $\alpha(T(S)) < \alpha(S)$, $\alpha(S) > 0$.

Note that contraction and completely continuous mappings are α-condensing, but the converse may not be true. The following generalization of Theorem

2.1 for α-condensing mappings in Banach spaces is well-known and will be used in the sequel.

Theorem 2.2. *Let U and $\overline{U}$ be respectively open and closed subsets of a Banach space X such that $0 \in U$. If $N(\overline{U})$ is bounded and $N : \overline{U} \to X$ a continuous and α-condensing map, then either*

(a) *N has a fixed point in $\overline{U}$; or,*
(b) *there is an element u of the boundary ∂U such that $u = \lambda N u$ for some real number $\lambda \in (0,1)$.*

Our main result of this section is

Theorem 2.3. *Let U and $\overline{U}$ be open-bounded and closed-bounded subsets of a Banach algebra X such that $0 \in U$ and let $A, B, C : \overline{U} \to X$ be three operators satisfying*

(a) *A and C are $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-functions ψ_A and ψ_C respectively,*
(b) *B is continuous and compact, and*
(c) *$M\psi_A(r) + \psi_C(r) < r$ for $r > 0$, where*

$$M = \|B(\overline{U})\| = sup\left\{\|B(x)\| : x \in \overline{U}\right\}.$$

Then either

(i) *the equation $AxBx + Cx = x$ has a solution in $\overline{U}$, or*
(ii) *there is an element $u \in \partial U$ such that $u = \lambda[AuBu + Cu]$ for some $\lambda \in (0,1)$, where ∂U is the boundary of U.*

Proof. Define a mapping $T : \overline{U} \to X$ by

$$Tx = AxBx + Cx, \; x \in \overline{U}. \tag{2.3}$$

First, we show that T is a continuous mapping on $\overline{U}$. Let $\{x_n\}$ be a sequence in $\overline{U}$ converging to a point $x_* \in \overline{U}$. Then, we have

$$\begin{aligned}
\|T(x_n) - T(x_*)\| &\le \|Ax_n\, Bx_n - Ax_*\, Bx_*\| + \|Cx_n - Cx^*\| \\
&\le \|Ax_n\, Bx_n - Ax_*\, Bx_n\| + \|Ax_*\, Bx_n - Ax_*\, Bx_*\| \\
&\quad + \|Cx_n - Cx^*\| \\
&\le \|Ax_n - Ax_*\|\|Bx_n\| + \|Ax_*\|\|Bx_n - Bx_*\| \\
&\quad + \|Cx_n - Cx^*\| \\
&\le M\psi_A(\|x_n - x_*\|) + \|Ax_*\|\|Bx_n - Bx_*\| \\
&\quad + \psi_C(\|x_n - x_*\|).
\end{aligned}$$

Taking the limit superior on both sides,

$$\begin{aligned}
&\limsup_{n\to\infty} \|T(x_n) - T(x_*)\| \\
&\le M \limsup_{n\to\infty} \psi_A(\|x_n - x_*\|) + \|Ax_*\| \limsup_{n\to\infty} \|Bx_n - Bx_*\| \\
&\quad + \limsup_{n\to\infty} \psi_C(\|x_n - x_*\|) \\
&\le M\psi_A \left(\limsup_{n\to\infty} \|x_n - x_*\| \right) + \|Ax_*\| \limsup_{n\to\infty} \|Bx_n - Bx_*\| \\
&\quad + \psi_C \left(\limsup_{n\to\infty} \|x_n - x_*\| \right) \\
&= 0.
\end{aligned}$$

Therefore, it follows that

$$\lim_{n\to\infty} \|T(x_n) - T(x_*)\| = 0,$$

and so, T is a continuous mapping on $\overline{U}$. Now the result follows immediately from Theorem 2.2, if we show that the operator T is α-condensing on $\overline{U}$. Let S be a set in $\overline{U}$. Then we have the following estimate concerning the operators A, B and C on $\overline{U}$. Let x^* be a fixed element of S. Then by hypothesis (a),

$$\|Ax\| \le \|Ax^*\| + \|Ax^* - Ax\| \le \|Ax^*\| + \psi_A(\|x^* - x\|) \le \beta$$

for all $x \in S$, where

$$\beta = \|Ax^*\| + \psi_A(\mathrm{diam}(S)) < \infty, \tag{2.4}$$

because S is bounded.

Now there are two cases :

Case I : If $\beta = 0$, then $\|Ax\| = 0$ and consequently, $Tx = Cx$. In this case, we show that T is a α-condensing mapping on $\overline{U}$. Let $\epsilon > 0$ be given and suppose that

$$S = \bigcup_{i=1}^{n} S_i$$

with

$$\mathrm{diam}(S_i) \le \alpha(S) + \epsilon$$

for all $i = 1, 2, ..., n$.

Now

$$C(S) \subseteq \bigcup_{i=1}^{n} C(S_i) = \bigcup_{i=1}^{n} Y_i.$$

If $w_0, w_1 \in Y_i$, for some i, then there exist $x_0, x_1 \in S_i$ such that $Cx_0 = w_0$ and $Cx_1 = w_1$. Since ϕ is nondecreasing, one has

$$\|Cx_0 - Cx_1\| \leq \psi_C(\|x_0 - x_1\|) \leq \psi_C(\mathrm{diam}(S_i)) < \phi(\alpha(S) + \epsilon).$$

This is true for every $w_0, w_1 \in Y_i$ and so

$$\mathrm{diam}(Y_i) < \psi_C(\alpha(S) + \epsilon),$$

for all $i = 1, 2, ..., n$. Thus we have

$$\alpha(C(S)) = \max_i \mathrm{diam}(Y_i) < \phi(\alpha(S) + \epsilon).$$

Since ϵ is arbitrary, we have

$$\alpha(C(S)) \leq \phi(\alpha(S)).$$

Now from (3.3) it follows that

$$\alpha(T(S)) = \alpha(C(S)) \leq \phi(\alpha(S)) < \alpha(S)$$

whenever $\alpha(S) > 0$. This shows that T is a α-condensing on $\overline{U}$.

Case II : Now suppose that $\beta \neq 0$. In this case also we show that T is a α-condensing on $\overline{U}$. Since B is compact, $B(S)$ is a precompact subset of X. Hence for $\eta > 0$, there exist subsets $G_1, G_2, \ldots, G_m$ of X such that

$$B(S) = \bigcup_{j=1}^{m} G_j \text{ and } \mathrm{diam}(G_j) < \frac{\eta}{\beta}.$$

This further gives that

$$S \subset \bigcup_{j=1}^{m} B^{-1}(G_j).$$

Let $\epsilon > 0$ be given and suppose that

$$S = \bigcup_{i=1}^{n} S_i$$

with

$$\mathrm{diam}(S_i) < \alpha(S) + \epsilon$$

for all $i = 1, 2, ..., n$. We put $F_{ij} = S_i \bigcap B^{-1}(G_j)$, then $S \subset \bigcup F_{ij}$.

Now

$$T(S) \subseteq \bigcup_{i,j} T(F_{ij}) = \bigcup_{i,j} T\left(S_i \bigcap B^{-1}(G_j)\right) = \bigcup_{i,j} Y_{ij}.$$

If $w_0, w_1 \in Y_{ij}$, for some $i = 1, \ldots, n$ and $j = 1, \ldots, m$, then there exist $x_0, x_1 \in F_{ij} = S_i \bigcap B^{-1}(G_j)$ such that $Tx_0 = w_0$ and $Tx_1 = w_1$.

Since ψ_A and ψ_A are nondecreasing, one has

$$\begin{aligned}\|Tx_0 - Tx_1\| &\leq \|Ax_0Bx_0 - Ax_1Bx_1\| + \|Cx_0 - Cx_1\| \\ &\leq \|Ax_0Bx_0 - Ax_1Bx_0\| + \|Ax_1Bx_0 - Ax_1Bx_1\| \\ &\quad + \|Cx_0 - Cx_1\| \\ &\leq \|Ax_0 - Ax_1\|\|Bx_0\| + \|Ax_1\|\|Bx_0 - Bx_1\| \\ &\quad + \|Cx_0 - Cx_1\| \\ &\leq \psi_A(\|x_0 - x_1\|)\|Bx_0\| + \|Ax_1\|\|Bx_0 - Bx_1\| \\ &\quad + \psi_C(\|x_0 - x_1\|) \\ &\leq \psi_A(\mathrm{diam}(F_{ij}))\|B(\overline{U})\| + \|A(S)\|\|Bx_0 - Bx_1\| \\ &\quad + \psi_C(\mathrm{diam}(F_{ij})) \\ &< M\psi_A(\mathrm{diam}(F_{ij})) + \psi_C(\mathrm{diam}(F_{ij})) + \eta.\end{aligned}$$

This further implies that

$$\begin{aligned}\|Tx_0 - Tx_1\| &\leq M\psi_A(\mathrm{diam}(S_i)) + \psi_C(\mathrm{diam}(S_i)) + \eta \\ &< M\,\psi_A(\alpha(S) + \epsilon) + \psi_C(\alpha(S) + \epsilon) + \eta.\end{aligned}$$

This is true for every $w_0, w_1 \in Y_{ij}$, and so

$$\mathrm{diam}(Y_{ij}) \leq M\psi_A(\alpha(S) + \epsilon) + \psi_C(\alpha(S) + \epsilon) + \eta,$$

for all $i = 1, 2, ..., n$. Thus we have

$$\alpha(T(S)) \leq \max_{i,j} \mathrm{diam}(Y_{ij}) \leq M\psi_A(\alpha(S) + \epsilon) + \psi_C(\alpha(S) + \epsilon) + \eta.$$

Since ϵ is arbitrary, one has

$$\alpha(T(S)) \leq M\psi_A(\alpha(S) + \epsilon) + \psi_C(\alpha(S) + \epsilon).$$

Since ϵ is arbitrary, we have

$$\alpha(T(S)) \leq M\psi_A(\alpha(S)) + \psi_C(\alpha(S)) < \alpha(S),$$

whenever $\alpha(S) > 0$.

