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PARAMETER IDENTIFICATION OF AN INTEGRODIFFERENTIAL EQUATION

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Abstract. In this paper, we study an inverse problem of reconstructing time independent coefficient of an integrodifferential equation from the final time overspecified data. The reconstruction of the parameter is transformed to a minimization problem through optimal control frame work. The stability estimate for the coefficient with the upper bound in terms of the final measurement derived through the minimization of the cost functional.

1. Introduction

The theory and applications of integrodifferential equations play an important role in the mathematical modeling of many fields: physical, biological phenomena and engineering sciences in which it is necessary to take into account the effect of real world problems. The advantage of the integrodifferential equation's representation for a variety of problems is witnessed by its increasing frequency in the literature and in many texts on method of advanced applied mathematics. Also, the suitability of the solution method for

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machine computation and the inherent simplicity of the structure of the subject combine to make the integrodifferential equation approach a very valuable one for many applications [21, 22].

Heat conduction problems have many applications in science and engineering, for example crystal growing [16], material structure control [26] and integrated circuit packaging. The heat spread over medium depends not only on the geometry and physical properties of medium, but also primarily on heat transfer from it and other external sources. The product of a convective heat transfer coefficient and the temperature difference between the medium surface and its surrounding circumferences is termed as the convective heat flux. The convective heat transfer coefficient is an important parameter in modelling the heat transfer processes. However the dynamical representation of the heat conduction process is modelled very well by including the past history of one or more variables through memory kernel. This phenomenon is governed by parabolic integrodifferential equations with time dependent memory kernel when the medium is homogeneous and space dependent memory kernel when the medium is heterogeneous. This type of dynamical system appear in chemical diffusion, thermoelasticity [4, 5], heat conduction process [23], nuclear reactor dynamics and medical sciences [24].

In the last few years, various methods are used for the identification of heat source terms, of unknown boundary conditions, of memory kernel or spatially varying function. The identification of the convective coefficient using the different kinds of temperature measurements by means of inverse heat conduction is important to understand the heat conduction of the medium. Various problems to identify time-dependent memory kernels in parabolic equations have been studied in a number of papers. In the present paper we study an inverse problem to determine the spatially varying parameter $q(x)$ in a divergence form parabolic integrodifferential equation governing the heat flow in materials with memory. It is interesting to note that analyzing the interior structure of that medium through recovering parameters of media by boundary measurements have a broad range of applications in geophysics, searching minerals, oil and gas, nondestructive evaluation of materials etc.

In this paper, we consider the inverse problem of the following integrodifferential equation [32]

$$\begin{cases} y_t - \nabla(d(x)\nabla y) + \int_0^t K(t,\tau) y(x,\tau)\,\mathrm{d}\tau + q(x)y = f(x,t), & (x,t) \in \Omega_T, \\ y(x,t) = 0, & (x,t) \in \Sigma, \\ y(x,0) = y_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega_T = \Omega \times (0, T]$, $\Sigma = \partial\Omega \times (0, T]$, the domain $\Omega = (0, \ell)$ and $T > 0$ is an arbitrary but fixed moment of time. The initial conditions $y_0(x)$ is sufficiently regular and the unknown coefficient $q(x)$ is assumed to be sufficiently smooth and shall be kept independent of time t. Here we assume that the diffusive coefficient d is dependent of space only. Suppose we assume that there is a possibility to provide the additional temperature for the inverse heat problems at final time

$$y(x, T) = m(x) \quad \text{for all } x \in \Omega, \tag{1.2}$$

where the given function $m(x)$ satisfy the homogeneous Dirichlet boundary conditions. The goal of this article is to obtain the stability estimates for this inverse problem of determining coefficient $q(x)$ in the integrodifferential equation with respect to the solution at final time. The key ingredient to these stability estimates is an optimal control framework to the system.

Different kinds of methods have been developed for the analysis of the inverse heat conduction and reported in the literature. Romanov [27] studied the identification of the spatially varying kernel in an integrodifferential equation of electrodynamics and also proved that for sufficiently large time interval conditional stability of the solution of inverse problem. Denisov and Shores [11] analyzed the existence and uniqueness of solutions to the inverse problem of reconstruction of spatially varying absorbing parameter in the one dimensional model of adsorption dynamics through monotone operator method by transforming the system of equations into integrodifferential equation from final time observations. In [29], the principle of invariant imbedding has been applied to the identification of reflection coefficient in the nonlinear integrodifferential model of scattering process. Xiao *et al.*, [31] studied the existence and uniqueness of solutions for the direct problem as well as the existence of quasisolutions of the inverse source problem of integrodifferential parabolic equations, which comes from nonlinear pollution problems in porous media in an appropriate class of admissible source functions. Colombo [6] established the local time existence and global time uniqueness for the solution of the identification of the convolution memory kernel K and the temperature in the evolution equation from nonlocal overspecified data by treating the evolution problem an abstract inverse problem. Baranibalan *et.al.*, [2] established the stability and uniqueness results for the coefficient q from a measurement of the solution with respect to the normal derivative on an arbitrary part of the boundary and certain spatial derivatives through the Carleman estimates and certain energy estimates for parabolic equations with memory. Favini and Lorenzi [14, 15] proved a global existence and uniqueness result for the recovery of unknown scalar kernels in linear singular first-order integrodifferential initial-boundary value problems and recovered unknown kernels, depending on time only, in linear singular first-order integrodifferential Cauchy problems in

