Nonlinear Functional Analysis and Applications
Vol. 17, No. 3 (2012), pp. 285-300

http://nfaa.kyungnam.ac.kr/jour-nfaa.htm

NEW INEQUALITIES FOR THE B-OPERATORS

N. A. Rather[1], K. Boubaker[2] and M. A. Shah[3]

[1]Department of Mathematics, Kashmir University
Hazratbal, Srinagar-190006, India
e-mail: dr.narather@gmail.com

[2]ESSTT, University of Tunis
Tunis, Tunisia
e-mail: mmbb11112000@yahoo.fr

[3]Department of Mathematics, Kashmir University
Hazratbal, Srinagar-190006, India
e-mail: mushtaqa022@gmail.com

Abstract. Let P_n be the class of polynomials $P(z)$ of degree n and B_n a family of operators that map P_n into itself. For $B \in B_n$, we investigate the dependence of

$$\left| B[P(Rz)] - \alpha B[P(rz)] + \beta \left\{ \left(\frac{R+1}{r+1} \right)^n - |\alpha| \right\} B[P(rz)] \right|$$

on the minimum and the maximum modulus of $P(z)$ on $|z| = 1$ for arbitrary real or complex numbers α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$ *and* $R > r \geq 1$ with or without restriction on the zeros of the polynomial $P(z)$ and present some new inequalities for B-operators yielding certain sharp compact generalizations of some well-known Bernstein-type inequalities for polynomials.

1. Introduction

Let $P_n(z)$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n. If $P \in P_n$, then

$$\underset{|z|=1}{Max} \left| P'(z) \right| \leq n \underset{|z|=1}{Max} \left| P(z) \right| \tag{1.1}$$

and

$$\underset{|z|=R>1}{Max} \left| P(z) \right| \leq R^n \underset{|z|=1}{Max} \left| P(z) \right|. \tag{1.2}$$

[0]Received May 14, 2012. Revised July 12, 2012.
[0]2000 Mathematics Subject Classification: 30A06, 30C10.
[0]Keywords: Polynomials, B-operators, inequalities, complex domain.

Inequality (1.1) is an immediate consequence of S. Bernstein's theorem (see [5], [8], [11]) on the derivative of a trigonometric polynomial. Inequality (1.2) is a simple deduction from the maximum modulus principle (see [9] or [12]). For the class of polynomials $P \in P_n$, having all their zeros in $|z| \leq 1$, we have

$$\underset{|z|=1}{Min} \left|P'(z)\right| \geq n \underset{|z|=1}{Min} |P(z)| \tag{1.3}$$

and

$$\underset{|z|=R>1}{Min} |P(z)| \geq R^n \underset{|z|=1}{Min} |P(z)| . \tag{1.4}$$

Inequalities (1.3) and (1.4) are due to A.Aziz and Q.M.Dawood [2]. Both the results are sharp and equality in (1.3) and (1.4) holds for $P(z) = \lambda z^n, \lambda \neq 0$.

For the class of polynomials $P \in P_n$ having no zero in $|z| < 1$, then (1.1) and (1.2) can be replaced by

$$\underset{|z|=1}{Max} \left|P'(z)\right| \leq \frac{n}{2} \underset{|z|=1}{Max} |P(z)| \tag{1.5}$$

and

$$\underset{|z|=R>1}{Max} |P(z)| \leq \frac{R^n + 1}{2} \underset{|z|=1}{Max} |P(z)| . \tag{1.6}$$

Equality in (1.5) and (1.6) holds for $P(z) = \lambda z^n + \mu, |\lambda| = |\mu| = 1$. Inequality (1.5) was conjectured by Erdös and later verified by Lax [6]. Ankeny and Rivlin [1] used inequality (1.5) to prove inequality (1.6).

A.Aziz and Q.M.Dawood [2] improved inequalities (1.5) and (1.6) and showed that if $P(z) \neq 0$ in $|z| < 1$, then

$$\underset{|z|=1}{Max} \left|P'(z)\right| \leq \frac{n}{2} \left\{ \underset{|z|=1}{Max} |P(z)| - \underset{|z|=1}{Min} |P(z)| \right\} \tag{1.7}$$

and

$$\underset{|z|=R>1}{Max} |P(z)| \leq \frac{R^n + 1}{2} \underset{|z|=1}{Max} |P(z)| - \frac{R^n - 1}{2} \underset{|z|=1}{Min} |P(z)| . \tag{1.8}$$

As a compact generalization of inequalities (1.5) and (1.6), Aziz and Rather [3] have shown that, if $P \in P_n$ and $P(z) \neq 0$ for $|z| < 1$, then for every real or complex number α with $|\alpha| \leq 1$, $R > 1$ and $|z| \geq 1$,

$$|P(Rz) - \alpha P(z)| \leq \frac{\{|R^n - \alpha| \, |z|^n + |1 - \alpha|\}}{2} \underset{|z|=1}{Max} |P(z)| . \tag{1.9}$$

The result is sharp and equality in (1.9) holds for $P(z) = az^n + b, |a| = |b| = 1$.

Rahman [10] (see also Rahman and Schemissier [11]) introduced a class B_n of operators B that carries a polynomial $P \in P_n$ into

$$B[P](z) = \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!}, \tag{1.10}$$

where λ_0, λ_1 and λ_2 are such that all the zeros of

$$u(z) = \lambda_0 + C(n,1)\lambda_1 z + C(n,2)\lambda_2 z^2 \tag{1.11}$$

lie in the half plane

$$|z| \leq |z - n/2|\,. \tag{1.12}$$

As a generalization of inequalities (1.1) and (1.2), Q.I.Rahman [10] proved that if $P \in P_n$, then

$$|P(z)| \leq |z|^n \underset{|z|=1}{Max} |P(z)| \quad for \quad |z| = 1$$

implies

$$|B[P](z)| \leq |B[z^n]| \underset{|z|=1}{Max} |P(z)| \quad for \quad |z| \geq 1, \tag{1.13}$$

where $B \in B_n$ (see [10], inequality (5.1)) and if $P(z) \neq 0$ for $|z| < 1$, then

$$|B[P](z)| \leq \frac{1}{2}\left\{|B[z^n]| + |\lambda_0|\right\} \underset{|z|=1}{Max} |P(z)| \quad for \quad |z| \geq 1. \tag{1.14}$$

where $B \in B_n$ (see [10], inequality (5.2) and (5.3) or [11]).