This shows that T is a α-condensing mapping on $\overline{U}$. Now the desired conclusion follows by an application of Theorem 2.2. This completes the proof. □

As a consequence of Theorem 2.3 we obtain the following corollary in its applicable form to nonlinear differential and integral equations.

Corollary 2.1. *Let $\mathcal{B}_r(0)$ and $\overline{\mathcal{B}_r(0)}$ be open and closed balls in a Banach algebra X centered at origin 0 of radius r, for some real number $r > 0$ and let $A, B, C : \overline{\mathcal{B}_r(0)} \to X$ be three operators satisfying*

(a) *A and C are Lipschitz with the Lipschitz constants α and β respectively,*
(b) *B is continuous and compact, and*
(c) $\alpha M + \beta < 1$, *where* $M = sup\Big\{\|B(x)\| : x \in \overline{\mathcal{B}_r(0)}\Big\}$.

Then either

(i) *the equation* $Ax\,Bx + Cx = x$ *has a solution in* $\overline{\mathcal{B}_r(0)}$, *or*
(ii) *there is an element* $u \in X$ *such that* $\|u\| = r$ *satisfying* $\lambda[Au\,Bu + Cu] = u$ *for some* $0 < \lambda < 1$.

Remark 2.1. Theorem 2.3 is an improvement of the nonlinear alternatives of Leray-Schauder type due to Dhage [2,6] and Dhage and O'Regan [11] under weaker conditions.

2.2. **Fixed point theory in ordered spaces.** A non-empty closed set K in a Banach algebra X is called a cone if (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}, \lambda \geq 0$ and (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of X. A cone K is called positive if (iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplication composition in X. We introduce an order relation $\leq$ in X as follows. Let $x, y \in X$. Then $x \leq y$ if and only if $y - x \in K$. A cone K is called normal if the norm $\|\cdot\|$ is semi-monotone increasing on K, that is, there is a constant $N > 0$ such that $\|x\| \leq N\|y\|$ for all $x, y \in K$ with $x \leq y$. It is known that if the cone K is normal in X, then every order-bounded set in X is norm-bounded. The details of cones and their properties appear in Guo and Lakshmikantham [16].

Lemma 2.1. (Dhage [3]) *Let K be a positive cone in a real Banach algebra X and let $u_1, u_2, v_1, v_2 \in K$ be such that $u_1 \leq v_1$ and $u_2 \leq v_2$. Then $u_1u_2 \leq v_1v_2$.*

For any $a, b \in X, a \leq b$, the order interval $[a, b]$ is a set in X given by

$$[a, b] = \{x \in X : a \leq x \leq b\}.$$

Definition 2.2. A mapping $T : [a, b] \to X$ is said to be nondecreasing or monotone increasing if $x \leq y$ implies $Tx \leq Ty$ for all $x, y \in [a, b]$.

We use the following three fixed point theorems of Dhage [2,6] for proving the existence of extremal solutions for the PBVP (1.1) under certain monotonicity conditions.

Theorem 2.4. (Dhage [2]) *Let K be a cone in the Banach algebra X and let $a, b \in X$ be such that $a \leq b$. Suppose that $A, B : [a, b] \to K$ and $C : X \to X$ are three operators such that*

(a) *A and C are Lipschitz with the Lipschitz constants α and β respectively,*
(b) *B is completely continuous,*

(c) *the elements* $a, b \in X$ *satisfy* $a \leq AaBa + Ca$ *and* $AbBb + Cb \leq b$, *and*
(d) A, B *and* C *are nondecreasing.*

Further if the cone K *is positive and normal, then the operator equation* $Ax\,Bx + Cx = x$ *has the least and the greatest solution in* $[a, b]$, *whenever* $\alpha M + \beta < 1$, *where* $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$.

Theorem 2.5. (Dhage [7]) *Let* K *be a cone in the Banach algebra* X *and let* $a, b \in X$ *be such that* $a \leq b$. *Suppose that* $A, B : [a, b] \to K$ *and* $C : X \to X$ *are three operators such that*

(a) A *is completely continuous,*
(b) B *and* C *are totally bounded,*
(c) *the elements* $a, b \in X$ *satisfy* $a \leq AaBa + Ca$ *and* $AbBb + Cb \leq b$, *and*
(d) A, B *and* C *are nondecreasing.*

Further, if the cone K *is positive and normal, then the operator equation* $Ax\,Bx + Cx = x$ *has the least and the greatest solution in* $[a, b]$.

Theorem 2.6. (Dhage [7]) *Let* K *be a cone in a Banach algebra* X *and let* $a, b \in X$ *be such that* $a \leq b$. *Suppose that* $A, B : [a, b] \to K$ *and* $C : X \to X$ *are three operators such that*

(a) A *and* C *are Lipschitz mappings with the Lipschitz constants* α *and* β *respectively,*
(b) B *is totally bounded,*
(c) *the elements* $a, b \in X$ *satisfy* $a \leq AaBa + Ca$ *and* $AbBb + Cb \leq b$, *and*
(d) A, B *and* C *are nondecreasing.*

Further, if the cone K *is positive and normal, then the operator equation* $Ax\,Bx + Cx = x$ *has the least and the greatest solution in* $[a, b]$, *whenever* $\alpha M + \beta < 1$, *where* $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$.

Next we prove an improvement of the following two fixed point theorems due to the present author for the mappings in ordered Banach algebras.

Theorem 2.7. (Dhage [7]) *Let* $[a, b]$ *be an order interval in an ordered Banach algebra* X *with the cone* K *and let* $A, B : [a, b] \to K$ *and* $C : [a, b] \to X$ *be three nondecreasing operators such that*

(a) A *is a Lipschitz mapping with the Lipschitz constant* $\alpha < 1/2$,
(b) B *is completely continuous,*
(c) C *is totally bounded, and*
(d) $Ax\,By + Cz \in [a, b]$ *for all* $x, y, z \in [a, b]$.

Further, if the cone K *is positive and normal, then the operator equation* $Ax\,Bx + Cx = x$ *has the least and the greatest solution in* $[a, b]$, *whenever* $\alpha M < 1$, *where* $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$.

Theorem 2.8. (Dhage [7]) *Let $[a,b]$ be an order interval in an ordered Banach algebra X with the cone K and let $A,B : [a,b] \to K$ and $C : [a,b] \to X$ be three nondecreasing operators such that*

(a) *A is completely continuous,*
(b) *B is totally bounded,*
(c) *C is a contraction with the contraction constant $\beta < 1/2$, and*
(d) *$Ax\,By + Cz \in [a,b]$ for all $x,y,z \in [a,b]$.*

Further, if the cone K is positive and normal, then the operator equation $Ax\,Bx + Cx = x$ has the least and the greatest solution in $[a,b]$.

We use the following two fixed point theorems in the sequel.