Banach spaces. Perthame and Zubelli [25] considered a size-structured model for cell division and address the question of determining the division (birth) rate from the measured stable size distribution of the population, which is executed by novel regularization technique on an integrodifferential equation on the half line along with the regular dependence theory for the solution in terms of the coefficients. Durdiev and Rashidov [12] studied the inverse problem of determining the multidimensional kernel of the integral term in a parabolic equation of second order. Wu and Yu [30] studied the inverse problem for an integrodifferential equation related to the Basset problem.

Apart from the literature mentioned above for inverse problems for parabolic integrodifferential equations, it should be noted that, to the best of our knowledge, the optimization technique plays a crucial role in proving the results for inverse problems and has the following advantages over other methods. The optimization technique is a classical tool to yield general solution for inverse problems without unique solution. The basic idea is to restrict the solution under consideration to some compact set and then take the minimizer of some cost functional as the general solution.The idea of the minimization of the cost functional can work well when good approximation for the exact solution is known in advance. As far as the method of optimal control for inverse problems is concerned, Hoffman and Jiang [19] investigated an inverse problem of reconstructing a source term in a phase field model for solidification. An inverse problem of recovering the implied volatility coefficient in the Black-Scholes type equation has been studied by Jiang and Tao [20]. In the absence of the memory term ($K = 0$), problem (1.1) has been studied by various authors and several results concerning inverse problem have been established. Chen and Liu [3] investigated the numerical reconstruction of the coefficient $q(x)$ in the parabolic equation $u_t - \Delta u + q(x)u = 0$ from the final measurement by using the optimization method combined with the finite element method. After these contributions to the study of various types of inverse problems via optimal control framework, lot of papers started appearing in the literature for single [7, 8, 9, 10, 33, 35] and coupled system of equations [17, 18, 28] from various type of overspecified measurement data.

The brief description of our work is as follows: Let $\tilde{y}(x,t)$ be the solution of the following equation

$$\begin{cases} \tilde{y}_t - \nabla(d(x)\nabla\tilde{y}) + \int\limits_0^t K(t,\tau)\tilde{y}(x,\tau)\,\mathrm{d}\tau + \tilde{q}(x)\tilde{y} = f(x,t), & (x,t) \in \Omega_T, \\ \tilde{y}(x,t) = 0, & (x,t) \in \Sigma, \\ \tilde{y}(x,0) = y_0(x), & x \in \Omega. \end{cases} \tag{1.3}$$

Set $Y = y - \tilde{y}$ and $Q = q - \tilde{q}$ so that the subtraction of (1.3) from (1.1) yields

$$\begin{cases} Y_t - \nabla(d(x)\nabla Y) + \int\limits_0^t K(t,\tau)Y(x,\tau)\,\mathrm{d}\tau + q(x)Y = -Q\tilde{y}, & (x,t)\in\Omega_T, \\ Y(x,t) = 0, & (x,t)\in\Sigma, \\ Y(x,0) = 0, & x\in\Omega. \end{cases} \tag{1.4}$$

More preciously, let y and $\tilde{y}$ be the solutions of the systems (1.1) and (1.3) respectively; then for all T, there exists a constant $C > 0$ depending only on T and Ω satisfying

$$\max |Q|^2 \le C \int\limits_\Omega |m - \tilde{m}|^2 \,\mathrm{d}x$$

where m and $\tilde{m}$ are the values of the solutions of the systems (1.1) and (1.3) at final time $t = T$, that is $y(x,T) = m(x)$ and $\tilde{y}(x,T) = \tilde{m}(x)$.

The outline of the paper is as follows: In Section 2, we transform the given problem into an optimal control problem by using the optimization theory. In Section 3 we prove the existence of minimizer and estabilished the first order necessary optimality conditions. Making use of the necessary conditions and energy estimates, we complete the stability result in Section 4.

