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VISCOSITY APPROXIMATION METHOD FOR THE SPLIT COMMON FIXED POINT PROBLEM OF QUASI-STRICT PSEUDO-CONTRACTIONS WITHOUT PRIOR KNOWLEDGE OF OPERATOR NORMS

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Abstract. Let H_1, H_2, H_3 be real Hilbert spaces, let $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. The split common fixed point problem (SCFP) in the infinite-dimensional Hilbert spaces introduced by Moudafi [10] is

$$\text{to find } \ x \in F(U), y \in F(T) \ \text{ such that } \ Ax = By, \tag{1}$$

where $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two nonlinear operators with nonempty fixed-point sets $F(U)$ and $F(T)$. Note that, by taking $B = I$ and $H_2 = H_3$ in (1), we recover the split fixed point problem originally introduced in Censor and Segal [7]. Recently, Moudafi introduced alternating algorithms [10] and simultaneous algorithms [12] with weak convergence for the SCFP (1) of firmly quasi-nonexpansive operators. However, to employ Moudafi's algorithms, one needs to know a prior norm (or at least an estimate of the norm) of the bounded linear operators. In this paper, we will continue to consider the SCFP (1) governed by the general class of quasi-strict pseudo-contractions in Hilbert space. We introduce a viscosity iterative algorithm with a way of selecting the stepsizes such that the implementation of the algorithm does not need any prior information about the operator norms. We prove the strong convergence of the algorithm.

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1. Introduction

The split common fixed point problem (SCFP) has been investigated recently, which is a generalization of the split feasibility problem and of the convex feasibility problem. The SCFP attracts many authors' attention due to its extraordinary utility and broad applicability in many areas of applied mathematics (most notably, fully-discretized models inverse problems which arise from phase retrievals and in medical image reconstruction [2]). Various algorithms have been invented to solve it (see [4, 11]). In this paper, our interest is in the study of the convergence of viscosity iterative algorithm for the following SCFP introduced by Moudafi [10] :

$$\text{find} \;\; x \in F(U), y \in F(T) \;\; \text{such that} \;\; Ax = By, \tag{1.1}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two nonlinear operators with nonempty fixed-point sets $F(U) = C$ and $F(T) = Q$. This allows asymmetric and partial relations between the variables x and y. The interest is to cover many situation, for instance in decomposition methods for PDE's, applications in game theory and in intensity-modulated radiation therapy (IMRT). In decision sciences, this allows to consider agents who interplay only via some components of their decision variables (see [1]). In (IMRT), this amounts to envisage a weak coupling between the vector of doses absorbed in all voxels and that of the radiation intensity (see [5]).

To begin with, let us recall that the split feasibility problem (SFP) is to find a point

$$x \in C \;\; \text{such that} \;\; Ax \in Q, \tag{1.2}$$

where C and Q are nonempty closed convex subset of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. The SFP in finite-dimensional Hilbert spaces was originally introduced by Censor and Elfving [6]. Many algorithms have been invented to solve it (see [3, 6], [14]–[18] and references therein).

To solve the (1.2), Byrne [2] proposed his CQ algorithm which generates a sequence $\{x_k\}$ by

$$x_{k+1} = P_C(I - \gamma A^*(I - P_Q)A)x_k, \; k \in \mathbb{N}, \tag{1.3}$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A.

Censor and Segal [7] consider the following split common fixed point problem (SCFP):

$$\text{find} \;\; x^* \in F(U) \;\; \text{such that} \;\; Ax^* \in F(T), \tag{1.4}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $U : H_1 \to H_1$ and $T : H_2 \to H_2$ are two nonexpansive operators with nonempty fixed point sets $F(U) = C$ and $F(T) = Q$.

To solve (1.4), Censor and Segal [7] proposed and proved, in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{k+1} = U(x_k + \gamma A^t(T - I)Ax_k), \quad k \in \mathbb{N},$$

where $\gamma \in (0, \frac{2}{\lambda})$, with λ being the largest eigenvalue of the matrix A^tA(A^t stands for matrix transposition).

For solving the SCFP (1.1), Moudafi [10] introduced the following alternating algorithm

$$\begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_{k+1} - By_k)), \end{cases} \tag{1.5}$$

for firmly quasi-nonexpansive operators U and T, where non-decreasing sequence $\gamma_k \in (\varepsilon, \min(\frac{1}{\lambda_A}, \frac{1}{\lambda_B}) - \varepsilon)$, λ_A, λ_B stand for the spectral radius of A^*A and B^*B respectively.

Very recently, Moudafi [12] introduced the following simultaneous iterative method to solve SCFP (1.1):

$$\begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)), \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_k - By_k)), \end{cases} \tag{1.6}$$

for firmly quasi-nonexpansive operators U and T, where $\gamma_k \in (\varepsilon, \frac{2}{\lambda_A+\lambda_B} - \varepsilon)$, λ_A, λ_B stand for the spectral radius of A^*A and B^*B respectively.

For solving SCFP (1.4) of quasi-nonexpansive operators, Moudafi [11] introduced the following relaxed algorithm:

$$x_{k+1} = \alpha_k u_k + (1 - \alpha_k)U(u_k), \quad k \in \mathbb{N}, \tag{1.7}$$

where $u_k = x_k + \gamma\beta A^*(T - I)Ax_k$, $\beta \in (0, 1)$, $\alpha_k \in (0, 1)$ and $\gamma \in (0, \frac{1}{\lambda\beta})$, with λ being the spectral radius of the operator A^*A. Moudafi proved weak convergence result of the algorithm in Hilbert spaces.