Theorem 2.9. *Let $[a,b]$ be a norm-bounded order interval in the ordered Banach space X and let $T : [a,b] \to [a,b]$ be a continuous and α-condensing mapping. If T is nondecreasing, then T has the least fixed point x_* and the greatest fixed point x^* in $[a,b]$ and the sequences $\{T^n(a)\}$ and $\{T^n(b)\}$ converge to x_* and x^* respectively.*

Proof. The proof is obtained using essentially the same arguments that given in Dhage [7] with appropriate modifications. Hence we omit the details. □

Theorem 2.10. (Heikkilä and Lakshmikantham [17]) *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a,b] \to [a,b]$ be a nondecreasing mapping. If each sequence $\{Qx_n\} \subseteq Q([a,b])$ converges, whenever $\{x_n\}$ is a monotone sequence in $[a,b]$, then the sequence of Q-iteration of a converges to the least fixed point x_* of Q and the sequence of Q-iteration of b converges to the greatest fixed point x^* of Q. Moreover,*

$$x_* = \min\{y \in [a,b] \mid y \geq Qy\} \quad \text{and} \quad x^* = \max\{y \in [a,b] \mid y \leq Qy\}.$$

Theorem 2.11. *Let $[a,b]$ be an order interval in an ordered real Banach algebra X with the cone K and let $A,B : [a,b] \to K$ and $C : [a,b] \to X$ be three nondecreasing operators such that*

(a) *A is $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ,*
(b) *B is completely continuous,*
(c) *every sequence $\{Cy_n\} \subseteq C([a,b])$ converges, whenever $\{y_n\}$ is a monotone sequence in $[a,b]$, and*
(d) *$Ax\,Bx + Cz \in [a,b]$ for all $x,z \in [a,b]$.*

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx + Cx = x$ has the least and the greatest solution in $[a,b]$, whenever $M\psi(r) < r$, if $r > 0$, where $M = \|B([a,b])\| := \sup\{\|Bx\| : x \in [a,b]\}$.

Proof. Let $y \in [a,b]$ be fixed and define a mapping $T_y : [a,b] \to X$ by

$$T_y(x) = Ax\,Bx + Cy.$$

First we show that T_y is a continuous mapping on $[a,b]$. Let $\{x_n\}$ be a sequence in $[a,b]$ converging to a point $x_* \in [a,b]$. Then, we have

$$\begin{aligned}\|T_y(x_n) - T_y(x_*)\| &= \|Ax_n\,Bx_n - Ax_*\,Bx_*\| \\ &\leq \|Ax_n\,Bx_n - Ax_*\,Bx_n\| + \|Ax_*\,Bx_n - Ax_*\,Bx_*\| \\ &\leq \|Ax_n - Ax_*\|\|Bx_n\| + \|Ax_*\|\|Bx_n - Bx_*\| \\ &\leq M\psi(\|x_n - x_*\|) + \beta\|Bx_n - Bx_*\|\end{aligned}$$

where $\beta = \|Ax_*\| < \infty$.

Passing to the limit superior as $n \to \infty$ in the above inequality yields

$$\lim_{n\to\infty} \|T_y(x_n) - T_y(x_*)\| = 0.$$

This shows that the mapping T_y is continuous on $[a,b]$. To show T_y is nondecreasing on $[a,b]$, let $x_1, x_2 \in [a,b]$ be such that $x_1 \leq x_2$. By the positivity of the cone K in X, we obtain

$$T_y(x_1) = Ax_1\,Bx_1 + Cy \leq Ax_2\,Bx_2 + Cy = T_y(x_2).$$

Therefore, hypothesis (d) implies that T_y defines a nondecreasing mapping $T_y : [a,b] \to [a,b]$. Now proceeding with the arguments similar to the proof of Theorem 2.3, it can be shown that T_y is a α-condensing mapping on $[a,b]$. Hence by Theorem 2.2, the operator T_y has the least fixed point x_* and the greatest fixed point x^* and the sequences $\{T_y^n(a)\}$ and $\{T_y^n(b)\}$ converge to x_* and x^* respectively.

Define a mapping $Q : [a,b] \to X$ by $Qy = z$, where z is a greatest solution to the operator equation $AzBz + Cy = z$ and which is obviously unique for each $y \in [a,b]$. We show that Q is a nondecreasing mapping on $[a,b]$. Let $y_1, y_2 \in [a,b]$ be such that $y_1 \leq y_2$. Then there are unique elements $z_1, z_2 \in [a,b]$ such that

$$Qy_1 = z_1 = Az_1\,Bz_1 + Cy_1 = T_{y_1}(z_1)$$

and

$$Qy_2 = z_2 = Az_2\,Bz_2 + Cy_2 = T_{y_2}(z_2).$$

From the monotonicity of C, it follows that

$$T_{y_1}(x) = Ax\,Bx + Cy_1 \leq Ax\,Bx + Cy_2 = T_{y_2}(x)$$

for all $x \in [a,b]$. Hence for any $x \in [a,b]$

$$T_{y_1}^n(x) \leq T_{y_2}^n(x)$$

for all $n \in \mathbb{N}$. In particular,

$$T_{y_1}^n(b) \leq T_{y_2}^n(b)$$

for all $n \in \mathbb{N}$. By Theorem 2.9,

$$z_1 = \lim_{n\to\infty} T_{y_1}^n(b) \leq \lim_{n\to\infty} T_{y_2}^n(b) = z_2.$$

This shows that Q defines a nondecreasing operator $Q : [a, b] \to [a, b]$. See also Dhage [7] and the references therein.

Next, let $\{y_n\}$be a monotone sequence in $[a, b]$. We will show that the sequence $\{Qy_n\}$ converges. By definition of Q, there is a monotone increasing sequence $\{z_n\}$ in $[a, b]$ such that $Q(y_n) = z_n = T_{y_n}(z_n),\ n \in \mathbb{N}$. Let $S = \{z_n\}$. Then S is a bounded and countable subset of $[a, b]$ such that $S \subseteq \bigcup_{n\in\mathbb{N}} T_{y_n}(S)$. Since the map $x \mapsto T_y(x)$ is countably condensing for each $y \in [a, b]$, one has

$$\alpha(S) \leq \alpha\big(\cup_{n\in\mathbb{N}} T_{y_n}(S)\big) = \max\{\alpha(T_{y_n}(S)) : n \in \mathbb{N}\} < \alpha(S)$$

for each $n \in \mathbb{N}$. If $\alpha(S) \neq 0$, then we get a contradiction. As a result $\alpha(S) = 0$ and that $\overline{S}$ is compact. Hence the sequence $\{z_n\}$ converges to a point, say z in $[a, b]$. Now, by hypothesis (c), the sequence $\{T_{y_n}(z)\}$ converges, say to the point $T_y(z)$ for some $y \in [a, b]$. Then, we have

$$\|T_{y_n}(z_n) - T_y(z)\| \leq \|T_{y_n}(z_n) - T_{y_n}(z)\| + \|T_{y_n}(z) - T_y(z)\|.$$

Passing the limit to $n \to \infty$ in the above inequality,

$$\lim_{n\to\infty} \|T_{y_n}(z_n) - T_y(z)\| = 0.$$

As a result, the sequence $\{Qy_n\} \subseteq Q([a, b])$ converges, whenever $\{y_n\}$ is a monotone sequence in $[a, b]$.

Hence, by Theorem 2.10, the operator Q has the least and the greatest fixed point in $[a, b]$. Now the greatest fixed point of Q is the greatest solution to the operator equation $Ax\,Bx + Cx = x$ in $[a, b]$. Similarly, it is proved that the operator equation $Ax\,Bx + Cx = x$ has the least solution in $[a, b]$. This completes the proof. □

Corollary 2.2. *Let $[a, b]$ be an order interval in an ordered real Banach algebra X with the cone K and let $A, B : [a, b] \to K$ and $C : [a, b] \to X$ be three nondecreasing operators such that*

(a) *A is a Lipschitz mapping with the Lipschitz constant α,*
(b) *B is completely continuous,*
(c) *C is totally bounded, and*
(d) *the elements $a, b \in X$ satisfy $a \leq Aa\,Ba + Ca$ and $Ab\,Bb + Cb \leq b$.*

Further if the cone K is positive and normal, then the operator equation $Ax\,Bx + Cx = x$ has the least and the greatest solution in $[a, b]$, whenever $\alpha M < 1$, where $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$.

Theorem 2.12. *Let $[a, b]$ be an order interval in an ordered real Banach algebra X with the cone K and let $A, B : [a, b] \to K$ and $C : [a, b] \to X$ be three nondecreasing operators such that*

(a) *A is completely continuous mapping,*
(b) *every sequence $\{By_n\} \subset B([a, b])$ converges, whenever $\{y_n\}$ is a monotone sequence in $[a, b]$,*
(c) *C is a nonlinear $\mathcal{D}$-contraction with the $\mathcal{D}$-function ψ_C, and*
(d) *$Ax\,By + Cx \in [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K is positive and normal, then the operator equation $Ax\,Bx + Cx = x$ has the least and the greatest solution in $[a, b]$.

Proof. Let $y \in [a, b]$ be fixed and define a mapping $T_y : [a, b] \to X$ by

$$T_y(x) = Ax\,By + Cx.$$

It is easy to see that the mapping T_y is continuous on $[a, b]$. To show T_y is nondecreasing on $[a, b]$, let $x_1, x_2 \in [a, b]$ be such that $x_1 \le x_2$. By the positivity of the cone K in X, we obtain

$$T_y(x_1) = Ax_1\,By + Cx_1 \le Ax_2\,By + Cx_2 = T_y(x_2).$$

Therefore, hypothesis (d) implies that T_y defines a nondecreasing mapping $T_y : [a, b] \to [a, b]$.