2. Optimal control problem

Consider the optimal control problem with the classes of admissible set

$$\mathcal{M} = \left\{ q(x) \mid 0 < q_0 \le q(x) \le q_1 < \infty \text{ and } \nabla q \in L^2(\Omega) \right\}. \tag{2.1}$$

The constants q_0 and q_1 are lower and upper bounds for the unknown $q(x)$. The coefficients $d(x)$ and $q(x)$, initial condition $y_0(x)$, source term $f(x,t)$ and memory kernel $K(t,\tau)$ satisfy the following assumptions:

Assumption 2.1. *For $\alpha > 0$,*

(A1) $q(x) \in \mathcal{C}^\alpha(\overline{\Omega}),\ K(t,\tau) \in \mathcal{C}^{\frac{\alpha}{2},\frac{\alpha}{2}}([0,T]\times[0,T]),\ f(x,t) \in \mathcal{C}^{\alpha,\frac{\alpha}{2}}(\Omega_T)$,
(A2) $m(x) \in L^2(\Omega),\ y_0(x) \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$,
(A3) $0 < d_0 \le d(x) \le d_1 < \infty$ *in* Ω *and* $d(x) \in L^\infty(\Omega)$.

For the definitions of the spaces $\mathcal{C}^\alpha(\Omega),\ \mathcal{C}^{\alpha,\beta}(\Omega_T),\ H^1(\Omega),\ L^2(\Omega)$ and $L^\infty(\Omega)$, one can refer the books [1] and [13].

Now we define the optimal control problem (P) as follows: Find $\tilde{q}(x) \in \mathcal{M}$ satisfying

$$\mathcal{J}(\tilde{q}) = \min_{q\in\mathcal{M}} \mathcal{J}(q), \tag{2.2}$$

where

$$\mathcal{J}(q) = \frac{1}{2}\int\limits_\Omega |y(x,T;q) - m(x)|^2 \,\mathrm{d}x + \frac{N}{2}\int\limits_\Omega |\nabla q|^2 \,\mathrm{d}x, \tag{2.3}$$

y is the solution of the equation (1.1) for the given coefficient $q(x) \in \mathcal{M}$ and N is the regularization parameter.

In order to analyze the inverse problem for the differential equations, the knowledge of the direct problem is essential. By the well-known Schauder's theory for parabolic equation, one can provide following existence result [34].

Theorem 2.2. *Let $0 < \alpha < 1$ and the coefficient $q(x) \in \mathcal{C}^{\alpha}(\Omega)$. Then the system has a unique solution $y(x,t) \in \mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\Omega_T)$ provided that $y_0(x) \in \mathcal{C}^{2+\alpha}(\Omega)$.*

The following theorem proves the existence of the optimal control $\tilde{q}(x) \in \mathcal{M}$ minimizing the cost functional $\mathcal{J}(q)$.

Theorem 2.3. *Given $y_0(x) \in \mathcal{C}^{2+\alpha}(\Omega)$ and $q(x) \in L^2(\Omega)$ there exists a solution $\tilde{q} \in \mathcal{M}$ and $\tilde{y}(x,t) \in \mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}(\Omega_T)$ of the minimization problem*

$$\mathcal{J}(\tilde{q}) = \min_{q \in \mathcal{M}} \mathcal{J}(q).$$

Proof. From the definition of $\mathcal{J}(q)$, note that the function is nonnegative and thus it has the greatest lower bound. Let $\{y_n, q_n\}$ be the minimizing sequence of the minimization problem.

$$A = \inf_{q \in \mathcal{M}} \mathcal{J}(q) \leq \frac{1}{2} \int_{\Omega} |y_n(x,T;q) - m(x)|^2 \, \mathrm{d}x + \frac{N}{2} \int_{\Omega} |\nabla q_n|^2 \, \mathrm{d}x < A + \frac{1}{n}.$$

From this, $\{q_n\}$ is bounded in $L^2(\Omega)$ and $\|\nabla q_n\|_{L^2(\Omega)} < C$, where C is independent of n. Then by the classical Sobolev imbedding $H^1(\Omega) \hookrightarrow \mathcal{C}^{\alpha}(\Omega)$, $\|q_n\|_{\mathcal{C}^{\frac{1}{2}}(\Omega)} \leq C$. Thus by classical solution of parabolic equation, using energy type estimate, we will show that the corresponding solution y_n is bounded on $\mathcal{C}^{2+\frac{1}{2},1+\frac{1}{4}}(\Omega_T)$. More precisely, consider the equation (1.1) with the state variable y_n and the coefficient q_n, we have

$$\|y_n\|_{\mathcal{C}^{2+\frac{1}{2},1+\frac{1}{4}}(\Omega_T)} \leq C.$$

Then there exists a subsequence $\{y_{n_l}, q_{n_l}\}$ such that

$$q_{n_l} \to \tilde{q} \in \mathcal{C}^{\frac{1}{2}}(\Omega) \quad \text{uniformly on } \mathcal{C}^{\alpha}(\Omega),$$
$$y_{n_l} \to \tilde{\phi} \quad \text{uniformly on } \mathcal{C}^{\alpha,\frac{\alpha}{2}}(\Omega_T) \cap \mathcal{C}^{2+\alpha,1+\frac{\alpha}{2}}_{\text{loc}}(\Omega_T).$$