In this paper we consider a problem of investigating the dependence of

$$|B[P(Rz)] + \phi\,(R, r, \alpha, \beta)\, B[P(rz)]|$$

where

$$\phi\,(R, r, \alpha, \beta) = \beta \left\{ \left(\frac{R+1}{r+1}\right)^n - |\alpha| \right\} - \alpha, \tag{1.15}$$

on the minimum and the maximum modulus of $P(z)$ on $|z| = 1$ for arbitrary real or complex numbers α, β with $|\alpha| \leq 1$,$|\beta| \leq 1$ and $R > r \geq 1$, and obtain certain compact generalizations and refinements of some well known polynomial inequalities.

2. Lemmas

For the proofs of main results, we need the following lemmas. First Lemma is due to Aziz and Rather [4].

Lemma 2.1. *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every $R \geq r \geq 1$ and $|z| = 1$,*

$$|P(Rz)| \geq \left(\frac{R+1}{r+1}\right)^n |P(rz)|\,.$$

The following Lemma follows from corollary 18.3 on page 65 of [7].

Lemma 2.2. *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then all the zeros of $B[P](z)$ also lie in $|z| \leq 1$.*

Lemma 2.3. *If $P \in P_n$ and $P(z)$ does not vanish in $|z| < 1$, then for arbitrary real or complex numbers α and β with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $|z| = 1$,*

$$\begin{aligned} &|B[P(Rz)] + \phi(R, r, \alpha, \beta) B[P(rz)]| \\ &\leq |B[Q(Rz)] + \phi(R, r, \alpha, \beta) B[Q(rz)]| \end{aligned} \tag{2.1}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$ and $\phi(R, r, \alpha, \beta)$ is defined by (1.15). The result is sharp and equality in (2.1) holds for $P(z) = z^n + 1$.

Proof. Since the nth degree polynomial $P(z)$ does not vanish in $|z| < 1$, all the zeros of the polynomial $Q(z) = z^n \overline{P(1/\overline{z})}$ of degree n lie in $|z| \leq 1$. Applying Theorem 1.1 with $F(z)$ replaced by $Q(z)$, it follows that

$$|B[P(Rz)] + \phi(R, r, \alpha, \beta) B[P(rz)]| \leq |B[Q(Rz)] + \phi(R, r, \alpha, \beta) B[Q(rz)]|$$

for $|z| \geq 1, |\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$. This proves the Lemma 2.3. □

Lemma 2.4. *If $P \in P_n$, then for arbitrary real or complex numbers α and β with $|\alpha| \leq 1, |\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} &|B[P(Rz)] + \phi(R, r, \alpha, \beta) B[P(rz)]| \\ &\quad + |B[Q(Rz)] + \phi(R, r, \alpha, \beta) B[Q(rz)]| \\ &\leq \{|R^n + \phi(R, r, \alpha, \beta) r^n| \, |B[z^n]| \\ &\quad + |1 + \phi(R, r, \alpha, \beta)| \, |\lambda_0|\} \, Max_{|z|=1} |P(z)| \end{aligned} \tag{2.2}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$ and $\phi(R, r, \alpha, \beta)$ is defined by (1.15). The result is sharp and equality in (2.2) holds for $P(z) = \lambda z^n, \lambda \neq 0$.

Proof. Let $M = Max_{|z|=1}|P(z)|$, then $|P(z)| \leq M$ for $|z| = 1$. If μ is any real or complex number with $|\mu| > 1$, then by Rouche's Theorem, the polynomial $f(z) = P(z) - \mu M$ does not vanish in $|z| < 1$. If $f^*(z) = z^n \overline{f(1/\overline{z})}$, then all the zeros of $f^*(z)$ lie in $|z| \leq 1$. Applying Lemma 2.3 with $P(z)$ replaced by $f(z)$ and $F(z)$ by $f^*(z)$, it follows that for all real or complex numbers α, β with $|\alpha| \leq 1$, $|\beta| \leq 1, R > r \geq 1$ and $|z| \geq 1$,

$$\begin{aligned} &|B[f(Rz)] + \phi(R, r, \alpha, \beta) B[f(rz)]| \\ &\leq |B[f^*(Rz)] + \phi(R, r, \alpha, \beta) B[f^*(rz)]| \,. \end{aligned} \tag{2.3}$$

Since $Q(z) = z^n \overline{P(1/\overline{z})}$, we have

$$f^*(z) = z^n \overline{f(1/\overline{z})} = z^n \overline{P(1/\overline{z})} - \overline{\mu} M z^n = Q(z) - \overline{\mu} M z^n.$$

Using the fact that B is a linear operator and $B[1] = \lambda_0$, we obtain from (2.3),

$$\begin{aligned}&|(B[P(Rz)] + \phi(R, r, \alpha, \beta) B[P(rz)]) - \mu(1 + \phi(R, r, \alpha, \beta)) \lambda_0 M|\\ &\leq |(B[Q(Rz)] + \phi(R, r, \alpha, \beta) B[Q(rz)]) - \overline{\mu}(R^n + \phi(R, r, \alpha, \beta) r^n) B[z^n]M|\end{aligned}$$

for all real or complex numbers α and β with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$. Now choosing the argument of μ such that

$$\begin{aligned}&|(B[Q(Rz)] + \phi(R, r, \alpha, \beta) B[Q(rz)]) - \overline{\mu}(R^n + \phi(R, r, \alpha, \beta) r^n) B[z^n]M|\\ &= |\mu| |R^n + \phi(R, r, \alpha, \beta) r^n| |B[z^n]| M - |B[Q(Rz)] + \phi(R, r, \alpha, \beta) B[Q(rz)]|,\end{aligned}$$

we get

$$\begin{aligned}&|B[P(Rz)] + \phi(R, r, \alpha, \beta) B[P(rz)]| - |\mu| |1 + \phi(R, r, \alpha, \beta)| |\lambda_0| M\\ &\leq |(B[Q(Rz)] + \phi(R, r, \alpha, \beta) B[Q(rz)]) - \overline{\mu}(R^n + \phi(R, r, \alpha, \beta) r^n) B[z^n]M|\\ &= |\mu| |R^n + \phi(R, r, \alpha, \beta) r^n| |B[z^n]| M - |B[Q(Rz)] + \phi(R, r, \alpha, \beta) B[Q(rz)]|\end{aligned}$$