Next, we show that the operator T_y is α-condensing on $[a, b]$. Let S be a set in $[a, b]$. Then we have the following estimate concerning he operators A, B and C on $[a, b]$. Now there are two cases :

Case I : If $\|By\| = 0$, then $T_y(x) = Cx$. Now proceeding with the arguments as in the case I of proof of Theorem 2.3 it can be shown that the mapping T_y is α-condensing on $[a, b]$.

Case II : If $\|By\| > 0$, then in this case also, we will show that T_y is α-condensing mapping on $[a, b]$. Since A is compact, $A(S)$ is a relatively compact subset of X. Hence for $\eta > 0$, there exist subsets $G_1, G_2, \ldots, G_m$ of X such that

$$A(S) = \bigcup_{j=1}^{m} G_j \quad \text{and} \quad \operatorname{diam}(G_j) < \frac{\eta}{\|By\|}.$$

This further gives that

$$S \subset \bigcup_{j=1}^{m} A^{-1}(G_j).$$

Let $\epsilon > 0$ be given and suppose that

$$S = \bigcup_{i=1}^{n} S_i$$

with

$$\mathrm{diam}(S_i) \le \alpha(S) + \epsilon$$

for all $i = 1, 2, ..., n$. We put $F_{ij} = S_i \bigcap A^{-1}(G_j)$, then $S \subset \bigcup F_{ij}$.

Now

$$T_y(S) \subseteq \bigcup_{i,j} T_y(F_{ij}) = \bigcup_{i,j} T_y \left(S_i \bigcap A^{-1}(G_j) \right) = \bigcup_{i,j} Y_{ij}.$$

If $w_0, w_1 \in Y_{ij}$, for some $i = 1, \dots, n$ and $j = 1, \dots, m$, then there exist $x_0, x_1 \in F_{ij} = S_i \bigcap A^{-1}(G_j)$ such that $T_y x_0 = w_0$ and $T_y x_1 = w_1$.

Since ψ_C is nondecreasing, one has

$$\begin{aligned} \|T_y(x_0) - T_y(x_1)\| &= \|Ax_0By - Ax_1By + Cx_0 - Cx_1\| \\ &\le \|Ax_0By - Ax_1By\| + \|Cx_0 - Cx_1\| \\ &\le \|Ax_0 - Ax_1\| \|By\| + \|Cx_0 - Cx_1\| \\ &\le \mathrm{diam}(G_j)\, \|By\| + \psi_C(\|x_0 - x_1\|) \\ &< \psi_C(\mathrm{diam}(F_{ij})) + \eta. \end{aligned}$$

This further implies that

$$\|T_y(x_0) - T_y(x_1)\| \le \psi_C(\mathrm{diam}(S_i)) + \eta < \psi_C(\alpha(S) + \epsilon) + \eta.$$

This is true for every $w_0, w_1 \in Y_{ij}$, and so

$$\mathrm{diam}(Y_{ij}) < \psi_C(\alpha(S) + \epsilon) + \eta,$$

for all $i = 1, 2, ..., n$. Thus we have

$$\alpha(T_y(S)) = \max_{i,j} \mathrm{diam}(Y_{ij}) \le \psi_C(\alpha(S) + \epsilon) + \eta.$$

Since ϵ is arbitrary, we have

$$\alpha(T_y(S)) \le \psi_C(\alpha(S)) < \alpha(S),$$

whenever $\alpha(S) > 0$.

This shows that T_y is a α-condensing mapping on $[a, b]$. Hence by Theorem 2.9, the operator T_y has the least fixed point x_* and the greatest fixed point x^* and the sequences $\{T_y^n(a)\}$ and $\{T_y^n(b)\}$ converge to x_* and x^* respectively.

Define a mapping $Q : [a, b] \to X$ by $Qy = z$, where z is the greatest solution to the operator equation $AzBy + Cz = z$ and which is obviously unique for each $y \in [a, b]$. We show that Q is a nondecreasing mapping on $[a, b]$. Let $y_1, y_2 \in [a, b]$ be such that $y_1 \le y_2$. Then there are unique elements $z_1, z_2 \in [a, b]$ such that

$$Qy_1 = z_1 = Az_1\, By_1 + Cz_1 = T_{y_1}(z_1)$$

and

$$Qy_2 = z_2 = Az_2\, By_2 + Cz_2 = T_{y_2}(z_2).$$

From the monotonicity of A and C, it follows that

$$T_{y_1}(x) = Ax\,By_1 + Cx \leq Ax\,By_2 + Cx = T_{y_2}(x)$$

for all $x \in [a, b]$. Hence for any $x \in [a, b]$

$$T_{y_1}^n(x) \leq T_{y_2}^n(x)$$

for all $n \in \mathbb{N}$. In particular,

$$T_{y_1}^n(b) \leq T_{y_2}^n(b)$$

for all $n \in \mathbb{N}$. By Theorem 2.9,

$$z_1 = \lim_{n\to\infty} T_{y_1}^n(b) \leq \lim_{n\to\infty} T_{y_2}^n(b) = z_2.$$

This shows that Q defines a nondecreasing operator $Q : [a, b] \to [a, b]$. See also Dhage [7] and the references therein.

Next, let $\{y_n\}$be a monotone sequence in $[a, b]$. We will show that the sequence $\{Qy_n\}$ converges. By definition of Q, there is a monotone increasing sequence $\{z_n\}$ in $[a, b]$ such that $Q(y_n) = z_n = T_{y_n}(z_n),\ n \in \mathbb{N}$. Again, in view of hypothesis (b), it can be shown as in the proof of Theorem 2.11 that every sequence $\{Q(y_n)\}$ converges in X, whenever $\{y_n\}$ is a monotone sequence in $[a, b]$. Hence by Theorem 2.10, the operator Q has the least and the greatest fixed point in $[a, b]$. Now the greatest fixed point of Q is the greatest solution to the operator equation $Ax\,Bx + Cx = x$ in $[a, b]$. Similarly, it is proved that the operator equation $Ax\,Bx + Cx = x$ has the least solution in $[a, b]$. This completes the proof. □

Corollary 2.3. *Let $[a, b]$ be an order interval in an ordered real Banach algebra X with the cone K and let $A, B : [a, b] \to K$ and $C : [a, b] \to X$ be three nondecreasing operators such that*

(a) *A is completely continuous,*
(b) *B is totally bounded,*
(c) *C is a contraction, and*
(d) *the elements $a, b \in X$ satisfy $a \leq Aa\,Ba + Ca$ and $Ab\,Bb + Cb \leq b$.*

Further, if the cone K is positive and normal, then the operator equation $Ax\,Bx + Cx = x$ has the least and the greatest solution in $[a, b]$.

In the following sections we prove the main existence results for the PBVP (1.1) under suitable conditions.

3. Existence theory

Let $B(J, \mathbb{R})$ denote the space of bounded real-valued functions on J. Let $C(J, \mathbb{R})$, denote the space of all continuous real-valued functions on J. Define a norm $\|\cdot\|$ and a multiplication " $\cdot$ " in $C(J,\mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \quad \text{and} \quad (x.y)(t) = x(t)y(t) \ \text{ for } \ t \in J.$$

Clearly $C(J,\mathbb{R})$ becomes a Banach algebra with respect to the above norm and multiplication. By $L^1(J,\mathbb{R})$ we denote the set of Lebesgue integrable functions on J and the norm $\|\cdot\|_{L^1}$ in $L^1(J,\mathbb{R})$ is defined by

$$\|x\|_{L^1} = \int_0^T |x(s)|\,ds.$$

The following useful lemma is obvious and may be found in Nieto [20].

Lemma 3.1. *For any $h \in L^1(J,\mathbb{R}^+)$ and $\sigma \in L^1(J,\mathbb{R})$, x is a solution to the differential equation*

$$\left.\begin{aligned} x'(t) + h(t)x(t) = \sigma(t) \ a.\ e.\ t \in J \\ x(0) = x(T), \end{aligned}\right\} \tag{3.1}$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^T G_h(t,s)\sigma(s)\,ds \tag{3.2}$$

where,

$$G_h(t,s) = \begin{cases} \dfrac{e^{H(s)-H(t)+H(T)}}{e^{H(T)}-1}, & 0 \le s \le t \le T, \\[2ex] \dfrac{e^{H(s)-H(t)}}{e^{H(T)}-1}, & 0 \le t < s \le T, \end{cases} \tag{3.3}$$

where $H(t) = \int_0^t h(s)\,ds$.