Recently, Zhao and He [20] introduced the following viscosity approximation algorithm for the SCFP (1.4) of quasi-nonexpansive operators :

$$x_{k+1} = \alpha_k f(x_k) + (1 - \alpha_k)((1 - \omega_k)x_k + \omega_k Tx_k), \quad k \in \mathbb{N}, \tag{1.8}$$

where $T = U(I + \gamma A^*(S - I)A)$, $A : H_1 \to H_2$ is a bounded linear operator, $U : H_1 \to H_1$ and $S : H_2 \to H_2$ are two quasi-nonexpansive operators with nonempty fixed point sets $F(U) = C$ and $F(T) = Q$, $f : H_1 \to H_1$ is a contraction of modulus $\rho \in [0, 1)$, $\omega_k \in (0, \frac{1}{2})$ such that $0 < \liminf_{k\to\infty} \omega_k \leq$

$\limsup_{k\to\infty} \omega_k < \frac{1}{2}$, $\gamma \in (0, \frac{1}{\lambda})$, with λ being the spectral radius of the operator A^*A and $\alpha_k \in (0,1)$.

Note that in the algorithms (1.3), (1.5), (1.6), (1.7) and (1.8) mentioned above, the determination of the stepsize $\{\gamma_k\}$ depends on the operator (matrix) norms $\|A\|$ and $\|B\|$ (or the largest eigenvalues of A^*A and B^*B). In order to implement the algorithm (1.3), (1.5), (1.6), (1.7) and (1.8), one has first to compute (or, at least, estimate) operator norms of A and B, which is in general not an easy work in practice. To overcome this difficulty, López *et al.* [9] and Zhao and Yang [19] presented a helpful method for estimating the stepsizes which don't need prior knowledge of the operator norms for solving the split feasibility problems and multiple-set split feasibility problems, respectively.

Inspired by them, in this paper, we introduce a new choice of the stepsize sequence $\{\gamma_k\}$ for the viscosity iterative algorithm to solve SCFP (1.1) governed by quasi-strict pseudo-contractions as follows

$$\gamma_k \in \left(0, \frac{2\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2}\right). \tag{1.9}$$

The advantage of our choice (1.9) of the stepsizes lies in the fact that no prior information about the operator norms of A and B is required, and still convergence is guaranteed. The organization of this paper is as follows. Some useful definitions and results are listed for the convergence analysis of the iterative algorithms in the Section 2. In Section 3, the strong convergence theorem of the proposed viscosity iterative algorithm is obtained.

2. Preliminaries

Thought this paper, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. Let I denote the identity operator on H. Let $T : H \to H$ be a operator. A point $x \in H$ is said to be a fixed point of T provided $Tx = x$. In this paper, we use $F(T)$ to denote the fixed point set of T. We use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. We use $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ stand for the weak ω-limit set of $\{x_k\}$ and use Γ stand for the solution set of the SCFP (1.1).

Definition 2.1. An operator $T : H \to H$ is said to be

(i) *nonexpansive mapping*, if $\|Tx - Ty\| \le \|x - y\|$, for all $x, y \in H$;
(ii) *quasi-nonexpansive mapping*, if $F(T) \neq \emptyset$ and if $\|Tx - q\| \le \|x - q\|$, for all $x \in H, q \in F(T)$;
(iii) *firmly nonexpansive mapping*, if $\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2$, for all $x, y \in H$;

(iv) *firmly quasi-nonexpansive mapping*, if $F(T) \neq \emptyset$ and if $\|Tx - q\|^2 \leq \|x - q\|^2 + \|(I - T)x\|^2$, for all $x \in H, q \in F(T)$;
(v) *k-strict pseudo-contraction*, if there exist some $k \in [0, 1)$ such that $\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2$, for all $x, y \in H$;
(vi) *k-quasi-strict pseudo-contraction*, if $F(T) \neq \emptyset$ and there exist some $k \in [0, 1)$ such that $\|Tx - q\|^2 \leq \|x - q\|^2 + k\|(I - T)x\|^2$, for all $x \in H, q \in F(T)$.

Remark 2.2. Note that, the class of strict pseudo-contractions strictly included the class of nonexpansive mappings. That is, T is nonexpansive if and only if T is 0-strict pseudo-contraction.

Definition 2.3. An operator $T : H \to H$ is called demiclosed at the origin if, for any sequence $\{x_n\}$ which weakly converges to x, and if the sequence $\{Tx_n\}$ strongly converges to 0, then $Tx = 0$.