for $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$. This implies

$$\begin{aligned}&|B[P(Rz)] + \phi(R, r, \alpha, \beta) B[P(rz)]| + |B[Q(Rz)] + \phi(R, r, \alpha, \beta) B[Q(rz)]|\\ &\leq |\mu| \{|R^n + \phi(R, r, \alpha, \beta) r^n| |B[z^n]| + |\lambda_0| |1 + \phi(R, r, \alpha, \beta)|\} M,\end{aligned}$$

for $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$. Letting $|\mu| \to 1$, we obtain

$$\begin{aligned}&|B[P(Rz)] + \phi(R, r, \alpha, \beta) B[P(rz)]| + |B[Q(Rz)] + \phi(R, r, \alpha, \beta) B[Q(rz)]|\\ &\leq \{|R^n + \phi(R, r, \alpha, \beta) r^n| |B[z^n]| + |\lambda_0| |1 + \phi(R, r, \alpha, \beta)|\} M.\end{aligned}$$

This proves Lemma 2.4. □

3. Main results

Theorem 3.1. *If $F \in P_n$ has all its zeros in $|z| \leq 1$ and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \quad \text{for} \quad |z| = 1,$$

then for all real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $|z| \geq 1$,

$$\begin{aligned}&|B[P(Rz)] + \phi(R, r, \alpha, \beta) B[P(rz)]|\\ &\leq |B[F(Rz)] + \phi(R, r, \alpha, \beta) B[F(rz)]|\end{aligned} \tag{3.1}$$

where $B \in B_n$ and $\phi(R, r, \alpha, \beta)$ is defined by (1.15).

Proof. By hypothesis, the polynomial $F(z)$ of degree n has all its zeros in $|z| \leq 1$ and $P(z)$ is a polynomial of degree at most n such that

$$|P(z)| \leq |F(z)| \quad for \quad |z| = 1, \tag{3.2}$$

therefore, if $F(z)$ has a zero of multiplicity s at $z = e^{i\theta_0}$, then $P(z)$ has a zero of multiplicity at least s at $z = e^{i\theta_0}$. If $P(z)/F(z)$ is a constant, then the inequality (3.1) is obvious. We now assume that $P(z)/F(z)$ is not a constant so that by the maximum modulus principle, it follows that

$$|P(z)| < |F(z)| \ \ for \ |z| > 1 \ .$$

Suppose $F(z)$ has m zeros on $|z| = 1$ where $0 \leq m \leq n$ so that we can write

$$F(z) = F_1(z)F_2(z)$$

where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z| = 1$ and $F_2(z)$ is a polynomial of degree exactly $n - m$ having all its zeros in $|z| < 1$. This implies with the help of inequality (3.2) that

$$P(z) = P_1(z)F_1(z)$$

where $P_1(z)$ is a polynomial of degree at most $n - m$. Now, from inequality (3.2), we get

$$|P_1(z)| \leq |F_2(z)| \ \ for \ |z| = 1$$

where $F_2(z) \neq 0 \ for \ |z| = 1$. Therefore for every real or complex number λ with $|\lambda| > 1$, a direct application of Rouche's theorem shows that the zeros of the polynomial $P_1(z) - \lambda F_2(z)$ of degree $n - m \geq 1$ lie in $|z| < 1$. Hence the polynomial

$$f(z) = F_1(z)\left(P_1(z) - \lambda F_2(z)\right) = P(z) - \lambda F(z)$$

has all its zeros in $|z| \leq 1$ with at least one zero in $|z| < 1$, so that we can write

$$f(z) = (z - te^{i\delta})H(z)$$

where $t < 1$ and $H(z)$ is a polynomial of degree $n - 1$ having all its zeros in $|z| \leq 1$. Applying Lemma 2.1 to the polynomial $f(z)$, we obtain for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned}
|f(Re^{i\theta})| &= |Re^{i\theta} - te^{i\delta}||H(Re^{i\theta})| \\
&\geq |Re^{i\theta} - te^{i\delta}| \left(\frac{R+1}{r+1}\right)^{n-1} |H(re^{i\theta})| \\
&= \left(\frac{R+1}{r+1}\right)^{n-1} \frac{|Re^{i\theta} - te^{i\delta}|}{|re^{i\theta} - te^{i\delta}|} |(re^{i\theta} - te^{i\delta})H(re^{i\theta})| \\
&\geq \left(\frac{R+1}{r+1}\right)^{n-1} \left(\frac{R+t}{r+t}\right) |f(re^{i\theta})|.
\end{aligned}$$

This implies for $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$\left(\frac{r+t}{R+t}\right) |f(Re^{i\theta})| \geq \left(\frac{R+1}{r+1}\right)^{n-1} |f(re^{i\theta})|. \tag{3.3}$$

Since $R > r \geq 1 > t$ so that $f(Re^{i\theta}) \neq 0$ for $0 \leq \theta < 2\pi$ and $\frac{1+r}{1+R} > \frac{r+t}{R+t}$, from inequality (3.3), we obtain

$$|f(Re^{i\theta}| > \left(\frac{R+1}{r+1}\right)^n |f(re^{i\theta})| \quad R > r \geq 1 \ \ and \ \ 0 \leq \theta < 2\pi. \tag{3.4}$$

Equivalently,

$$|f(Rz)| > \left(\frac{R+1}{r+1}\right)^n |f(rz)|$$

for $|z| = 1$ and $R > r \geq 1$. Hence for every real or complex number α with $|\alpha| \leq 1$ and $R > r \geq 1$, we have

$$\begin{aligned} |f(Rz) - \alpha f(rz)| &\geq |f(Rz)| - |\alpha||f(rz)| \\ &> \left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\} |f(rz)|, \quad |z| = 1. \end{aligned} \tag{3.5}$$