Proof. The proof is well-known, but for sake of completeness, we give the details of it. If h is not identically zero, then $H(t) \ne 0$ for all $t > 0$. If h is identically zero, then $H(T) = 0$. We assume that h is not identically zero on J. Multiplying both sides of linear differential equation (3.1) by integrating factor $e^{H(t)}$, we obtain

$$\begin{aligned} \left(e^{H(t)}x(t)\right)' &= e^{H(t)}\sigma(t) \\ x(0) &= x(T). \end{aligned}$$

On integration the above equation yields

$$e^{H(t)}x(t) = x(0) + \int_0^t e^{H(s)}\sigma(s)\,ds. \tag{3.4}$$

Substituting $t = T$ in the above equation (3.4), we obtain

$$e^{H(T)}x(T) = x(T) + \int_0^T e^{H(s)}\sigma(s)\,ds.$$

Therefore,

$$x(T) = \int_0^T \frac{e^{H(s)}}{e^{H(T)} - 1}\sigma(s)\,ds.$$

Substituting this value in (3.4), we obtain

$$\begin{aligned} x(t) &= \int_0^T \frac{e^{H(s)-H(t)}}{e^{H(T)} - 1}\sigma(s)\,ds + \int_0^t e^{H(s)-H(t)}\sigma(s)\,ds \\ &= \int_0^t e^{H(s)-H(t)}\Big(\frac{1}{e^{H(T)} - 1} + 1\Big)\sigma(s)\,ds + \int_t^T \Big(\frac{e^{H(s)-H(t)}}{e^{H(T)} - 1}\Big)\sigma(s)\,ds \\ &= \int_0^t \Big(\frac{e^{H(s)-H(t)+H(T)}}{e^{H(T)} - 1}\Big)\sigma(s)\,ds + \int_t^T \Big(\frac{e^{H(s)-H(t)}}{e^{H(T)} - 1}\Big)\sigma(s)\,ds \\ &= \int_0^T G_h(t,s)\sigma(s)\,ds \end{aligned}$$

where G_h is a function on $J \times J$ defined by (3.3). The proof of the lemma is complete. □

Notice that the Green's function G_h is nonnegative on $J \times J$ and the number

$$M_h := \max\{\, |G_h(t,s)| \,:\, t,s \in [0,T]\,\}$$

exists for all $h \in L^1(J, \mathbb{R}^+)$.

We need the following definition in the sequel.

Definition 3.1. A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ is said to be Carathéodory if

(i) $t \mapsto \beta(t,x)$ is measurable for each $x \in \mathbb{R}$, and
(ii) $x \mapsto \beta(t,x)$ is continuous almost everywhere for $t \in J$.

Again, a Carathéodory function $\beta(t,x)$ is called L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a function $q_r \in L^1(J, \mathbb{R})$ such that

$$|\beta(t,x)| \le q_r(t), \quad a.e.\ t \in J$$

for all $x \in \mathbb{R}$ with $|x| \le r$.

Finally, a Carathéodory function $\beta(t,x)$ is called $L^1_{\mathbb{R}}$-Carathéodory if

(iv) there exists a function $q \in L^1(J, \mathbb{R})$ such that

$$|\beta(t,x)| \le q(t), \quad a.e.\ t \in J$$

for all $x \in \mathbb{R}$.

For convenience, the function q is referred to as a bound function of β.

We will use the following hypotheses in the sequel.

(A_0) The functions $t \mapsto f(t,x)$ and $t \mapsto k(t,x)$ are periodic of period T for all $x \in \mathbb{R}$.

(A_1) The mapping $x \mapsto \dfrac{x-k(0,x)}{f(0,x)}$ is injective on $\mathbb{R}$.

(A_2) The function $f : J \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous and there exists a function $\ell_1 \in B(J,\mathbb{R})$ such that

$$|f(t,x) - f(t,y)| \le \ell_1(t)\,|x-y| \ \text{ a.e. } \ t \in J$$

for all $x, y \in \mathbb{R}$.

(A_3) The function $k : J \times \mathbb{R} \to \mathbb{R}$ is continuous and there exists a function $\ell_2 \in B(J,\mathbb{R})$ such that

$$|k(t,x) - k(t,y)| \le \ell_2(t)\,|x-y| \ \text{ a.e. } \ t \in J$$

for all $x, y \in \mathbb{R}$.

(A_4) The function g is Carathéodory.

Note that hypotheses (A_2) through (A_4) are much common in the literature on the theory of nonlinear differential equations. Hypothesis (A_1) is somewhat new, but has been used in the literature for discussing the periodic solutions of differential and integral equations. Actually the function $f : J\times\mathbb{R} \to \mathbb{R}$ defined by $f(t,x) = \alpha + \beta|x|$ for some $\alpha, \beta \in \mathbb{R}$ satisfies the hypotheses (A_1)-(A_2) if $\alpha + \beta|x| \neq 0$ for all $x \in \mathbb{R}$.

Now consider the PBVP

$$\begin{cases} \Big(\dfrac{x(t)-k(t,x(t))}{f(t,x(t))}\Big)' + h(t)\Big(\dfrac{x(t)-k(t,x(t))}{f(t,x(t))}\Big) \\ \qquad\qquad = g_h(t,x(t)) \ \text{ a.e. } t \in J \\ x(0) = x(T) \end{cases} \tag{3.5}$$

where $h \in L^1(J,\mathbb{R}^+)$ is bounded and the function $g_h : J \times \mathbb{R} \to \mathbb{R}$ is defined by

$$g_h(t,x) = g(t,x) + h(t)\Big(\frac{x-k(t,x)}{f(t,x)}\Big). \tag{3.6}$$

Remark 3.1. Note that the PBVP (1.1) is equivalent to the PBVP (3.5) and a solution of the PBVP (1.1) is the solution for the PBVP (3.5) on J and vice versa.

Remark 3.2. Assume that hypotheses (A_3) and (A_4) hold. Then the function g_h defined by (3.6) is Carathéodory on $J \times \mathbb{R}$.

Lemma 3.2. *Assume that hypothesis* (A_0)-(A_1) *holds. Then for any* $h \in L^1(J, \mathbb{R}^+)$, *x is a solution to the differential equation (3.5) if and only if it is a solution of the integral equation*

$$x(t) = k(t, x(t)) + \big[f(t, x(t))\big]\Big(\int_0^T G_h(t,s) g_h(s, x(s))\,ds\Big), \tag{3.7}$$

where, the Green's function $G_h(t,s)$ *is defined by (3.3).*

Proof. Let $y(t) = \dfrac{x(t) - k(t, x(t))}{f(t, x(t))}$. Since $f(t,x)$ and $k(t,x)$ are periodic in t with period T for all $x \in \mathbb{R}$, we have

$$y(0) = \frac{x(0) - k(0, x(0))}{f(0, x(0))} = \frac{x(T) - k(T, x(T))}{f(T, x(T))} = y(T).$$

Now an application of Lemma 3.1 yields that the solution to differential equation (3.5) is the solution to integral equation (3.7). Conversely, suppose that x is any solution to the integral equation (3.7), then

$$y(0) = \frac{x(0) - k(0, x(0))}{f(0, x(0))} = y(T)$$

and

$$y(T) = \frac{x(T) - k(T, x(T))}{f(T, x(T))} = \frac{x(T) - k(0, x(T))}{f(0, x(T))}.$$

Since the function $x \mapsto \dfrac{x - k(0,x)}{f(0,x)}$ is injective on $\mathbb{R}$, one has $x(0) = x(T)$ and so, x is a solution to PBVP (1.1). The proof of the lemma is complete. □

We make use of the following hypothesis in the sequel.

(A_5) There is a continuous and nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and a function $\gamma \in L^1(J, \mathbb{R})$ such that $\gamma(t) > 0$, a.e. $t \in J$ satisfying

$$|g_h(t,x)| \le \gamma(t)\psi\big(|x|\big), \quad \text{a. e. } t \in J,$$

for all $x \in \mathbb{R}$.

Theorem 3.1. *Assume that the hypotheses* (A_0)-(A_5) *hold. Suppose that there exists a real number* $r > 0$ *such that* $L_1[M_h\|\gamma\|_{L^1}\psi(r) + L_2 < 1$ *and*

$$r > \frac{K + FM_h\|\gamma\|_{L^1}\psi(r)}{1 - [L_1 M_h\|\gamma\|_{L^1}\psi(r) + L_2]} \tag{3.8}$$

where, $F = \sup_{t\in[0,T]} |f(t,0)|$, $K = \sup_{t\in[0,T]} |k(t,0)|$, $L_1 = \max_{t\in J} \ell_1(t)$ *and* $L_2 = \max_{t\in J} \ell_2(t)$. *Then the PBVP (1.1) has a solution on J.*

Proof. Let $X = C(J, \mathbb{R})$. Define an open ball $\mathcal{B}_r(0)$ centered at origin 0 of radius r, where, the real number r satisfies the inequality (3.8). Define three mappings A, B and C on $\overline{\mathcal{B}_r(0)}$ by

$$Ax(t) = f(t, x(t)), \quad t \in J, \tag{3.9}$$

$$Bx(t) = \int_0^T G_h(t,s) g_h(s, x(s))\, ds, \quad t \in J \tag{3.10}$$

and

$$Cx(t) = k(t, x(t)), \quad t \in J, \tag{3.11}$$

Obviously A, B and C define the operators $A, B, C : \overline{\mathcal{B}_r(0)} \to X$. Then the integral equation (3.7) is equivalent to the operator equation

$$Ax(t)\, Bx(t) + Cx(t) = x(t), \quad t \in J. \tag{3.12}$$

We shall show that the operators A, B and C satisfy all the hypotheses of Theorem 2.3.