Hence replacing (y,q) in (1.1) by (y_{n_l}, q_{n_l}) and passing the limits one can see that $(\tilde{\phi}, \tilde{q})$ satisfies (1.1). Now consider

$$\int_{\Omega} |\nabla(q_{n_l} - \tilde{q})|^2 \, \mathrm{d}x \quad \geq \quad 0.$$

It follows that

$$\begin{aligned}\lim_{l\to\infty}\mathcal{J}(q_{n_l}) &= \lim_{l\to\infty}\frac{1}{2}\int_\Omega |y_{n_l}-m(x)|^2\,\mathrm{d}x+\lim_{l\to\infty}\frac{N}{2}\int_\Omega |\nabla q_{n_l}|^2\,\mathrm{d}x\\ &\geq \frac{1}{2}\int_\Omega |y(x,T,q)-m(x)|^2\,\mathrm{d}x+\frac{N}{2}\int_\Omega|\nabla\tilde q|^2\,\mathrm{d}x=\mathcal{J}(\tilde q).\end{aligned}$$

From this observation

$$\min_{q\in\mathcal{M}}\mathcal{J}(q)\leq\mathcal{J}(\tilde q)\leq\lim_{n\to\infty}\inf\mathcal{J}(q_{n_l})=\min_{q\in\mathcal{M}}\mathcal{J}(q).$$

Hence

$$\mathcal{J}(\tilde q)=\min_{q\in\mathcal{M}}\mathcal{J}(q).$$

Thus, $\tilde q = q$ is an optimal solution of the control problem. □

Now we obtain the necessary optimality conditions which have to be satisfied by each optimal control q. Suppose p is the solution of the adjoint system associated with (1.1) of the form

$$\begin{cases}-p_t-\nabla(d(x)\nabla p)+\int\limits_t^T K(t,\tau)p(x,\tau)\,\mathrm{d}\tau+qp=0, & (x,t)\in\Omega_T,\\ p(x,t)=0, & (x,t)\in\Sigma,\\ p(x,T)=y(x,T)-m(x), & x\in\Omega,\end{cases}\tag{2.4}$$

where m is the value of the solution of the system (1.1) at the final time $t=T$.

Theorem 2.4. *Let $\tilde q$ be the solution of the optimal control problem (P). Then there exists a set of functions (y,p,q) satisfying*

$$\int_0^T\int_\Omega py(q-h)\,\mathrm{d}x\,\mathrm{d}t+N\int_\Omega\nabla q\cdot\nabla(h-q)\,\mathrm{d}x\geq 0,\tag{2.5}$$

for any $h\in\mathcal{M}$.

Proof. For any $h\in\mathcal{M}$ and $0\leq\delta\leq 1$, set

$$q_\delta=(1-\delta)q+\delta h\in\mathcal{M}.$$

Then there exists a solution y_δ of the equation (1.1) with the coefficient $q=q_\delta$ satisfying

$$\mathcal{J}_\delta=\mathcal{J}(q_\delta)=\frac{1}{2}\int_\Omega|y(x,T;q_\delta)-m(x)|^2\,\mathrm{d}x+\frac{N}{2}\int_\Omega|\nabla q_\delta|^2\,\mathrm{d}x.\tag{2.6}$$

Now taking Fréchet derivative of $\mathcal{J}_\delta$ with optimal solution $\tilde{q}$ we have

$$\frac{d\mathcal{J}_\delta}{d\delta}\bigg|_{\delta=0} = \int_\Omega [y(x,T)-m(x)]\frac{\partial y}{\partial \delta}\bigg|_{\delta=0} \mathrm{d}x + N\int_\Omega \nabla q\cdot\nabla(h-q)\,\mathrm{d}x \geq 0. \quad (2.7)$$

If we take $\overline{y_\delta} = \dfrac{\partial y_\delta}{\partial \delta}$, then $\overline{y_\delta}$ satisfies the following system with the coefficient q_δ,

$$\begin{cases} (\overline{y_\delta})_t - \nabla(d(x)\nabla\overline{y_\delta}) + \int_0^t K(t,\tau)\overline{y_\delta}\,\mathrm{d}\tau + q\overline{y_\delta} = (q-h)\overline{y_\delta}, & (x,t)\in\Omega_T,\\ \overline{y_\delta}(x,t) = 0, & (x,t)\in\Sigma,\\ \overline{y_\delta}(x,0) = 0, & x\in\Omega. \end{cases}$$

Taking $\zeta = \overline{y_\delta}|_{\delta=0}$, we see that ζ satisfies the following system

$$\begin{cases} \zeta_t - \nabla(d(x)\nabla\zeta) + \int_0^t K(t,\tau)\zeta\,\mathrm{d}\tau + q\zeta = (q-h)y, & (x,t)\in\Omega_T,\\ \zeta(x,t) = 0, & (x,t)\in\Sigma,\\ \zeta(x,0) = 0, & x\in\Omega. \end{cases} \quad (2.8)$$