Also, inequality (3.4) can be written in the form

$$|f(re^{i\theta)}| < \left(\frac{r+1}{R+1}\right)^n |f(Re^{i\theta})| \tag{3.6}$$

for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$. Since $f(Re^{i\theta}) \neq 0$ and $\left(\frac{r+1}{R+1}\right)^n < 1$, from inequality (3.6), we obtain for $0 \leq \theta < 2\pi$ and $R > r \geq 1$,

$$|f(re^{i\theta}| < |f(Re^{i\theta}).$$

Equivalently,

$$|f(rz)| < |f(Rz)| \ \ for \ \ |z| = 1.$$

Since all the zeros of $f(Rz)$ lie in $|z| \leq (1/R) < 1$, a direct application of Rouche's theorem shows that the polynomial $f(Rz) - \alpha f(rz)$ has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| \leq 1$. Applying Rouche's theorem again, it follows from (3.5) that for arbitrary real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$, all the zeros of the polynomial

$$\begin{aligned} T(z) &= f(Rz) - \alpha f(rz) + \beta \left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\} f(rz) \\ &= \left[P(Rz) - \alpha P(rz) + \beta \left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\} P(rz)\right] \\ &\quad - \lambda \left[F(Rz) - \alpha F(rz) + \beta \left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\} F(rz)\right] \\ &= [P(Rz) + \phi(R, r, \alpha, \beta)P(rz)] \\ &\quad - \lambda [F(Rz) + \phi(R, r, \alpha, \beta)F(rz)] \end{aligned}$$

lie in $|z| < 1$ where $|\lambda| > 1$. Using Lemma 2.2 and the fact that the operator B is linear, we conclude that all the zeros of polynomial

$$\begin{aligned} W(z) &= B[T(z)] \\ &= (B[P(Rz)] + \phi(R,r,\alpha,\beta)B[P(rz)]) \\ &\quad - \lambda(B[F(Rz)] + \phi(R,r,\alpha,\beta)B[F(rz)]) \end{aligned}$$

also lie in $|z| < 1$ for every λ with $|\lambda| > 1$. This implies

$$|B[P(Rz)]+\phi(R,r,\alpha,\beta)B[P(rz)]| \leq |B[F(Rz)]+\phi(R,r,\alpha,\beta)B[F(rz)]| \quad (3.7)$$

for $|z| \geq 1$ and $R > r \geq 1$. If inequality (3.7) is not true, then exist a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\begin{aligned} &|B[P(Rz)] + \phi(R,r,\alpha,\beta)B[P(rz)]|_{z=z_0} \\ &> |B[F(Rz)] + \phi(R,r,\alpha,\beta)B[F(rz)]|_{z=z_0}. \end{aligned}$$

But all the zeros of $F(Rz)$ lie in $|z| < 1$, therefore, it follows (as in case of $f(z)$) that all the zeros of $F(Rz) + \phi(R,r,\alpha,\beta)F(rz)$ lie in $|z| < 1$. Hence by Lemma 2.2, all the zeros of $B[F(Rz)]+\phi(R,r,\alpha,\beta)B[F(rz)]$ also lie in $|z| < 1$, which shows that

$$\{B[F(Rz)] + \phi(R,r,\alpha,\beta)B[F(rz)]\}_{z=z_0} \neq 0$$

with $|z_0| \geq 1$. We take

$$\lambda = \frac{[B[P(Rz)] + \phi(R,r,\alpha,\beta)B[P(rz)]]_{z=z_0}}{[B[F(Rz)] + \phi(R,r,\alpha,\beta)B[F(rz)]]_{z=z_0}},$$

then λ is a well defined real or complex number with $|\lambda| > 1$ and with this choice of λ, we obtain $W(z_0) = 0$ where $|z_0| \geq 1$. This contradicts the fact that all the zeros of $W(z)$ lie in $|z| < 1$. Thus

$$|B[P(Rz)] + \phi(R,r,\alpha,\beta)B[P(rz)]| \leq |B[F(Rz)] + \phi(R,r,\alpha,\beta)B[F(rz)]|$$

for $|z| \geq 1$ and $R > r \geq 1$. This proves the Theorem 3.1. □

A variety of interesting results can be deduced from Theorem 3.1 as special cases. Here we mentiopn a few of these.

The following interesting result, which is a compact generalization of the inequalities (1.1), (1.2) and (1.13), follows from Theorem 3.1 by taking

$$F(z) = z^n \underset{|z|=1}{Max} |P(z)|.$$

Corollary 3.2. *If $P \in P_n$, then for all real or complex numbers α and β with $|\alpha| \leq 1, |\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} &|B[P(Rz)] + \phi(R,r,\alpha,\beta)\, B[P(rz)]| \\ &\leq |R^n + \phi(R,r,\alpha,\beta)\, r^n||B[z^n]| \underset{|z|=1}{Max} |P(z)| \end{aligned} \tag{3.8}$$

where $B \in B_n$ and $\phi(R,r,\alpha,\beta)$ is defined by (1.15)*. The result is best possible and equality in* (3.8) *holds for $P(z) = \lambda z^n, \lambda \neq 0$.*

The case $B[P(z)] = P(z)$ of Corollary 3.2 leads to:

Corollary 3.3. *If $P \in P_n$, then for all real or complex numbers α and β with $|\alpha| \leq 1, |\beta| \leq 1$, $R > r \geq 1$, and $|z| \geq 1$,*

$$|P(Rz) + \phi(R,r,\alpha,\beta)\, P(rz)| \leq |R^n + \phi(R,r,\alpha,\beta)\, r^n||z|^n| \underset{|z|=1}{Max} |P(z)|\,, \tag{3.9}$$

where $\phi(R,r,\alpha,\beta)$ is defined by (1.15)*. The result is best possible and equality in* (3.9) *holds for $P(z) = \lambda z^n, \lambda \neq 0$.*

Remark 3.4. For $\alpha = \beta = 0$ and $|z| = 1$, inequality (3.8) reduces to inequality (13). Further, if we take $\alpha = 1$ and divide the two sides of (3.9) by $R - r$, and make $R \to r$, we get for $r \geq 1$, $|\beta| \leq 1$ and $|z| \geq 1$,

$$\left| zP'(rz) + n\frac{\beta}{r+1} P(rz) \right| \leq n \left| r^{n-1} + \frac{\beta r^n}{r+1} \right| |z|^n \underset{|z|=1}{Max} |P(z)|\,,$$

which, in particular, includes inequality (1.1) as a special case.