We first show that A is a Lipschitz mapping on X. Let $x, y \in X$. Then by (A_3),

$$\begin{aligned} |Ax(t) - Ay(t)| &= |f(t, x(t)) - f(t, y(t))| \\ &\le \ell_1(t)\, |x(t) - y(t)| \\ &\le L_1\, \|x - y\| \end{aligned}$$

for all $t \in J$. Taking the supremum over t, we obtain

$$\|Ax - Ay\| \le L_1 \|x - y\|$$

for all $x, y \in X$. So A is a Lipschitz mapping on X with the Lipschitz constant L_1. Similarly, it can be shown that C is also a Lipschitz mapping with the Lipschitz constant L_2.

Next we show that B is completely continuous on X. Using the standard arguments as in Granas *et al.* [15], it is shown that B is a continuous operator on X. We shall show that $B(\overline{\mathcal{B}_r(0)})$ is a uniformly bounded and equicontinuous set in X. Let $x \in \overline{\mathcal{B}_r(0)}$ be arbitrary. Since g is Carathéodory, we have

$$\begin{aligned} |Bx(t)| &= \Big| \int_0^T G_h(t,s) g_h(s, x(s))\, ds \Big| \\ &\le M_h \int_0^T \big[\gamma(s)\psi(|x(s)|)\big]\, ds \\ &= M_h \int_0^T \gamma(s)\psi(|x(s)|)\, ds \\ &\le M_h \|\gamma\|_{L^1} \psi(r). \end{aligned}$$

Taking the supremum over t, we obtain $\|Bx\| \leq M$ for all $x \in \overline{\mathcal{B}_r(0)}$, where $M = M_h\|\gamma\|_{L^1}\psi(r)$ This shows that $B(\overline{\mathcal{B}_r(0)})$ is a uniformly bounded set in X. Next we show that $B(\overline{\mathcal{B}_r(0)})$ is an equicontinuous set. To finish, it is enough to show that $y' = (Bx)'$ is bounded on $[0,T]$. Now for any $t \in [0,T]$, one has

$$\begin{aligned}
|y'(t)| &= \Big|\int_0^T \frac{\partial}{\partial t} G_h(t,s) g_h(s,x(s))\,ds\Big| \\
&= \Big|\int_0^T (-h(s)) G_h(t,s) g_h(s,x(s))\,ds\Big| \\
&\leq H\, M_h\|\gamma\|_{L^1}\psi(r) \\
&= c,
\end{aligned}$$

where $H = \max_{t\in J} h(t)$. Hence for any $t, \tau \in [0,T]$ one has

$$|Bx(t) - Bx(\tau)| \leq c\,|t-\tau| \to 0 \quad \text{as} \quad t \to \tau.$$

This shows that $B(\overline{\mathcal{B}_r(0)})$ is an equi-continuous set in X. Now the set $B(\overline{\mathcal{B}_r(0)})$ is uniformly bounded and equi-continuous, so it is relatively compact by Arzelà-Ascoli theorem. As a result B is a compact and continuous operator on $\overline{\mathcal{B}_r(0)}$. Thus all the conditions of Theorem 2.3 are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let $u \in X$ be a solution to $x = \lambda(Ax\,Bx + Cx)$ such that $\|u\| = r$. Then for any $\lambda \in (0,1)$, we have

$$u(t) = \lambda\, k(t,u(t)) + \lambda\,[f(t,u(t))]\left(\int_0^T G_h(t,s)g_h(s,u(s))\,ds\right)$$

for $t \in J$. Therefore,

$$\begin{aligned}
|u(t)| &\leq \lambda\,|k(t,u(t)| + \lambda\,|f(t,u(t)|\left(\Big|\int_0^T G_h(t,s)g_h(s,u(s))\,ds\Big|\right) \\
&\leq \lambda\Big(\big|k(t,u(t)) - k(t,0)\big| + \big|k(t,0)\big|\Big) \\
&\quad + \lambda\big[f(t,u(t))\big]\left(\int_0^T G_h(t,s)|g_h(s,u(s))|\,ds\right) \\
&\leq [\ell_2(t)\,|u(t)| + K] + [\ell_1(t)\,|u(t)| + F]\left(\int_0^T M_h|g_h(s,u(s))|\,ds\right) \\
&\leq L_2|u(t)| + K + L_1M_h\,|u(t)|\left(\int_0^T \gamma(s)\psi(|u(s)|)\,ds\right) \\
&\quad + FM_h\left(\int_0^T \gamma(s)\psi(|u(s|)\,ds\right)
\end{aligned}$$

$$\leq \Big[L_1 M_h \|\gamma\|_{L^1}\psi(\|u\|) + L_2\Big]|u(t)| + K + FM_h\|\gamma\|_{L^1}\psi(\|u\|). \tag{3.13}$$

Taking the supremum over t in the above inequality (3.13) yields

$$\|u\| \leq \frac{K + FM_h\|\gamma\|_{L^1}\psi(\|u\|)}{1 - \big[L_1 M_h\|\gamma\|_{L^1}\psi(\|u\|) + L_2\big]}.$$

Substituting $\|u\| = r$ in above inequality yields

$$r \leq \frac{K + FM_h\|\gamma\|_{L^1}\psi(r)}{1 - \big[L_1 M_h\|\gamma\|_{L^1}\psi(r) + L_2\big]}.$$

This is a contradiction to (3.8). Hence the conclusion (ii) of Corollary 2.1 does not hold. Therefore the operator equation $Ax\,Bx + Cx = x$ and consequently the PBVP (1.1) has a solution on J. This completes the proof. □

Remark 3.3. We note that in Theorem 3, we only require the hypothesis (A_1) to hold in $[-r, r]$.

4. Existence of extremal solutions

We equip the space $C(J,\mathbb{R})$ with the order relation $\leq$ with the help of the cone defined by

$$K = \{x \in C(J,\mathbb{R}) : x(t) \geq 0, \forall\, t \in J\}. \tag{4.1}$$

It is well known that the cone K is positive and normal in $C(J,\mathbb{R})$. We need the following definitions in the sequel.

Definition 4.1. A function $a \in AC(J,\mathbb{R})$ is called a lower solution of the PBVP (1.1) on J if the function $t \mapsto \Big(\dfrac{a(t) - k(t,a(t))}{f(t,a(t))}\Big)$ is absolutely continuous on J, and

$$\left.\begin{aligned} \frac{d}{dt}\Big[\frac{a(t) - k(t,a(t))}{f(t,a(t))}\Big] &\leq g(t,a(t)) \text{ a.e. } t \in J \\ a(0) &\leq a(T). \end{aligned}\right\}$$

Again, a function $b \in AC(J,\mathbb{R})$ is called an upper solution of the PBVP (1.1) on J if the function $t \mapsto \Big(\dfrac{b(t) - k(t,b(t))}{f(t,b(t))}\Big)$ is absolutely continuous on J, and

$$\left.\begin{aligned} \frac{d}{dt}\Big[\frac{b(t) - k(t,b(t))}{f(t,b(t))}\Big] &\geq g(t,b(t)) \text{ a.e. } t \in J \\ b(0) &\geq b(T). \end{aligned}\right\}$$

Definition 4.2. A solution x_M of the PBVP (1.1) is said to be maximal if for any other solution x to PBVP (1.1) one has $x(t) \leq x_M(t)$ for all $t \in J$. Similarly, a solution x_m of the PBVP (1.1) is said to be minimal if $x_m(t) \leq x(t)$ for all $t \in J$, where x is any solution of the PBVP (1.1) on J.

Remark 4.1. The upper and lower solutions of the PBVP (1.1) are respectively the upper and lower solutions of the PBVP (3.5) and vice-versa. Similarly, the maximal and minimal solutions of the PBVP (1.1) are respectively the maximal and minimal solutions of the PBVP (3.5) and vice-versa.

4.1. **Carathéodory case.** We need the following definition in the sequel.

Definition 4.3. A function $\beta : \mathbb{R} \to \mathbb{R}$ is called nondecreasing if $f(x) \leq f(y)$ for all $x, y \in \mathbb{R}$ with $x \leq y$. Similarly, $\beta(x)$ is called increasing in x if $f(x) < f(y)$ for all $x, y \in \mathbb{R}$ with $x < y$.

We consider the following set of assumptions:

(B_0) $f : J \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$, $g_h : J \times \mathbb{R} \to \mathbb{R}^+$.