Then the optimality condition becomes

$$\int_\Omega [y(x,T)-m(x)]\zeta(x,T)\,\mathrm{d}x + N\int_\Omega \nabla q\cdot\nabla(h-q)\,\mathrm{d}x \geq 0. \quad (2.9)$$

Let $L\zeta \equiv \zeta_t - \nabla(d(x)\nabla\zeta) + \int_0^t K(t,\tau)\zeta d\tau + q\zeta$ and suppose p is the solution of the following problem

$$\begin{cases} L^*p \equiv -p_t - \nabla(d(x)\nabla p) + \int_t^T K(\tau,t)p\ \mathrm{d}t + qp = 0, & (x,t)\in\Omega_T,\\ p(x,t) = 0, & (x,t)\in\Sigma,\\ p(x,T) = y(x,T)-m(x), & x\in\Omega, \end{cases} \quad (2.10)$$

where L^* is the adjoint operator of L. From (2.8) we have

$$
\begin{aligned}
0 &= \int_0^T \int_\Omega \zeta L^* p \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_\Omega \zeta(x,T)[y(x,T)-m(x)]\,\mathrm{d}x + \int_0^T\int_\Omega pL\zeta\,\mathrm{d}x\,\mathrm{d}t, \\
0 &= -\int_\Omega \zeta(x,T)[u(x,T)-m(x)]\,\mathrm{d}x + \int_0^T\int_\Omega py(q-h)\,\mathrm{d}x\,\mathrm{d}t. \qquad (2.11)
\end{aligned}
$$

Substituting (2.11) into (2.9) leads to

$$
\int_0^T\int_\Omega py(q-h)\,\mathrm{d}x\,\mathrm{d}t + N\int\int_\Omega \nabla q\cdot\nabla(h-q)\,\mathrm{d}x \geq 0. \qquad (2.12)
$$

Hence the theorem. □

3. Main results

In this section, we establish a stability estimate for the inverse problem of retrieving the smooth coefficient $q(x)$ in the given system. Main theorem estimates the discrepancy in the coefficients $q(x)$ of the material with an upper bound given by some Sobolev norms of the solutions at $t=T$. The optimal control problem established in the previous section will be the key ingredient in the proof of such a stability estimate.

The following lemmas form the most fundamental tool in proving the main result of this section.

Lemma 3.1. *Let y be the solution of the system* (1.1). *Then we have the following estimate:*

$$
\begin{aligned}
&\max_{0\leq t\leq T}\int_\Omega |y|^2\,\mathrm{d}x + 2d_1\int_0^T\int_\Omega |\nabla y|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\leq M_1\left(\int_\Omega |y_0|^2\,\mathrm{d}x + \int_0^T\int_\Omega |f|^2\,\mathrm{d}x\,\mathrm{d}t\right),
\end{aligned} \qquad (3.1)
$$

where the constant $M_1 = \exp\left[2T(2q_1+2+\|K\|^2_{L^\infty}T^2)\right]$.

Proof. Multiply the equation (1.1) by y and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int\limits_{\Omega}|y|^2\,\mathrm{d}x + d_1\int\limits_{\Omega}|\nabla y|^2\,\mathrm{d}x = -\int\limits_{\Omega}q|y|^2\,\mathrm{d}x + \int\limits_{\Omega}fy\,\mathrm{d}x - \int\limits_{\Omega}y(K_0^t * y)\,\mathrm{d}x,$$

where $K_0^t * y = \int_0^t K(t,\tau)y\,d\tau$ and satisfies

$$\int\limits_0^t\int\limits_{\Omega}|K_0^t * y|^2\,\mathrm{d}x\,\mathrm{d}t \le \|K\|_{L^\infty}^2 T^2\int\limits_0^t\int\limits_{\Omega}|y|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Using the assumption on the coefficient q and applying Cauchy's inequality, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{\mathrm{d}t}\int\limits_{\Omega}|y|^2\,\mathrm{d}x + d\int\limits_{\Omega}|\nabla y|^2\,\mathrm{d}x\\
&\le (q_1+1)\int\limits_{\Omega}|y|^2\,\mathrm{d}x + \frac{1}{2}\int\limits_{\Omega}|f|^2\,\mathrm{d}x + \frac{1}{2}\int\limits_{\Omega}\int\limits_0^t\int\limits_0^t|K(t,\tau)|^2|y|^2\,\mathrm{d}\tau\,\mathrm{d}\tau\,\mathrm{d}x\\
&\le \left[q_1+1+\frac{1}{2}\|K\|_{L^\infty}^2T^2\right]\int\limits_{\Omega}|y|^2\,\mathrm{d}x + \frac{1}{2}\int\limits_{\Omega}|f|^2\,\mathrm{d}x.
\end{aligned}$$