Setting $\alpha = 0$ in (3.8), we obtain:

Corollary 3.5. *If $P \in P_n$, then for every real or complex number β with $|\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} &\left| B[P(Rz)] + \beta \left(\frac{R+1}{r+1}\right)^n B[P(rz)] \right| \\ &\leq \left| R^n + \beta \left(\frac{R+1}{r+1}\right)^n r^n \right| |B[z^n]| \underset{|z|=1}{Max} |P(z)|\,. \end{aligned} \tag{3.10}$$

where $B \in B_n$ and $\phi(R,r,\alpha,\beta)$ is defined by (1.15)*. The result is best possible and equality in* (3.10) *holds for $P(z) = \lambda z^n, \lambda \neq 0$.*

For $\beta = 0$, it follows from Corollary 3.2 that if $P \in P_n$, then for every real or complex number α *with* $|\alpha| \leq 1$, $R > r \geq 1$, and $|z| \geq 1$,

$$|B[P(rz)] - \alpha B[P(rz)]| \leq |R^n - \alpha r^n||B[z^n]| \underset{|z|=1}{Max} |P(z)|\,. \tag{3.11}$$

where $B \in B_n$. The result is best possible and equality in (3.11) holds for $P(z) = \lambda z^n, \lambda \neq 0$.

Next we establish the following result.

Theorem 3.6. *If $P \in P_n$, and $P(z)$ has all its zeros in $|z| \leq 1$, then for arbitrary real or complex numbers α and β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$*

$$\begin{aligned} &|B[P(Rz)] + \phi\,(R, r, \alpha, \beta)\, B[P(rz)]| \\ &\geq |R^n + \phi\,(R, r, \alpha, \beta)\, r^n|\, |B[z^n]| \underset{|z|=1}{Min} |P(z)| \quad for \;\; |z| \geq 1 \end{aligned} \tag{3.12}$$

where $B \in B_n$ and $\phi\,(R, r, \alpha, \beta)$ is defined by (1.15)*. The result is best possible and equality in* (3.12) *holds for $P(z) = \lambda z^n, \lambda \neq 0$.*

Proof. The result is clear if $P(z)$ has a zero on $|z| = 1$, for then $m = Min_{|z|=1} |P(z)| = 0$. We now assume that $P(z)$ has all its zeros in $|z| < 1$ so that $m > 0$ and

$$m \leq |P(z)| \qquad for \quad |z| = 1.$$

This gives for every λ with $|\lambda| < 1$,

$$|\lambda z^n|\, m \leq |P(z)| \qquad for \quad |z| = 1.$$

By Rouche's Theorem, it follows that all the zeros of the polynomial $F(z) = P(z) - \lambda m z^n$ lie in $|z| < 1$ for every real or complex number λ with $|\lambda| < 1$. Therefore(as in proof of the Theorem 1.1), we conclude that all the zeros of the polynomial $G(z) = F(Rz) + \phi\,(R, r, \alpha, \beta)\, F(rz)$ lie in $|z| < 1$ for arbitrary real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$. Hence by Lemma 2.2, all the zeros of the polynomial

$$\begin{aligned} S(z) = B[G(z)] &= B[F(Rz) + \phi\,(R, r, \alpha, \beta)\, F(rz)] \\ &= B[P(Rz)] + \phi\,(R, r, \alpha, \beta)\, B[P(rz)] \\ &\quad - \lambda\,(R^n + \phi\,(R, r, \alpha, \beta)\, r^n)\, B[z^n] m \end{aligned} \tag{3.13}$$

lie in $|z| < 1$ for all real or complex numbers α, λ with $|\alpha| \leq 1, |\lambda| < 1$ and $R > r \geq 1$. This implies for $|z| \geq 1$ and $R > r \geq 1$,

$$\begin{aligned} &|B[P(Rz)] + \phi\,(R, r, \alpha, \beta)\, B[P(rz)]| \\ &\geq |R^n + \phi\,(R, r, \alpha, \beta)\, r^n|\, |B[z^n]|\, m. \end{aligned} \tag{3.14}$$

If inequality (3.14) is not true, then there is a point $z = w$ with $|w| \geq 1$ such that

$$\begin{aligned} &|\{B[P(Rz)] + \phi\,(R, r, \alpha, \beta)\, B[P(rz)]\}_{z=w}| \\ &< |R^n + \phi\,(R, r, \alpha, \beta)\, r^n|\, |\{B[z^n]\}_{z=w}|\, m. \end{aligned}$$

Since $\{B[z^n]\}_{z=w} \neq 0$, we take

$$\lambda = \frac{\{B[P(Rz)] + \phi\,(R, r, \alpha, \beta)\, B[P(rz)]\}_{z=w}}{m\,(R^n + \phi\,(R, r, \alpha, \beta)\, r^n)\, \{B[z^n]\}_{z=w}}$$

so that λ is a well defined real or complex number with $|\lambda| < 1$ and with this choice of λ, from (3.13), we get $S(w) = 0$ with $|w| \geq 1$. This contradicts the fact that all the zeros of $S(z)$ lie in $|z|<1$. Thus for arbitrary real or complex numbers α,β with $|\alpha|\le 1, |\beta|\le 1, R>r\ge 1$ and $|z|\ge 1$,

$$\begin{aligned}&|B[P(Rz)]+\phi\,(R,r,\alpha,\beta)\,B[P(rz)]|\\ &\ge |R^n+\phi\,(R,r,\alpha,\beta)\,r^n|\,|B[z^n]|\underset{|z|=1}{Min}|P(z)|\,.\end{aligned}$$

This completes the proof of Theorem 3.6 □

The case $B[P(z)]=P(z)$ of Theorem 3.6 yields:

Corollary 3.7. *If $P\in P_n$, and $P(z)$ has all its zeros in $|z|\le 1$, then for arbitrary real or complex numbers α and β with $|\alpha|\le 1, |\beta|\le 1$ and $R>r\ge 1$*

$$\begin{aligned}&|P(Rz)+\phi\,(R,r,\alpha,\beta)\,P(rz)|\\ &\ge |R^n+\phi\,(R,r,\alpha,\beta)\,r^n|\,|z|^n\underset{|z|=1}{Min}|P(z)| \quad for\ \ |z|\ge 1\end{aligned} \tag{3.15}$$

where $\phi\,(R,r,\alpha,\beta)$ is defined by (1.15)*. The result is best possible and equality in* (3.15) *holds for $P(z)=\lambda z^n, \lambda\neq 0$.*

If we divide the two sides of (3.15) by $R-r$ with $\alpha=1$ and let $R\to r$, we get for $P(z)=0$ in $|z|\le 1$, $|\beta|\le 1$, and $r\ge 1$

$$\underset{|z|=1}{Min}\left|zP'(rz)+n\frac{\beta}{r+1}P(rz)\right|\ge n\left|r^{n-1}+\frac{\beta r^n}{r+1}\right|\underset{|z|=1}{Min}|P(z)|\,.$$

The result is best possible.