(B_1) The mapping $x \mapsto \dfrac{x - k(0,x)}{f(0,x)}$ is increasing in the real interval $\big[\min_{t\in J} a(t), \max_{t\in J} b(t)\big]$.

(B_2) The functions $f(t,x)$, $k(t,x)$ and $g_h(t,x)$ are nondecreasing in x almost everywhere for $t \in J$.

(B_3) The PBVP (1.1) has a lower solution a and an upper solution b on J with $a \leq b$.

(B_4) The function $q : J \to \mathbb{R}$ defined by
$$q(t) = g_h(t, b(t)),$$
is Lebesgue integrable.

We remark that hypothesis (B_3) holds in particular if f is continuous and g is L^1-Carathéodory on $J \times \mathbb{R}$.

Remark 4.2. Assume that hypotheses (B_0)-(B_4) hold. Then the function $t \mapsto g_h(t, x(t))$ is Lebesgue integrable on J and
$$|g_h(t, x(t))| = g_h(t, x(t)) \leq q(t) \text{ a.e. } t \in J,$$
for all $x \in [a, b]$.

Remark 4.3. If the hypothesis (B_0) and (B_1) holds, then the map $x \mapsto \dfrac{x - k(0,x)}{f(0,x)}$ is injective and
$$\frac{a(0) - k(0, a(0)}{f(0, a(0))} \leq \frac{a(T) - k(T, a(T))}{f(T, a(T))}$$

and
$$\frac{b(0)-k(0,b(0))}{f(0,b(0))} \geq \frac{b(T)-k(T,b(T))}{f(T,b(T))}.$$

Theorem 4.1. *Suppose that the assumptions (A_0)–(A_4) and (B_0)-(B_4) hold. Further if $L_1 M_h \|h\|_{L^1} + L_2 < 1$, where q is given in Remark 4, $L_1 = \max_{t\in J} \ell_1(t)$ and $L_2 = \max_{t\in J} \ell_2(t)$, then PBVP (1.1) has a minimal and a maximal solution on J.*

Proof. Now PBVP (1.1) is equivalent to integral equation (3.7) on J. Let $X = C(J,\mathbb{R})$. Define three operators A, B and C on X by (3.9), (3.10) and (3.11) respectively. Then integral equation (3.7) is transformed into an operator equation $Ax(t)Bx(t) + Cx(t) = x(t)$ in a Banach algebra X. Notice that (B_0) implies $A, B : [a, b] \to K$ and $C : [a, b] \to X$. Note that condition (B_1) provides $a \leq Aa\,Ba + Ca$ and $Ab\,Bb + Cb \leq b$. Since the cone K in X is normal, $[a, b]$ is a norm-bounded set in X. Now it is shown, as in the proof of Theorem 3, that A and C are Lipschitz mappings with the Lipschitz constants L_1 and L_2 respectively. Also B is completely continuous operator on $[a, b]$. Again the hypothesis (B_2) implies that A, B and C are nondecreasing on $[a, b]$. To see this, let $x, y \in [a, b]$ be such that $x \leq y$. Then by (B_2),
$$Ax(t) = f(t, x(t)) \leq f(t, y(t)) = Ay(t)$$
for all $t \in J$. Again, we have
$$\begin{aligned} Bx(t) &= \int_0^T G_h(t,s) g_h(s, x(s))\, ds \\ &\leq \int_0^T G_h(t,s) g_h(s, x(s))\, ds \\ &= By(t) \end{aligned}$$
for all $t \in J$. Similarly,
$$Cx(t) = k(t, x(t)) \leq k(t, y(t)) = Cy(t)$$
So A, B and C are nondecreasing operators, A and B on $[a, b]$ and C on X. Again, Lemma 3.1 and hypotheses (B_1)-(B_2) together imply that
$$\begin{aligned} a(t) &\leq k(t, a(t)) + [f(t, a(t))] \left(\int_0^T G_h(t,s) g_h(s, a(s)))\, ds \right) \\ &\leq k(t, x(t)) + [f(t, x(t))] \left(\int_0^T G_h(t,s) g_h(s, x(s))\, ds \right) \\ &\leq k(t, b(t)) + [f(t, b(t))] \left(\int_0^T G_h(t,s) g_h(s, b(s)))\, ds \right) \\ &\leq b(t), \end{aligned}$$

for all $t \in J$ and $x \in [a,b]$. As a result $a(t) \le Ax(t)Bx(t) + Cx(t) \le b(t)$ for all $t \in J$ and $x \in [a,b]$. Hence, $Ax\,Bx + Cx \in [a,b]$ for all $x \in [a,b]$. Again,

$$\begin{aligned} M &= \|B([a,b])\| \\ &= \sup\{\|Bx\| : x \in [a,b]\} \\ &\le \sup\left\{\sup_{t\in J}\int_0^T G_h(t,s)|g_h(s,x(s))|\,ds \;\Big|\; x \in [a,b]\right\} \\ &\le M_h \int_0^T q(s)\,ds \\ &= M_h\|q\|_{L^1}. \end{aligned}$$

Since $L_1 M_h\|q\|_{L^1} + L_2 < 1$, we apply Theorem 2.4 to the operator equation $Ax\,Bx + Cx = x$ to yield that the PBVP (1.1) has a minimal and a maximal solution in $[a,b]$ defined on J. This completes the proof. □

4.2. **Discontinuous case.** We need the following definition in the sequel.

Definition 4.4. A mapping $\beta : J \times \mathbb{R} \to \mathbb{R}$ is said to be Chandrabhan if

(i) $t \mapsto \beta(t, x(t))$ is measurable for each $x \in C(J,\mathbb{R})$, and
(ii) $x \mapsto \beta(t,x)$ is nondecreasing almost everywhere for $t \in J$.

Again, a Chandrabhan function $\beta(t,x)$ is called L^1-Chandrabhan if

(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(J,\mathbb{R})$ such that
$$|\beta(t,x)| \le q_r(t), \quad a.e.\ t \in J$$
for all $x \in \mathbb{R}$ with $|x| \le r$.

Finally, a Chandrabhan mapping β is called $L^1_{\mathbb{R}}$-Chandrabhan if

(iv) there exists a function $q \in L^1(J,\mathbb{R})$ such that
$$|\beta(t,x)| \le q(t), \quad a.e.\ t \in I$$
for all $x \in \mathbb{R}$.

For convenience, the function q is referred to as a bound function of β.

We consider the following hypotheses in the sequel.

(C_1) The function $f : J \times \mathbb{R} \to \mathbb{R} - \{0\}$ is continuous.
(C_2) The function $k : J \times \mathbb{R} \to \mathbb{R}$ is continuous.
(C_3) The function $f(t,x)$ and $k(t,x)$ are nondecreasing in x almost everywhere for $t \in J$.
(C_4) The function g_h defined by (3.6) is Chandrabhan.

Theorem 4.2. *Suppose that the assumptions (A_0), (B_0)-(B_1), (B_3)-(B_4) and (C_1)-(C_4) hold. Then PBVP (1.1) has a minimal and a maximal solution on J.*

Proof. Now PBVP (1.1) is equivalent to integral equation (3.7) on J. Let $X = C(J, \mathbb{R})$. Define two operators A and B on X by (3.9), (3.10) and (3.11) respectively. Then integral equation (3.7) is transformed into an operator equation $Ax(t)\,Bx(t) + Cx(t) = x(t)$ in a Banach algebra X. Notice that (B_0) implies $A, B : [a, b] \to K$. Note that condition (B_1) provides $a \leq Aa\,Ba + Ca$ and $Ab\,Bb + Cb \leq b$. Since the cone K in X is normal, $[a, b]$ is a norm bounded set in X.

Step I : First we show that A is completely continuous on $[a, b]$. Now the cone K in X is normal, so the order interval $[a, b]$ is norm-bounded in X. Hence there exists a constant $r > 0$ such that $\|x\| \leq r$ for all $x \in [a, b]$. As f is continuous on compact $J \times [-r, r]$, it attains its maximum, say M. Therefore, for any subset S of $[a, b]$ we have

$$\begin{aligned}\|A(S)\|_{\mathcal{P}} &= \sup\{\|Ax\| : x \in S\} \\ &= \sup\left\{\sup_{t\in J} |f(t, x(t))| : x \in S\right\} \\ &\leq \sup\left\{\sup_{t\in J} |f(t, x)| : x \in [-r, r]\right\} \\ &\leq M.\end{aligned}$$

This shows that $A(S)$ is a uniformly bounded subset of X.