Integrate with respect to t from 0 to t, we obtain

$$\begin{aligned}
&\int\limits_{\Omega}|y|^2\,\mathrm{d}x + 2d_1\int\limits_0^t\int\limits_{\Omega}|\nabla y|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\le 2(q_1+1+\frac{1}{2}\|K\|_{L^\infty}^2T^2)\int\limits_0^t\int\limits_{\Omega}|y|^2\,\mathrm{d}x + \int\limits_0^t\int\limits_{\Omega}|f|^2\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega}|y_0|^2\,\mathrm{d}x.
\end{aligned}$$

Using Gronwall's inequality, the proof of Lemma 3.1 is completed. □

Lemma 3.2. *Let p be the solution of the system* (2.4)*. Then we have the following estimate:*

$$\max_{0\le t\le T}\int\limits_{\Omega}|p|^2\,\mathrm{d}x + 2d_1\int\limits_0^T\int\limits_{\Omega}|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}t \le M_2\int\limits_{\Omega}|u(x,T)-m(x)|^2\,\mathrm{d}x, \tag{3.2}$$

where the constant $M_2 = \exp\left[2T(2q_1+1+\|K\|_{L^\infty}^2T^2)\right]$.

Proof. Multiply the equation (2.4) by p and integrate over Ω to obtain

$$-\frac{1}{2}\frac{d}{\mathrm{d}t}\int_{\Omega}|p|^2\,\mathrm{d}x+d_1\int_{\Omega}|\nabla p|^2\,\mathrm{d}x=-\int_{\Omega}q|p|^2\,\mathrm{d}x-\int_{\Omega}p\,\left(K_t^T*p\right)\,\mathrm{d}x,$$

where $K_t^T*p=\int_t^T K(t,\tau)p\,d\tau$. Using the assumption on the coefficient q and applying Cauchy's inequality, we have

$$\begin{aligned}&-\frac{1}{2}\frac{d}{\mathrm{d}t}\int_{\Omega}|p|^2\,\mathrm{d}x+d_1\int_{\Omega}|\nabla p|^2\,\mathrm{d}x\\&\le(q_1+\frac{1}{2})\int_{\Omega}|p|^2\,\mathrm{d}x+\frac{1}{2}\|K\|_{L^\infty}^2T^2\int_{\Omega}|p|^2\,\mathrm{d}x.\end{aligned}$$

Integrate with respect to t from t to T, we obtain

$$\begin{aligned}&\int_{\Omega}|p|^2\,\mathrm{d}x+2d_1\int_t^T\int_{\Omega}|\nabla p|^2\,\mathrm{d}x\,\mathrm{d}t\\&\le\left(2q_1+1+\|K\|_{L^\infty}^2T^2\right)\int_t^T\int_{\Omega}|p|^2\,\mathrm{d}x\,\mathrm{d}t+\int_{\Omega}|u(x,T)-m(x)|^2\,\mathrm{d}x.\end{aligned}$$

Using Gronwall's inequality, the proof of Lemma 3.2 is thus completed. □

Lemma 3.3. *Let Y be the solution of the system* (1.4). *Then we have the following estimate:*

$$\max_{0\le t\le T}\int_{\Omega}|Y|^2\,\mathrm{d}x+2d_1\int_0^T\int_{\Omega}|\nabla Y|^2\,\mathrm{d}x\,\mathrm{d}t\le M_1\max|Q|^2\int_0^T\int_{\Omega}|\tilde{y}|^2\,\mathrm{d}x\,\mathrm{d}t,\quad(3.3)$$

where M_1 is the constant defined in Lemma 3.1.

Proof. Multiply the equation (1.4) by Y and integrate over Ω to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{\mathrm{d}t}\int_{\Omega}|Y|^2\,\mathrm{d}x+d_1\int_{\Omega}|\nabla Y|^2\,\mathrm{d}x\\&=-\int_{\Omega}\left[(K_0^t*Y)Y+q|Y|^2+Q\tilde{y}Y\right]\,\mathrm{d}x.\end{aligned}$$

Using the assumption on the coefficient q and applying Cauchy's inequality, we have

$$\frac{1}{2}\frac{d}{\mathrm{d}t}\int_\Omega |Y|^2\,\mathrm{d}x + d_1\int_\Omega |\nabla Y|^2\,\mathrm{d}x$$
$$\leq q_1\int_\Omega |Y|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |Y|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |Y|^2\,\mathrm{d}x$$
$$+\frac{1}{2}\max|Q|^2\int_\Omega |\tilde{y}|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |K_0^t * Y|^2\,\mathrm{d}x.$$