The next corollary follows by taking $\beta=0$ in (3.12).

Corollary 3.8. *If $P\in P_n$ and $P(z)$ has all its zeros in $|z|\le 1$, then for every real or complex number α with $|\alpha|\le 1$, $R>r\ge 1$ and $|z|\ge 1$,*

$$|B[P(Rz)]-\alpha B[P(rz)]|\ge |R^n-\alpha r^n|\,|B[z^n]|\underset{|z|=1}{Min}|P(z)| \tag{3.16}$$

where $B\in B_n$. The result is best possible and equality in (3.16) holds for $P(z)=\lambda z^n, \lambda\neq 0$.

For $\alpha=0$, it follows from Corollary 3.8 that *if $P\in P_n$ and $P(z)$ has all its zeros in $|z|\le 1$, then for every real or complex number α with $|\alpha|\le 1$, $R>r\ge 1$ and $|z|\ge 1$,*

$$|B[P(Rz)]|\ge |B[R^n z^n]|\underset{|z|=1}{Min}|P(z)| \tag{3.17}$$

where $B\in B_n$. The result is sharp.

Remark 3.9. For the choice $\beta = \lambda_1 = \lambda_2 = 0$ in (3.12), we obtain for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,

$$|P(Rz) - \alpha P(rz)| \geq |R^n - \alpha r^n| \, |z|^n \underset{|z|=1}{Min} |P(z)| \,, \tag{3.18}$$

which, in particular, includes a compact generalization of the inequalities (1.3) and (1.4) as a special case.

Next, for the choice $\alpha = 0$ in (3.12), we get the following result.

Corollary 3.10. *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every real or complex number β with $|\beta| \leq 1$, $|z| = 1$ and $R > r \geq 1$,*

$$\begin{aligned} &\underset{|z|=1}{Min} \left| B[P(Rz)] + \beta \left(\frac{R+1}{r+1} \right)^n B[P(rz)] \right| \\ &\geq \left| R^n + \beta \left(\frac{R+1}{r+1} \right)^n r^n \right| |B[z^n]| \underset{|z|=1}{Max} |P(z)| \end{aligned} \tag{3.19}$$

where $B \in B_n$. The result is best possible.

Setting $\lambda_0 = \lambda_2 = 0$ in (3.12) and noting that all the zeros of $u(z)$ defined by (1.11) lie in the half plane (1.12), we get

Corollary 3.11. *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for arbitrary real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} &\left| RP'(Rz) + \phi\left(R, r, \alpha, \beta\right) rP'(rz) \right| \\ &\geq n \left| R^n + \phi\left(R, r, \alpha, \beta\right) r^n \right| |z|^{n-1} \underset{|z|=1}{Min} |P(z)| \,. \end{aligned} \tag{3.20}$$

The result is sharp and the extremal polynomial is $P(z) = \lambda z^n, \lambda \neq 0$.

Finally we prove the following compact generalization of the inequalities (1.3), (1.4), (1.5) and (1.6), which also include refinements of the inequalities (1.9) and (1.14) as special cases.

Theorem 3.12. *If $P \in P_n$ and $P(z) \neq 0$ for $|z| < 1$, then for arbitrary real or complex numbers α and β with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $|z| \geq 1$.*

$$\begin{aligned} &|B[P(Rz)] + \phi(R, r, \alpha, \beta) B[P(rz)]| \\ &\leq \frac{1}{2}\{|R^n + \phi(R, r, \alpha, \beta) r^n| |B[z^n]| \\ &\quad + |1 + \phi(R, r, \alpha, \beta)| |\lambda_0|\} \underset{|z|=1}{Max} |P(z)| \\ &\quad - \frac{1}{2}\{|R^n + \phi(R, r, \alpha, \beta) r^n| |B[z^n]| \\ &\quad - |1 + \phi(R, r, \alpha, \beta)| |\lambda_0|\} \underset{|z|=1}{Min} |P(z)| \end{aligned} \tag{3.21}$$

where $B \in B_n$. The result is sharp and equality in (3.21) *holds for $P(z) = az^n + b$, $|a| = |b| = 1$.*

Proof. By hypothesis, the polynomial $P(z)$ does not vanish in $|z| < 1$, therefore if $m = Min_{|z|=1}|P(z)|$, then $m \leq |P(z)|$ for $|z| \leq 1$. We first show that for every real or complex number δ with $|\delta| \leq 1$, the polynomial $H(z) = P(z) + m\delta z^n$ does not vanish in $|z| < 1$. This is obvious if $m = 0$ and for $m > 0$, we prove it by a contradiction. Assume that $H(z)$ has a zero in $|z| < 1$ say at $z = w$ with $|w| < 1$, then we have $P(w) + m\delta w^n = H(w) = 0$. This gives

$$|P(w)| = |m\delta w^n| \leq m|w|^n < m,$$

which is clearly a contradiction to the minimum modulus principle. Hence $H(z)$ has no zero in $|z| < 1$ for every real or complex number δ with $|\delta| \leq 1$. If $G(z) = z^n\overline{H(1/\overline{z})}$,then all the zeros of nth degree polynomial $G(z)$ lie in $|z| \leq 1$. Applying Lemma 2.3 with $P(z)$ replaced by $H(z)$ and $F(z)$ by $G(z)$, we obtain for arbitrary real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $|z| \geq 1$,