In real Hilbert space, we easily get the following equality:

$$\begin{aligned} 2\langle x, y\rangle &= \|x\|^2 + \|y\|^2 - \|x - y\|^2 \\ &= \|x + y\|^2 - \|x\|^2 - \|y\|^2, \quad \forall\, x, y \in H. \end{aligned} \tag{2.1}$$

Lemma 2.4. *Let C be a closed convex subset of real Hilbert space H. Given $x \in H$ and $z \in C$, then $z = P_C x$ if and only if there holds the relation: $\langle x - z, y - z\rangle \leq 0$, for all $y \in C$.*

Lemma 2.5. ([13]) *Let H be a real Hilbert space. Then for all $t \in [0, 1]$ and $x, y \in H$, $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$.*

Lemma 2.6. ([8]) *Assume $\{s_k\}$ is a sequence of nonnegative real numbers such that*

$$\begin{cases} s_{k+1} \leq (1 - \lambda_k)s_k + \lambda_k\delta_k, & k \geq 0, \\ s_{k+1} \leq s_k - \eta_k + \mu_k, & k \geq 0, \end{cases}$$

where $\{\lambda_k\}$ is a sequence in $(0, 1)$, $\{\eta_k\}$ is a sequence of nonnegative real numbers and $\{\delta_k\}$ and $\{\mu_k\}$ are two sequences in $\mathbb{R}$ such that

(i) $\Sigma_{k=1}^{\infty}\lambda_k = \infty$;
(ii) $\lim_{k\to\infty} \mu_k = 0$;
(iii) $\lim_{l\to\infty} \eta_{k_l} = 0$ *implies* $\limsup_{l\to\infty} \delta_{k_l} \leq 0$, *for any subsequence* $\{k_l\} \subset \{k\}$.

Then $\lim_{k\to\infty} s_k = 0$.

Lemma 2.7. *Assume C is a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a self-mapping of C, if T is a k-strict pseudo-contraction, then the mapping $I - T$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup \tilde{x}$ and $(I-T)x_n \to 0$, then $(I-T)\tilde{x} = 0$.*

3. Main Results

In this section, we introduce a viscosity iterative algorithm where the stepsizes don't depends on the operator norms $\|A\|$ and $\|B\|$, and prove strong convergence of the algorithm for SCFP (1.1) of quasi-strict pseudo-contractions.

Algorithm 3.1. *Let $f_1 : H_1 \to H_1$ and $f_2 : H_2 \to H_2$ be two contractions with constants ρ_1, $\rho_2 \in [0,1)$ and $\alpha_k \in [0,1]$. Choose an initial guess $x_0 \in H_1$, $y_0 \in H_2$ arbitrarily. Assume that the kth iterate $x_k \in H_1$, $y_k \in H_2$ has been constructed; then we calculate the (k + 1)th iterate (x_{k+1}, y_{k+1}) via the formula:*

$$\begin{cases} u_k = x_k - \gamma_k A^*(Ax_k - By_k), \\ x_{k+1} = \alpha_k f_1(x_k) + (1-\alpha_k)((1-\omega_k)u_k + \omega_k U(u_k)), \\ v_k = y_k + \gamma_k B^*(Ax_k - By_k), \\ y_{k+1} = \alpha_k f_2(y_k) + (1-\alpha_k)((1-\omega_k)v_k + \omega_k T(v_k)). \end{cases} \tag{3.1}$$

Assume for small enough $\varepsilon > 0$, the stepsize γ_k is chosen in such a way that

$$\gamma_k \in \left(\varepsilon, \frac{2\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2} - \varepsilon\right), \quad k \in \Omega, \tag{3.2}$$

otherwise, $\gamma_k = \gamma$ (γ being any nonnegative value), where the set of indexes $\Omega = \{k : Ax_k - By_k \neq 0\}$.

Remark 3.2. Note that in (3.2) the choice of the stepsizes γ_k is independent of the norms $\|A\|$, $\|B\|$. The value of γ does not influence of the considered algorithm, but it was introduced just for the sake of clarity. Furthermore, we will see from Lemma 3.3 that γ_k is well-defined.

Lemma 3.3. *Assume the solution set Γ of (1.1) is nonempty. Then γ_k defined by (3.2) is well-defined.*

Proof. Take $(x, y) \in \Gamma$, i.e., $x \in F(U)$; $y \in F(T)$ and $Ax = By$. We have

$$\langle A^*(Ax_k - By_k), x_k - x\rangle = \langle Ax_k - By_k, Ax_k - Ax\rangle$$

and

$$\langle B^*(Ax_k - By_k), y - y_k\rangle = \langle Ax_k - By_k, By - By_k\rangle.$$

By adding the two above equalities and by taking into account the fact that $Ax = By$, we obtain

$$\begin{aligned}\|Ax_k - By_k\|^2 &= \langle A^*(Ax_k - By_k), x_k - x\rangle + \langle B^*(Ax_k - By_k), y - y_k\rangle \\ &\leq \|A^*(Ax_k - By_k)\| \cdot \|x_k - x\| + \|B^*(Ax_k - By_k)\| \cdot \|y - y_k\|.\end{aligned}$$

Consequently, for $k \in \Omega$, that is, $\|Ax_k - By_k\| > 0$, we have $\|A^*(Ax_k - By_k)\| \neq 0$ or $\|B^*(Ax_k - By_k)\| \neq 0$. This leads that γ_k is well-defined. □

Theorem 3.4. *Let H_1, H_2, H_3 be real Hilbert spaces. Given two bounded linear operators $A : H_1 \to H_3$, $B : H_2 \to H_3$, let $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi-strict pseudo-contractions with constants t_1, t_2 where $t_1, t_2 \in [0, 1)$, and the solution set Γ of* (1.1) *is nonempty. Let $t = \max\{t_1, t_2\}$ and the sequence $\{(x_k, y_k)\}$ is generated by Algorithm* 3.1. *Assume that the following conditions are satisfied:*

(i) $\rho_1, \rho_2 \in [0, \frac{1}{\sqrt{2}})$;
(ii) $\lim_{k\to\infty} \alpha_k = 0$, $\sum_{k=0}^{\infty} \alpha_k = \infty$;
(iii) *$U - I$ and $T - I$ are demiclosed at origin;*
(iv) *$\omega_k \in (0, 1)$ such that* $0 < \liminf_{k\to\infty} \omega_k \leq \limsup_{k\to\infty} \omega_k < 1 - t$.