Integrate with respect to t from 0 to t, we obtain

$$\int_\Omega |Y|^2\,\mathrm{d}x + 2d_1\int_0^t\int_\Omega |\nabla Y|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq (2q_1 + 2 + \|K\|^2_{L^\infty})\int_0^t\int_\Omega |Y|^2\,\mathrm{d}x\,\mathrm{d}t + \max|Q|^2\int_0^t\int_\Omega |\tilde{y}|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Using Gronwall's inequality, the proof of Lemma 3.3 is thus completed. □

Suppose that p, $\tilde{p}$ are the solutions of the adjoint system with coefficients q and $\tilde{q}$ respectively. By setting $P = p - \tilde{p}$, we get

$$\begin{cases} -P_t - \nabla(d(x)\nabla P) + \int_t^T K(\tau,t)P(x,\tau)d\tau + qP = -Q\tilde{p}, & (x,t)\in\Omega_T,\\ P(x,t) = 0, & (x,t)\in\Sigma,\\ P(x,T) = Y(x,T) - (m-\tilde{m}), & x\in\Omega. \end{cases} \tag{3.4}$$

Lemma 3.4. *Let P be the solution of the system* (3.4). *Then we have the following estimate:*

$$\max_{0\leq t\leq T}\int_\Omega |P|^2\,\mathrm{d}x + 2d_1\int_0^T\int_\Omega |\nabla P|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq M_1\left\{\max|Q|^2\int_0^T\int_\Omega \left[|\tilde{p}|^2 + M_1\max|Q|^2|\tilde{y}|^2\right]\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega |m-\tilde{m}|^2\,\mathrm{d}x\right\} \tag{3.5}$$

where M_1 is the constant, defined in Lemma 3.1.

Proof. Multiply the equation (3.4) by P and integrate over Ω

$$\frac{-1}{2}\frac{d}{\mathrm{d}t}\int_\Omega |P|^2\,\mathrm{d}x + d_1\int_\Omega |\nabla P|^2\,\mathrm{d}x = -\int_\Omega \left[q|P|^2 + P(K_t^T * P) + Q\tilde{p}P\right]\,\mathrm{d}x.$$

Using the assumption on the coefficient q and applying Cauchy's inequality, we have

$$\frac{-1}{2}\frac{d}{\mathrm{d}t}\int_\Omega |P|^2\,\mathrm{d}x + d_1\int_\Omega |\nabla P|^2\,\mathrm{d}x$$
$$\leq q_1\int_\Omega |P|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |P|^2\,\mathrm{d}x + \frac{1}{2}\int_\Omega |K_t^T * P|^2\,\mathrm{d}x + \frac{1}{2}\max|Q|^2\int_\Omega |\tilde{p}|^2\,\mathrm{d}x.$$

Integrate with respect to t from t to T and rearranging terms, we obtain

$$\int_\Omega |P|^2\,\mathrm{d}x + 2d_1\int_t^T\int_\Omega |\nabla P|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leq (2q_1 + 2 + \|K\|^2_{L^\infty}T^2)\int_t^T\int_\Omega |P|^2\,\mathrm{d}x\,\mathrm{d}t + \max|Q|^2\int_t^T\int_\Omega |\tilde{p}|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_\Omega |Y(x,T) - (m-\tilde{m})|^2\,\mathrm{d}x.$$

Using Gronwall's inequality, one can easily conclude the proof by applying Lemma 3.3. □

Now we are ready to state and prove the main result of this article with the help of the lemmas that we have established in the preceding section.

Theorem 3.5. *Let y and p be the solutions of the equation* (1.1) *and the adjoint equation* (2.4) *respectively and suppose there exists a point $x_0 \in \Omega$ such that $q(x_0) = \tilde{q}(x_0)$. Then there exists an instant of time T_0 such that, for $T \geq T_0$, there exists a constant $C > 0$, independent of q_0, satisfying the following estimate*

$$\max_{x\in\Omega}|q-\tilde{q}|^2 \leq \frac{CT}{2N}\ell M_1 M_2\int_\Omega |m-\tilde{m}|^2\,\mathrm{d}x \tag{3.6}$$

where the constants M_1 and M_2 are defined in Lemma 3.1 *and Lemma* 3.2.

Proof. If y, p are the solutions of the equation (1.1) and the adjoint equation (2.4) respectively with the coefficient q then for the choice $h = \tilde{q} \in \mathcal{M}$ in the first order necessary optimality condition (2.5) gives

$$\int_0^T\int_\Omega py(q-\tilde{q})\,\mathrm{d}x\,\mathrm{d}t + N\int_\Omega \nabla q\cdot\nabla(\tilde{q}-q)\,\mathrm{d}x \geq 0. \tag{3.7}$$

Similarly, if $\tilde{y}, \tilde{p}$ are the solutions of the equation (1.3) and its adjoint equation respectively with the coefficient $\tilde{q}$ then for the choice $h = q \in \mathcal{M}$ in the first order necessary optimality condition (2.5) gives

$$\int_0^T\int_\Omega \tilde{p}\tilde{y}(\tilde{q} - q)\,\mathrm{d}x\,\mathrm{d}t + N\int_\Omega \nabla\tilde{q}\cdot\nabla(q - \tilde{q})\,\mathrm{d}x \geq 0. \tag{3.8}$$