Next we note that the function $f(t, x)$ is uniformly continuous on $[0, T] \times [-r, r]$. Therefore for any $t, \tau \in [0, T]$, we have

$$|f(t, x) - f(\tau, x)| \to 0 \text{ as } t \to \tau$$

for all $x \in [-r, r]$. Similarly, for any $x, y \in [-r, r]$

$$|f(t, x) - f(t, y)| \to 0 \text{ as } x \to y$$

for all $t \in [0, T]$. Hence for any $t, \tau \in [0, T]$ and for any $x \in S$ one has

$$\begin{aligned}|Ax(t) - Ax(\tau)| &= |f(t, x(t)) - f(\tau, x(\tau))| \\ &\leq |f(t, x(t)) - f(\tau, x(t)| + |f(\tau, x(t)) - f(\tau, x(\tau))| \\ &\to 0 \quad \text{as} \quad t \to \tau.\end{aligned}$$

This shows that $A(S)$ is an equi-continuous set in X. Now an application of Arzelà-Ascoli theorem yields that A is a completely continuous operator on $[a, b]$.

Step II : Next we show that B is a totally bounded operator on $[a, b]$. To finish, we shall show that $B(S)$ is uniformly bounded and equi-continuous set in X for any subset S of $[a, b]$. Since the cone K in X is normal, the order interval $[a, b]$ is norm-bounded. Let $y \in B(S)$ be arbitrary. Then,

$$y(t) = \int_0^T G_h(t,s) g_h(s, x(s))\, ds$$

for some $x \in S$. By hypothesis (B_2) one has

$$\begin{aligned} |y(t)| &= \int_0^T G_h(t,s) |g_h(s, x(s))|\, ds \\ &\leq M_h \int_0^T q(s)\, ds \\ &\leq M_h \|h\|_{L^1}. \end{aligned}$$

Taking the supremum over t,

$$\|y\| \leq M_h \|q\|_{L^1},$$

which shows that $B(S)$ is a uniformly bounded set in X. Similarly, let $t, \tau \in J$. To finish, it is enough to show that y' is bounded on $[0, T]$. Now for any $t \in [0, T]$,

$$\begin{aligned} |y'(t)| &\leq \Big| \int_0^T \frac{\partial}{\partial t} G_h(t,s) |g_h(s, x(s))|\, ds \Big| \\ &= \Big| \int_0^T (-h(s)) G_h(t,s) |g_h(s, x(s))|\, ds \Big| \\ &\leq K\, M_h \|q\|_{L^1} \\ &= c. \end{aligned}$$

where $H = \max_{t \in J} |h(t)|$. Hence for any $t, \tau \in [0, T]$, one has

$$|y(t) - y(\tau)| \leq c\, |t - \tau| \to 0 \quad \text{as} \quad t \to \tau.$$

This shows that $B(S)$ is an equi-continuous set of functions in $[a, b]$ for all $S \subset [a, b]$. Now $B(S)$ is a uniformly bounded and equi-continuous, so it is totally bounded by Arzelà-Ascoli theorem. It can be shown as in the case of operator A that the operator C is also totally bounded. Thus all the conditions of Theorem 2.5 are satisfied and hence an application of it yields that the PBVP (1.1) has a maximal and a minimal solution in $[a, b]$ defined on J. □

Theorem 4.3. *Suppose that the assumptions (A_0), (B_0)-(B_3) and (C_3)-(C_4) hold. Further if*

$$L_1 M_h \|q\|_{L^1} + L_2 < 1,$$

where q is given in (B_4), $L_1 = \max_{t \in J} \ell_1(t)$ and $L_2 = \max_{t \in J} \ell_2(t)$, then the PBVP (1.1) has a minimal and a maximal solution in $[a, b]$ defined on J.

Proof. Now PBVP (1.1) is equivalent to integral equation (3.7) on J. Let $X = C(J,\mathbb{R})$. Define three operators A, B and C on X by (3.9), (3.10) and (3.11) respectively. Then the integral equation (3.7) is transformed into an operator equation $Ax(t)\,Bx(t)+Cx(t) = x(t)$ in the Banach algebra X. Notice that (B_0) implies $A, B : [a, b] \to K$. Note that condition (B_1) provides $a \le Aa\,Ba + Ca$ and $Ab\,Bb+Cb \le b$. Since the cone K in X is normal, $[a, b]$ is a norm bounded set in X. Now it can be shown as in the proofs of Theorem 3 and Theorem 4.1 that the operators A and C are Lipschitz with the Lipschitz constants $\alpha = L_1$ and $\beta = L_2$ respectively. Again, B is a totally bounded operator with $M = \|B([a, b])\| = M_h\|q\|_{L^1}$. Since $\alpha M + \beta = L_1\,M_h\|q\|_{L^1} + L_2 < 1$, the desired conclusion follows by an application of Theorem 2.4. □

Theorem 4.4. *Suppose that the assumptions (A_1), (A_2), (B_0),(B_2)-(B_3) and (C_2)-(C_4) hold. Further, if $L_1M_h\|q\|_{L^1} < 1$, where q is given in (B_4) and $L_1 = \max_{t\in J}\ell_1(t)$, then the PBVP (1.1) has a minimal and a maximal solution in $[a, b]$ defined on J.*

Proof. Now PBVP (1.1) is equivalent to the integral equation (3.7) on J. Let $X = C(J,\mathbb{R})$. Define three operators A, B and C on X by (3.9), (3.10) and (3.11) respectively. Then the integral equation (3.7) is transformed into an operator equation $Ax(t)\,Bx(t)+Cx(t) = x(t)$ in the Banach algebra X. Notice that (B_0) implies $A, B : [a, b] \to K$. Since the cone K in X is normal, $[a, b]$ is a norm bounded set in X. Now it can be shown as in the proof of Theorem 3 that the operator A is Lipschitz mapping with the Lipschitz constant $\alpha = L_1$ and B is a completely continuous operator on $[a, b]$ with $M = \|B([a, b])\| = M_h\|q\|_{L^1}$. Again, following the arguments similar to Step I in the proof of Theorem 4.2, it is shown that C is a totally bounded operator on $[a, b]$. Since $\alpha M = L_1\,M_h\|q\|_{L^1} < 1$, the desired conclusion follows by an application of Theorem 2.11. □

Theorem 4.5. *Suppose that the assumptions (A_0), (A_3), (B_0)-(B_1), (B_3) and (C_1), (C_3)-(C_4) hold. Then the PBVP (1.1) has a minimal and a maximal solution in $[a, b]$ defined on J.*

Proof. The proof is similar to Theorem 4.2 and now the desired conclusion follows by an application of Theorem 2.12. □

5. An example

Given the closed and bounded interval $J = [0, 1]$ in $\mathbb{R}$, consider the nonlinear PBVP

$$\begin{cases} \dfrac{d}{dt}\left[\dfrac{x(t) - k(t, x(t))}{f(t, x(t))}\right] = \dfrac{|x(t)|}{32} - \dfrac{x(t) - \frac{1}{32}\sin x(t)}{1 + |x(t)|}, \quad \text{a.e. } t \in J \\ \qquad\qquad x(0) = x(1), \end{cases} \tag{5.1}$$

where the functions $f : J \times \mathbb{R} \to \mathbb{R} - \{0\}$, $k : J \times \mathbb{R} \to \mathbb{R}$ are defined by

$$f(t,x) = 1 + |x|, \quad \text{and} \quad k(t,x) = \frac{1}{32}\sin x.$$

Obviously $f : J \times \mathbb{R} \to \mathbb{R}^+ - \{0\}$ is continuous and the functions $t \mapsto f(t,x)$ and $t \mapsto k(t,x)$ are periodic of period $T = 1$ for all $x \in \mathbb{R}$. Define a function $g : J \times \mathbb{R} \to \mathbb{R}$ by

$$g(t,x) = \frac{|x|}{32} - \frac{x - \frac{1}{32}\sin x}{1+|x|}.$$

Now consider the PBVP

$$\begin{cases} \left(\dfrac{x(t) - \frac{1}{32}\sin x(t)}{1+|x(t)|}\right)' + \dfrac{x(t) - \frac{1}{32}\sin x(t)}{1+|x(t)|} = \dfrac{|x(t)|}{32}, \quad \text{a. e. } t \in J \\ x(0) = x(1). \end{cases} \tag{5.2}$$

It is easy to verify that f is continuous and Lipschitz on $J \times \mathbb{R}$ with a Lipschitz function $\ell_1(t) = 1$ for all $t \in J$. Here $h(t) = 1$, and so $M_h = \dfrac{e}{e-1}$. Also, here we have $F = \sup_{t \in J} |f(t,0)| = 1$. Now the real number $r = 4$ satisfies condition (3.8) of Theorem 3 with $\gamma(t) = \frac{1}{32}$ for all $t \in J$ and $\psi(r) = r$ for all $r \in \mathbb{R}^+$. Note that, in this case, $K = \sup_{t \in [0,T]} |k(t,0)| = 0$, h is bounded and besides,

$$x \mapsto \frac{x - k(0,x)}{f(0,x)} = \frac{x - \frac{1}{32}\sin x}{1+|x|}$$

is (strictly) increasing in the interval $[-4, 4]$. Therefore, an application of Theorem 3 yields that the PBVP (5.1) has a solution u on J with $\|u\| \le 4$.

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