Then the sequence $\{(x_k, y_k)\}$ strongly converges to a solution (x^*, y^*) of (1.1) *which solves the variational inequality problem:*

$$\begin{cases} \langle (I - f_1)x^*, x - x^*\rangle \geq 0, \\ \langle (I - f_2)y^*, y - y^*\rangle \geq 0, \end{cases} \quad (x, y) \in \Gamma. \tag{3.3}$$

Proof. From the condition on γ_k, we have

$$\inf\left\{\frac{2\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2} - \gamma_k\right\} > 0. \tag{3.4}$$

On the other hand, from $\|A^*(Ax_k - By_k)\|^2 \leq \|A^*\|^2\|Ax_k - By_k\|^2$ and $\|B^*(Ax_k - By_k)\|^2 \leq \|B^*\|^2\|Ax_k - By_k\|^2$, we obtain

$$\frac{2\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2}$$

is lower bounded by $\frac{2}{\|A\|^2 + \|B\|^2}$ and so

$$\inf\left\{\frac{2\|Ax_k - By_k\|^2}{\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2}\right\} > -\infty.$$

It follow from (3.4) that $\sup_{k\in\Omega} \gamma_k < +\infty$ and $\{\gamma_k\}_{k\geq 1}$ is bounded.

Let $(x^*, y^*) \in \Gamma$ be the solution of the variational inequality problem (3.3). Then $x^* \in F(U)$, $y^* \in F(T)$ and $Ax^* = By^*$. We have

$$\begin{aligned}
&\|u_k - x^*\|^2 \\
&= \|x_k - \gamma_k A^*(Ax_k - By_k) - x^*\|^2 \\
&= \|x_k - x^*\|^2 - 2\gamma_k \langle x_k - x^*, A^*(Ax_k - By_k)\rangle + \gamma_k^2 \|A^*(Ax_k - By_k)\|^2.
\end{aligned} \tag{3.5}$$

Using the equality (2.1), we have

$$\begin{aligned}
&- 2\langle x_k - x^*, A^*(Ax_k - By_k)\rangle \\
&= -2\langle Ax_k - Ax^*, Ax_k - By_k\rangle \\
&= -\|Ax_k - Ax^*\|^2 - \|Ax_k - By_k\|^2 + \|By_k - Ax^*\|^2.
\end{aligned} \tag{3.6}$$

By (3.5) and (3.6) we obtain

$$\begin{aligned}
\|u_k - x^*\|^2 \leq &\|x_k - x^*\|^2 - \gamma_k \|Ax_k - By_k\|^2 - \gamma_k \|Ax_k - Ax^*\|^2 \\
&+ \gamma_k \|By_k - Ax^*\|^2 + \gamma_k^2 \|A^*(Ax_k - By_k)\|^2.
\end{aligned} \tag{3.7}$$

Similarly, we have

$$\begin{aligned}
\|v_k - y^*\|^2 \leq &\|y_k - y^*\|^2 - \gamma_k \|Ax_k - By_k\|^2 - \gamma_k \|By_k - By^*\|^2 \\
&+ \gamma_k \|Ax_k - By^*\|^2 + \gamma_k^2 \|B^*(Ax_k - By_k)\|^2.
\end{aligned} \tag{3.8}$$

By adding the two last inequalities and by taking into account the fact that $Ax^* = By^*$, we obtain

$$\begin{aligned}
&\|u_k - x^*\|^2 + \|v_k - y^*\|^2 \\
&\leq \|x_k - x^*\|^2 + \|y_k - y^*\|^2 - \gamma_k [2\|Ax_k - By_k\|^2 \\
&\quad - \gamma_k (\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)].
\end{aligned} \tag{3.9}$$

With assumption on γ_k we obtain

$$\|u_k - x^*\|^2 + \|v_k - y^*\|^2 \leq \|x_k - x^*\|^2 + \|y_k - y^*\|^2. \tag{3.10}$$

Setting $\rho = \max\{\rho_1, \rho_2\}$, we have $\rho \in [0, \frac{1}{\sqrt{2}})$. By U and T are quasi-strict pseudo-contractions, we obtain