Adding the above two optimality condition we have the following inequality

$$N\int_\Omega |\nabla Q|^2\,\mathrm{d}x \leq \int_0^T\int_\Omega Q(py - \tilde{p}\tilde{y})\,\mathrm{d}x\,\mathrm{d}t \leq \int_0^T\int_\Omega [QpY + QP\tilde{y}]\,\mathrm{d}x\,\mathrm{d}t.$$

Applying Cauchy's Schwartz inequality and the estimates in Lemmas 3.1-3.3

$$\begin{aligned}
&N\int_\Omega |\nabla Q|^2\,\mathrm{d}x \\
&\leq \frac{1}{2}\max|Q|^2\int_0^T\int_\Omega |p|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T\int_\Omega |Y|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{1}{2}\max|Q|^2\int_0^T\int_\Omega |\tilde{y}|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\int_0^T\int_\Omega |P|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\leq \frac{1}{2}\max|Q|^2\left\{\int_0^T\int_\Omega |\tilde{p}|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega |\tilde{y}|^2\,\mathrm{d}x\,\mathrm{d}t\right\} \\
&\quad + \frac{1}{2}\left\{\int_0^T\int_\Omega |Y|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega |P|^2\,\mathrm{d}x\,\mathrm{d}t\right\} \\
&\leq \frac{1}{2}\max|Q|^2\left\{\int_0^T\int_\Omega |\tilde{y}|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega |\tilde{p}|^2\,\mathrm{d}x\,\mathrm{d}t\right\} \\
&\quad + \frac{M_1T}{2}\max|Q|^2\int_0^T\int_\Omega |\tilde{y}|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{M_1T}{2}\max|Q|^2\int_0^T\int_\Omega |\tilde{p}|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{M_1^2T}{2}\int_0^T\int_\Omega |\tilde{y}|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{M_1T}{2}\int_\Omega |m - \tilde{m}|^2\,\mathrm{d}x.
\end{aligned} \tag{3.9}$$

Besides, from Lemma 3.1 and an analogue of Lemma 3.2, there exists a constant $\Gamma > 0$ such that

$$\int_0^T\int_\Omega |\tilde{y}|^2 \,\mathrm{d}x\,\mathrm{d}t \le M_1 T\Gamma \quad \text{and} \quad \int_0^T\int_\Omega |\tilde{p}|^2 \,\mathrm{d}x\,\mathrm{d}t \le T^2 M_1 M_2 \Gamma. \tag{3.10}$$

Moreover, taking into account and applying Hölder's inequality, we get

$$|Q(x)| = \left| \int_{x_0}^{x} [Q(z)]' \,\mathrm{d}z \right| \le \sqrt{\ell} \left[\int_{x_0}^{x} |\nabla Q|^2 \,\mathrm{d}z \right]^{\frac{1}{2}}$$

so that

$$\max_{x\in\Omega} |Q(x)| \le \sqrt{\ell} \|\nabla Q(x)\|_{L^2(\Omega)}, \quad \forall x \in \Omega. \tag{3.11}$$

Combining the preceding estimates with (3.9), we arrive at

$$\begin{aligned} \max_{x\in\Omega} |Q|^2 &\le \ell \|\nabla Q(x)\|^2_{L^2(\Omega)} \\ &\le C_T \max |Q|^2 + \frac{C M_1 M_2 T \ell}{2N} \int_\Omega |m - \tilde{m}|^2 \,\mathrm{d}x \end{aligned}$$

where the constant $C_T = \dfrac{1}{2N} M_1 T \Gamma \ell \left[1 + M_2 T + M_1 T + M_1 M_2 T^2 + M_1^2 T \right]$. Now by choosing $T_0 > 0$ such that $C_{T_0} < 1$, one can complete the proof. □

Remark 3.6. From Theorem 3.5, we can easily see that if the final measurements of the systems (1.1) and (1.3) are equal, that is $y(x,T) = \tilde{y}(x,T)$ then the data q can be determined uniquely, that is, $q(x) = \tilde{q}(x)$ in Ω, for some small $T_0 > 0$. In fact, from (3.9)-(3.11), one indeed gets

$$\int_\Omega |\nabla Q|^2 \,\mathrm{d}x \le C_T \int_\Omega |\nabla Q|^2 \,\mathrm{d}x.$$

Again, choosing $T_0 > 0$ such that $C_{T_0} < 1$; one can conclude that

$$\int_\Omega |\nabla Q|^2 \,\mathrm{d}x \le 0.$$

Taking the assumption $q(x_0) = \tilde{q}(x_0)$ into account that, we deduce that $q(x) = \tilde{q}(x)$ for all $x \in \Omega$.

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