$$|B[H(Rz)] + \phi(R, r, \alpha, \beta) B[H(rz)]| \leq |B[G(Rz)] + \phi(R, r, \alpha, \beta) B[G(rz)]|,$$

where now $G(z) = z^n\overline{P(1/\overline{z})} - m\overline{\delta} = Q(z) - m\overline{\delta}, Q(z) = z^n\overline{P(1/\overline{z})}$. Equivalently,

$$\begin{aligned} &|B[P(Rz)] + \phi(R, r, \alpha, \beta) B[P(rz)] - m\delta (R^n + \phi(R, r, \alpha, \beta) r^n) B[z^n]| \\ &\leq \left|B[Q(Rz)] + \phi(R, r, \alpha, \beta) B[Q(rz)] - m\overline{\delta}(1 + \phi(R, r, \alpha, \beta)) \lambda_0\right| \end{aligned} \tag{3.22}$$

for all real or complex numbers α, β, δ with $|\alpha| \leq 1, |\beta| \leq 1, |\delta| \leq 1$ and $R > r \geq 1$. Now choosing the argument of δ such that

$$\begin{aligned} &|B[P(Rz)] + \phi(R, r, \alpha, \beta) B[P(rz)] - m\delta (R^n + \phi(R, r, \alpha, \beta) r^n) B[z^n]| \\ &= |B[P(Rz)] + \phi(R, r, \alpha, \beta) B[P(rz)]| + m|\delta| |R^n + \phi(R, r, \alpha, \beta) r^n| |B[z^n]|, \end{aligned}$$

we obtain from (3.22),

$$\begin{aligned}&|B[P(Rz)]+\phi(R,r,\alpha,\beta)B[P(rz)]|+m|\delta|\,|R^n+\phi(R,r,\alpha,\beta)r^n|\,|B[z^n]|\\&\le|B[Q(Rz)]+\phi(R,r,\alpha,\beta)B[Q(rz)]|+m|\delta|\,|1+\phi(R,r,\alpha,\beta)|\,|\lambda_0|,\end{aligned}$$

for $|z|\ge 1$. Equivalently,

$$\begin{aligned}&|B[P(Rz)]+\phi(R,r,\alpha,\beta)B[P(rz)]|\\&\quad+|\delta|\left(|R^n+\phi(R,r,\alpha,\beta)r^n|\,|B[z^n]|-|1+\phi(R,r,\alpha,\beta)|\,|\lambda_0|\right)m\\&\le|B[Q(Rz)]+\phi(R,r,\alpha,\beta)B[Q(rz)]|,\end{aligned}$$

for $|\alpha|\le 1, |\beta|\le 1, |\delta|\le 1$ and $R>r\ge 1$. Letting $|\delta|\to 1$, we get

$$\begin{aligned}&|B[P(Rz)]+\phi(R,r,\alpha,\beta)B[P(rz)]|\\&\quad+\left(|R^n+\phi(R,r,\alpha,\beta)r^n|\,|B[z^n]|-|1+\phi(R,r,\alpha,\beta)|\,|\lambda_0|\right)m\\&\le|B[Q(Rz)]+\phi(R,r,\alpha,\beta)B[Q(rz)]|\end{aligned}\tag{3.23}$$

for $|\alpha|\le 1, |\beta|\le 1, |\delta|\le 1$ and $R>r\ge 1$. Combining this inequality with Lemma 2.4, we get for all real or complex numbers α,β with $|\alpha|\le 1, |\beta|\le 1, R>r\ge 1$ and $|z|\ge 1$,

$$\begin{aligned}&2\,|B[P(Rz)]+\phi(R,r,\alpha,\beta)B[P(rz)]|\\&\quad+\left(|R^n+\phi(R,r,\alpha,\beta)r^n|\,|B[z^n]|-|1+\phi(R,r,\alpha,\beta)|\,|\lambda_0|\right)m\\&\le|B[P(Rz)]+\phi(R,r,\alpha,\beta)B[P(rz)]|+|B[Q(Rz)]+\phi(R,r,\alpha,\beta)B[Q(rz)]|,\\&\le\left(|R^n+\phi(R,r,\alpha,\beta)r^n|\,|B[z^n]|+|1+\phi(R,r,\alpha,\beta)|\,|\lambda_0|\right)Max_{|z|=1}|P(z)|.\end{aligned}$$

Equivalently,

$$\begin{aligned}&|B[P(Rz)]+\phi(R,r,\alpha,\beta)B[P(rz)]|\\&\le\frac{1}{2}\left\{|R^n+\phi(R,r,\alpha,\beta)r^n|\,|B[z^n]|+|1+\phi(R,r,\alpha,\beta)|\,|\lambda_0|\right\}Max_{|z|=1}|P(z)|\\&\quad-\frac{1}{2}\left\{|R^n+\phi(R,r,\alpha,\beta)r^n|\,|B[z^n]|-|1+\phi(R,r,\alpha,\beta)|\,|\lambda_0|\right\}Min_{|z|=1}|P(z)|.\end{aligned}$$

This completes the proof of Theorem 3.12 □

The following result is an immediate consequence of Theorem 3.12.

Corollary 3.13. *If $P\in P_n$ and $P(z)\ne 0$ for $|z|<1$, then for every real or complex number α with $|\alpha|\le 1, R>r\ge 1$ and $|z|\ge 1$,*

$$\begin{aligned}&|B[P(Rz)]-\alpha B[P(rz)]|\\&\le\frac{1}{2}\left\{|R^n-\alpha r^n|\,|B[z^n]|+|1-\alpha|\,|\lambda_0|\right\}\underset{|z|=1}{Max}|P(z)|\\&\quad-\frac{1}{2}\left\{|R^n-\alpha r^n|\,|B[z^n]|-|1-\alpha|\,|\lambda_0|\right\}\underset{|z|=1}{Min}|P(z)|\end{aligned}\tag{3.24}$$

where $B \in B_n$. The result is best possible.

Taking $\alpha = 0$ in Corollary 3.13, it follows that *if $P \in P_n$ and $P(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} |B[P(Rz)]| \leq & \frac{1}{2}\left(|B[R^n z^n]| + |\lambda_0|\right) \underset{|z|=1}{Max}|P(z)| \\ & - \frac{1}{2}\left(|B[R^n z^n]| - |\lambda_0|\right) \underset{|z|=1}{Min}|P(z)|. \end{aligned} \tag{3.25}$$

The result is best possible. Clearly (3.25) is a refinement of inequality (1.14).