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 \\
&\leq \alpha_k \|f_1(x_k) - x^*\|^2 + (1-\alpha_k)\|(1-\omega_k)u_k + \omega_k U(u_k) - x^*\|^2 \\
&\leq \alpha_k (\|f_1(x_k) - f_1(x^*)\| + \|f_1(x^*) - x^*\|)^2 + (1-\alpha_k)(1-\omega_k)\|u_k - x^*\|^2 \\
&\quad + (1-\alpha_k)\omega_k \|U(u_k) - x^*\|^2 - (1-\alpha_k)\omega_k(1-\omega_k)\|u_k - U(u_k)\|^2 \\
&\leq 2\alpha_k (\|f_1(x_k) - f_1(x^*)\|^2 + \|f_1(x^*) - x^*\|^2) + (1-\alpha_k)(1-\omega_k)\|u_k - x^*\|^2 \\
&\quad + (1-\alpha_k)\omega_k[\|u_k - x^*\|^2 + t\|u_k - U(u_k)\|^2] \\
&\quad - (1-\alpha_k)\omega_k(1-\omega_k)\|u_k - U(u_k)\|^2 \\
&\leq 2\alpha_k \rho_1^2 \|x_k - x^*\|^2 + 2\alpha_k \|f_1(x^*) - x^*\|^2 + (1-\alpha_k)\|u_k - x^*\|^2 \\
&\quad - (1-\alpha_k)\omega_k(1-\omega_k - t)\|u_k - U(u_k)\|^2
\end{aligned}$$

and

$$\begin{aligned}
&\|y_{k+1} - y^*\|^2 \\
&\leq 2\alpha_k \rho_2^2 \|y_k - y^*\|^2 + 2\alpha_k \|f_2(y^*) - y^*\|^2 + (1-\alpha_k)\|v_k - y^*\|^2 \\
&\quad - (1-\alpha_k)\omega_k(1-\omega_k - t)\|v_k - T(v_k)\|^2.
\end{aligned}$$

Adding up the last two inequalities and using (3.10), setting $s_k = \|x_k - x^*\|^2 + \|y_k - y^*\|^2$, we get

$$\begin{aligned}
s_{k+1} \leq& [1 - \alpha_k(1-2\rho^2)]s_k + 2\alpha_k(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2) \\
&- (1-\alpha_k)\omega_k(1-\omega_k - t)(\|u_k - U(u_k)\|^2 + \|v_k - T(v_k)\|^2) \\
\leq& [1 - \alpha_k(1-2\rho^2)]s_k \\
&+ \alpha_k(1-2\rho^2)\frac{2}{1-2\rho^2}(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2).
\end{aligned} \tag{3.11}$$

It follows from induction that

$$s_k \leq \max\left\{s_0, \frac{2}{1-2\rho^2}(\|f_1(x^*) - x^*\|^2 + \|f_2(y^*) - y^*\|^2)\right\}, \quad k \geq 0,$$

which implies that $\{x_k\}$ and $\{y_k\}$ are bounded. It follows that $\{u_k\}$, $\{v_k\}$, $\{f_1(x_k)\}$ and $\{f_2(y_k)\}$ are bounded.

Setting $\tilde{u}_k = (1-\omega_k)u_k + \omega_k U(u_k)$, $\tilde{v}_k = (1-\omega_k)v_k + \omega_k T(v_k)$, note that U is a quasi-strict pseudo-contraction, we have

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 \\
=&\alpha_k^2\|f_1(x_k) - x^*\|^2 + 2\alpha_k(1-\alpha_k)\langle f_1(x_k) - x^*, \tilde{u}_k - x^*\rangle \\
&+ (1-\alpha_k)^2\|\tilde{u}_k - x^*\|^2 \\
=&\alpha_k^2\|f_1(x_k) - x^*\|^2 + 2\alpha_k(1-\alpha_k)\langle f_1(x_k) - f_1(x^*), \tilde{u}_k - x^*\rangle \\
&+ 2\alpha_k(1-\alpha_k)\langle f_1(x^*) - x^*, \tilde{u}_k - x^*\rangle + (1-\alpha_k)^2\|\tilde{u}_k - x^*\|^2 \\
\leq&\alpha_k^2\|f_1(x_k) - x^*\|^2 + \alpha_k(1-\alpha_k)(\|f_1(x_k) - f_1(x^*)\|^2 + \|\tilde{u}_k - x^*\|^2) \\
&+ 2\alpha_k(1-\alpha_k)\langle f_1(x^*) - x^*, \tilde{u}_k - x^*\rangle + (1-\alpha_k)^2\|\tilde{u}_k - x^*\|^2 \\
\leq&\alpha_k^2\|f_1(x_k) - x^*\|^2 + \alpha_k(1-\alpha_k)\rho_1^2\|x_k - x^*\|^2 + \alpha_k(1-\alpha_k)\|\tilde{u}_k - x^*\|^2 \\
&+ 2\alpha_k(1-\alpha_k)\langle f_1(x^*) - x^*, \tilde{u}_k - x^*\rangle + (1-\alpha_k)^2\|\tilde{u}_k - x^*\|^2 \\
\leq&(1-\alpha_k)\|\tilde{u}_k - x^*\|^2 + \alpha_k(1-\alpha_k)\rho_1^2\|x_k - x^*\|^2 \\
&+ \alpha_k[\alpha_k\|f_1(x_k) - x^*\|^2 + 2(1-\alpha_k)\langle f_1(x^*) - x^*, \tilde{u}_k - x^*\rangle] \\
\leq&(1-\alpha_k)\|(1-\omega_k)u_k + \omega_k U(u_k) - x^*\|^2 + \alpha_k(1-\alpha_k)\rho_1^2\|x_k - x^*\|^2 \\
&+ \alpha_k[\alpha_k\|f_1(x_k) - x^*\|^2 + 2(1-\alpha_k)\langle f_1(x^*) - x^*, \tilde{u}_k - x^*\rangle] \\
\leq&(1-\alpha_k)(1-\omega_k)\|u_k - x^*\|^2 - (1-\alpha_k)\omega_k(1-\omega_k)\|u_k - U(u_k)\|^2 \\
&+(1-\alpha_k)\omega_k[\|u_k - x^*\|^2 + t\|u_k - U(u_k)\|^2] + \alpha_k(1-\alpha_k)\rho_1^2\|x_k - x^*\|^2 \\
&+ \alpha_k[\alpha_k\|f_1(x_k) - x^*\|^2 + 2(1-\alpha_k)\langle f_1(x^*) - x^*, \tilde{u}_k - x^*\rangle] \\
=&(1-\alpha_k)\|u_k - x^*\|^2 - (1-\alpha_k)\omega_k(1-\omega_k-t)\|u_k - U(u_k)\|^2 \\
&+ \alpha_k[\alpha_k\|f_1(x_k) - x^*\|^2 + 2(1-\alpha_k)\langle f_1(x^*) - x^*, \tilde{u}_k - x^*\rangle] \\
&+ \alpha_k(1-\alpha_k)\rho_1^2\|x_k - x^*\|^2.
\end{aligned} \tag{3.12}$$

Similarly, we have

$$\begin{aligned}
&\|y_{k+1} - y^*\|^2 \\
\leq&(1-\alpha_k)\|v_k - x^*\|^2 - (1-\alpha_k)\omega_k(1-\omega_k-t)\|v_k - T(v_k)\|^2 \\
&+ \alpha_k(1-\alpha_k)\rho_2^2\|y_k - y^*\|^2 + \alpha_k[\alpha_k\|f_2(y_k) - y^*\|^2 \\
&+ 2(1-\alpha_k)\langle f_2(y^*) - y^*, \tilde{v}_k - y^*\rangle].
\end{aligned} \tag{3.13}$$

So, by (3.10), (3.12), and (3.13) we obtain

$$\begin{aligned}
s_{k+1} \leq& (1-\alpha_k)s_k + \alpha_k(1-\alpha_k)\rho^2 s_k \\
&+ \alpha_k[\alpha_k(\|f_1(x_k) - x^*\|^2 + \|f_2(y_k) - y^*\|^2) \\
&+ 2(1-\alpha_k)(\langle f_1(x^*) - x^*, \tilde{u}_k - x^*\rangle + \langle f_2(y^*) - y^*, \tilde{v}_k - y^*\rangle)] \\
=& (1-\lambda_k)s_k + \lambda_k\delta_k,
\end{aligned} \tag{3.14}$$

where

$$\lambda_k = \alpha_k(1-(1-\alpha_k)\rho^2),$$

$$\delta_k = \frac{2(1-\alpha_k)(\langle f_1(x^*)-x^*, \tilde{u}_k - x^*\rangle + \langle f_2(y^*)-y^*, \tilde{v}_k - y^*\rangle)}{1-(1-\alpha_k)\rho^2} + \frac{\alpha_k(\|f_1(x_k)-x^*\|^2 + \|f_2(y_k)-y^*\|^2)}{1-(1-\alpha_k)\rho^2}.$$

On the other hand, from (3.1) we have

$$\begin{aligned}
&\|x_{k+1}-x^*\|^2\\
\le&\alpha_k\|f_1(x_k)-x^*\|^2 + (1-\alpha_k)\|(1-\omega_k)u_k + \omega_k U(u_k) - x^*\|^2\\
\le&\alpha_k\|f_1(x_k)-x^*\|^2 + (1-\alpha_k)(1-\omega_k)\|u_k - x^*\|^2\\
&+ (1-\alpha_k)\omega_k[\|u_k - x^*\|^2 + t\|u_k - U(u_k)\|^2]\\
&- (1-\alpha_k)\omega_k(1-\omega_k)\|u_k - U(u_k)\|^2\\
=&\alpha_k\|f_1(x_k)-x^*\|^2 + (1-\alpha_k)\|u_k - x^*\|^2\\
&- (1-\alpha_k)\omega_k(1-\omega_k - t)\|u_k - U(u_k)\|^2
\end{aligned}$$

and

$$\begin{aligned}
\|y_{k+1}-y^*\|^2 \le&\alpha_k\|f_2(y_k)-y^*\|^2 + (1-\alpha_k)\|v_k - y^*\|^2\\
&- (1-\alpha_k)\omega_k(1-\omega_k - t)\|v_k - T(v_k)\|^2.
\end{aligned}$$

Using (3.9), we obtain

$$\begin{aligned}
&s_{k+1}\\
\le&\|u_k - x^*\|^2 + \|v_k - y^*\|^2 + \alpha_k(\|f_1(x_k)-x^*\|^2 + \|f_2(y_k)-y^*\|^2)\\
&- (1-\alpha_k)\omega_k(1-\omega_k - t)(\|U(u_k)-u_k\|^2 + \|T(v_k)-v_k\|^2)\\
\le&\|x_k - x^*\|^2 + \|y_k - y^*\|^2 - \gamma_k[2\|Ax_k - By_k\|^2\\
&- \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)]\\
&+ \alpha_k(\|f_1(x_k)-x^*\|^2 + \|f_2(y_k)-y^*\|^2)\\
&- (1-\alpha_k)\omega_k(1-\omega_k - t)(\|U(u_k)-u_k\|^2 + \|T(v_k)-v_k\|^2).
\end{aligned} \tag{3.15}$$