Next, if we choose $\lambda_0 = \lambda_2 = 0$ in (3.21) and note that all the zeros of $u(z)$ defined by (1.11) lie in the half plane (1.12), we get for $|\alpha| \leq 1, |\beta \leq 1, |z| \geq 1$ and $R > r \geq 1$,

$$\begin{aligned} & \left|RP'(Rz) + \phi(R, r, \alpha, \beta)\, rP'(rz)\right| \\ & \leq \frac{n}{2}\left|R^n + \phi(R, r, \alpha, \beta)\, r^n\right| |z|^{n-1} \left\{\underset{|z|=1}{Max}|P(z)| - \underset{|z|=1}{Min}|P(z)|\right\}. \end{aligned} \tag{3.26}$$

which, in particular, gives inequality (1.7). For $\beta = 0$ (3.26) reduces to

$$\left|RP'(Rz) - \alpha rP'(rz)\right| \leq \frac{n}{2}\left|R^n - \alpha r^n\right| |z|^{n-1} \left\{\underset{|z|=1}{Max}|P(z)| - \underset{|z|=1}{Min}|P(z)|\right\}.$$

Also for $\alpha = 0$, Theorem 3.12 yields the following result.

Corollary 3.14. *If $P \in P_n$ and $P(z) \neq 0$ for $|z| < 1$, then for every real or complex number β with $|\beta| \leq 1, R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} & \left|B[P(Rz)] + \beta\left(\frac{R+1}{r+1}\right)^n B[P(rz)\right| \\ & \leq \frac{n}{2}\left\{\left|R^n + \beta\left(\frac{R+1}{r+1}\right)^n r^n\right| |B[z^n]| + \left|1 + \beta\left(\frac{R+1}{r+1}\right)^n\right| |\lambda_0|\right\} \underset{|z|=1}{Max}|P(z)| \\ & \quad - \frac{n}{2}\left\{\left|R^n + \beta\left(\frac{R+1}{r+1}\right)^n r^n\right| |B[z^n]| - \left|1 + \beta\left(\frac{R+1}{r+1}\right)^n\right| |\lambda_0|\right\} \underset{|z|=1}{Min}|P(z)|. \end{aligned}$$

Next choosing $\lambda_1 = \lambda_2 = 0$ in (3.21), we immediately get the following result, which is a refinemnet of inequality (1.9).

Corollary 3.15. *If $P \in P_n$ and $P(z) \neq 0$ for $|z| < 1$, then for all real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq 1$ and $|z| \geq 1$.*

$$\begin{aligned} &|P(Rz) + \phi(R,r,\alpha,\beta) P(rz)| \\ &\leq \frac{1}{2}\{|R^n + \phi(R,r,\alpha,\beta) r^n| |z|^n + |1 + \phi(R,r,\alpha,\beta)|\} \underset{|z|=1}{Max}|P(z)| \\ &\quad - \frac{1}{2}\{|R^n + \phi(R,r,\alpha,\beta) r^n| |z|^n - |1 + \phi(R,r,\alpha,\beta)|\} \underset{|z|=1}{Min}|P(z)|. \end{aligned} \tag{3.27}$$

The result is sharp and equality in (3.27) *holds for $P(z) = az^n + b, |a| = |b| = 1$.*

Dividing the two sides (3.27) by $R - r$ with $\alpha = 1$ and making $R \to r$, we obtain for $|\beta| \leq 1$, $|z| \geq 1$ and $r \geq 1$,

$$\begin{aligned} \left|zP'(rz) + n\frac{\beta}{r+1}P(rz)\right| &\leq \frac{n}{2}\left\{\left|r^{n-1} + n\beta\frac{r^n}{r+1}\right||z|^n + \left|\frac{\beta}{r+1}\right|\right\}\underset{|z|=1}{Max}|P(z)| \\ &\quad - \frac{n}{2}\left\{\left|r^{n-1} + n\beta\frac{r^n}{r+1}\right||z|^n - \left|\frac{\beta}{r+1}\right|\right\}\underset{|z|=1}{Min}|P(z)|. \end{aligned}$$

This inequality reduces to inequality (1.7) for $\beta = 0$ and $r = 1$.

References

[1] N.C. Ankeny and T.J. Rivlin, *On a theorm of S.Bernstein*, Pacific J. Math., **5** (1955), 849–852.

[2] A. Aziz and Q.M. Dawood, *Inequality for polynomials and its derivative*, Approx. Theory, **54** (1988), 306–313.

[3] A. Aziz and N.A. Rather, *On an inequality of S. Bernstein and Gauss-Lucas theorem*, Analytic and Geometric inequalities, Kluwer Acad. Pub., (1999), 29–35.

[4] A. Aziz and N.A. Rather, *Some New Generalizations of Zygmund-Type Inequalities for Polynomials*, Mathematical Inequalities and Applications, **15** (2011), 469–486.

[5] S. Bernstein, *Sur ĺ ordre de la meilleure approximation des fonctions par des polynomes de degré donné*, Momoires dé Í Académic Royal de Belgique, **4** (1912), 1–103.

[6] P.D. Lax, *Proof of a conjecture of P.Erdös on the derivative of a polynomial*, Bull. Amer. Math. Soc., **50** (1944), 509–513.

[7] M. Marden, *Geometry of polynomials*, Math. Surveys, No. **3**, Amer. Math. Soc. Providence, RI 1949.

[8] G.V. Milovanovic, D.S. Mitrinovic and Th.M. Rassias, *Topics in Polynomials: Extremal Properties, Inequalities, Zeros*, World scientific Publishing Co., Singapore, (1994).

[9] G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Vol.**1**, Springer New York, 1972.

[10] Q.I. Rahman, *Functions of exponential type*, Trans. Amer. Soc., **135** (1969), 295–309.

[11] Q.I. Rahman and G. Schmessier, *Analytic theory of polynomials*, Claredon Press, Oxford, 2002.

[12] M. Riesz, *Über einen satz des Herrn Serge Bernstein*, Proc. Acta Math., **40** (1916), 337–347.