Now, by setting

$$\mu_k = \alpha_k(\|f_1(x_k)-x^*\|^2 + \|f_2(y_k)-y^*\|^2),$$

$$\begin{aligned}
\eta_k =& \gamma_k[2\|Ax_k - By_k\|^2 - \gamma_k(\|A^*(Ax_k - By_k)\|^2 + \|B^*(Ax_k - By_k)\|^2)]\\
&+ (1-\alpha_k)\omega_k(1-\omega_k - t)(\|U(u_k)-u_k\|^2 + \|T(v_k)-v_k\|^2),
\end{aligned}$$

(3.15) can be rewritten as the following form,

$$s_{k+1} \le s_k - \eta_k + \mu_k, \quad k \ge 0. \tag{3.16}$$

By the assumption on α_k, we get $\sum_{k=0}^{\infty} \lambda_k = \infty$ and $\lim_{k\to\infty} \mu_k = 0$ which thanks to the boundedness of $\{x_k\}$ and $\{y_k\}$.

To use Lemma 2.6, it suffices to verify that, for all subsequence $\{k_l\} \subset \{k\}$, $\lim_{l\to\infty} \eta_{k_l} = 0$ implies

$$\limsup_{l\to\infty} \delta_{k_l} \leq 0. \tag{3.17}$$

It follows from $\lim_{l\to\infty} \eta_{k_l} = 0$ that

$$\lim_{l\to\infty} \gamma_{k_l}[2\|Ax_{k_l} - By_{k_l}\|^2 - \gamma_{k_l}(\|A^*(Ax_{k_l} - By_{k_l})\|^2 + \|B^*(Ax_{k_l} - By_{k_l})\|^2)] = 0,$$

and

$$\lim_{l\to\infty} (1-\alpha_{k_l})\omega_{k_l}(1-\omega_{k_l}-t)(\|U(u_{k_l}) - u_{k_l}\|^2 + \|T(v_{k_l}) - v_{k_l}\|^2) = 0.$$

In light of the assumption on γ_k, $\alpha_k \to 0$ and

$$0 < \liminf_{k\to\infty} \omega_k \leq \limsup_{k\to\infty} \omega_k < 1-t,$$

we obtain

$$\lim_{l\to\infty} \|Ax_{k_l} - By_{k_l}\| = \lim_{l\to\infty} \|U(u_{k_l}) - u_{k_l}\| = \lim_{l\to\infty} \|T(v_{k_l}) - v_{k_l}\| = 0. \tag{3.18}$$

So,

$$\lim_{l\to\infty} \|u_{k_l} - x_{k_l}\| = \lim_{l\to\infty} \gamma_{k_l}\|A^*(Ax_{k_l} - By_{k_l})\| = 0 \tag{3.19}$$

and

$$\lim_{l\to\infty} \|v_{k_l} - y_{k_l}\| = \lim_{l\to\infty} \gamma_{k_l}\|B^*(Ax_{k_l} - By_{k_l})\| = 0. \tag{3.20}$$

Taking $(\tilde{x}, \tilde{y}) \in \omega_w(x_{k_l}, y_{k_l})$, from (3.19) and (3.20) we have $(\tilde{x}, \tilde{y}) \in \omega_w(u_{k_l}, v_{k_l})$. Combined with the demiclosednesses of $U - I$ and $T - I$ at 0, (3.18) yields $U\tilde{x} = \tilde{x}$ and $T\tilde{y} = \tilde{y}$. So $\tilde{x} \in F(U)$ and $\tilde{y} \in F(T)$. On the other hand, $A\tilde{x} - B\tilde{y} \in \omega_w(Ax_{k_l} - By_{k_l})$ and weakly lower semicontinuity of the norm imply

$$\|A\tilde{x} - B\tilde{y}\| \leq \liminf_{l\to\infty} \|Ax_{k_l} - By_{k_l}\| = 0,$$

hence $(\tilde{x}, \tilde{y}) \in \Gamma$. So $\omega_w(x_{k_l}, y_{k_l}) \subset \Gamma$. Since $\lim_{k\to\infty} \alpha_k(\|f_1(x_k) - x^*\|^2 + \|f_2(y_k) - y^*\|^2) = 0$ and $\lim_{k\to\infty}(1-(1-\alpha_k)\rho^2) = 1-\rho^2$, to get (3.17), we only need to verify

$$\limsup_{l\to\infty}(\langle f_1(x^*) - x^*, \tilde{u}_{k_l} - x^*\rangle + \langle f_2(y^*) - y^*, \tilde{v}_{k_l} - y^*\rangle) \leq 0.$$

Indeed, from (3.18), (3.19) and (3.20) we have

$$\begin{aligned}
&\limsup_{l\to\infty}\big(\langle f_1(x^*)-x^*,\tilde{u}_{k_l}-x^*\rangle+\langle f_2(y^*)-y^*,\tilde{v}_{k_l}-y^*\rangle\big)\\
&=\limsup_{l\to\infty}\big(\langle f_1(x^*)-x^*,(1-\omega_{k_l})u_{k_l}+\omega_{k_l}U(u_{k_l})-x^*\rangle\\
&\quad+\langle f_2(y^*)-y^*,(1-\omega_{k_l})v_{k_l}+\omega_{k_l}T(v_{k_l})-y^*\rangle\big)\\
&=\limsup_{l\to\infty}\big(\langle f_1(x^*)-x^*,x_{k_l}-x^*\rangle+\langle f_2(y^*)-y^*,y_{k_l}-y^*\rangle\big)\\
&=-\liminf_{l\to\infty}\big(\langle (I-f_1)x^*,x_{k_l}-x^*\rangle+\langle (I-f_2)y^*,y_{k_l}-y^*\rangle\big).
\end{aligned}\tag{3.21}$$

We can take subsequence $\{(x_{k_{l_j}},y_{k_{l_j}})\}$ of $\{(x_{k_l},y_{k_l})\}$ such that $(x_{k_{l_j}},y_{k_{l_j}})\rightharpoonup(\tilde{x},\tilde{y})$ as $j\to\infty$ and

$$\begin{aligned}
&-\liminf_{l\to\infty}\big(\langle (I-f_1)x^*,x_{k_l}-x^*\rangle+\langle (I-f_2)y^*,y_{k_l}-y^*\rangle\big)\\
&=-\lim_{j\to\infty}\big(\langle (I-f_1)x^*,x_{k_{l_j}}-x^*\rangle+\langle (I-f_2)y^*,y_{k_{l_j}}-y^*\rangle\big)\\
&=-\big(\langle (I-f_1)x^*,\tilde{x}-x^*\rangle+\langle (I-f_2)y^*,\tilde{y}-y^*\rangle\big).
\end{aligned}\tag{3.22}$$

Since $\omega_w(x_{k_l},y_{k_l})\subset\Gamma$ and (x^*,y^*) is the solution of the variational inequality problem (3.3), from (3.21) and (3.22) we obtain

$$\limsup_{l\to\infty}(\langle f_1(x^*)-x^*,\tilde{u}_{k_l}-x^*\rangle+\langle f_2(y^*)-y^*,\tilde{v}_{k_l}-y^*\rangle)\le 0.$$

From Lemma 2.6, it follows

$$\lim_{k\to\infty}(\|x_k-x^*\|^2+\|y_k-y^*\|^2)=0,$$

which implies that $x_k\to x^*$ and $y_k\to y^*$. So, the sequence $\{(x_k,y_k)\}$ strongly converges to the solution (x^*,y^*) of (1.1) which solves the variational inequality problem (3.3). □

Corollary 3.5. *Let H_1, H_2, H_3 be real Hilbert spaces. Given two bounded linear operators $A:H_1\to H_3$, $B:H_2\to H_3$, let $U:H_1\to H_1$ and $T:H_2\to H_2$ be quasi-nonexpansive mapping with constants t_1,t_2 where $t_1,t_2\in[0,1)$, and the solution set Γ of (1.1) is nonempty. Let $t=\max\{t_1,t_2\}$ and the sequence $\{(x_k,y_k)\}$ is generated by Algorithm 3.1. Assume that the following conditions are satisfied:*

(i) $\rho_1,\rho_2\in[0,\frac{1}{\sqrt{2}})$;
(ii) $\lim_{k\to\infty}\alpha_k=0$, $\sum_{k=0}^{\infty}\alpha_k=\infty$;
(iii) *$U-I$ and $T-I$ are demiclosed at origin;*
(v) $\omega_k\in(0,1)$ *such that* $0<\liminf_{k\to\infty}\omega_k\le\limsup_{k\to\infty}\omega_k<1-t$.

Then the sequence $\{(x_k, y_k)\}$ *strongly converges to a solution* (x^*, y^*) *of* (1.1) *which solves the following variational inequality problem:*

$$\begin{cases} \langle (I - f_1)x^*, x - x^* \rangle \geq 0, \\ \langle (I - f_2)y^*, y - y^* \rangle \geq 0, \end{cases} \quad (x, y) \in \Gamma.$$

Corollary 3.6. *Let* H_1*,* H_2*,* H_3 *be real Hilbert spaces. Given two bounded linear operators* $A : H_1 \to H_3$*,* $B : H_2 \to H_3$*, let* $U : H_1 \to H_1$ *and* $T : H_2 \to H_2$ *be strict pseudo-contractions with constants* t_1, t_2 *where* $t_1, t_2 \in [0, 1)$*, and the solution set* Γ *of* (1.1) *is nonempty. Let* $t = \max\{t_1, t_2\}$ *and the sequence* $\{(x_k, y_k)\}$ *is generated by Algorithm* 3.1*. Assume that the following conditions are satisfied:*

(i) $\rho_1, \rho_2 \in [0, \frac{1}{\sqrt{2}})$*;*
(ii) $\lim_{k\to\infty} \alpha_k = 0$*,* $\sum_{k=0}^{\infty} \alpha_k = \infty$*;*
(iii) $\omega_k \in (0, 1)$ *such that* $0 < \liminf_{k\to\infty} \omega_k \leq \limsup_{k\to\infty} \omega_k < 1 - t$*.*

Then the sequence $\{(x_k, y_k)\}$ *strongly converges to a solution* (x^*, y^*) *of* (1.1) *which solves the following variational inequality problem:*

$$\begin{cases} \langle (I - f_1)x^*, x - x^* \rangle \geq 0, \\ \langle (I - f_2)y^*, y - y^* \rangle \geq 0, \end{cases} \quad (x, y) \in \Gamma.$$

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